INFINITARY COMPUTATIONAL
COMPLEXITY WITH INFINITE TIME
TURING MACHINES

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Abstract. Infinite Time Turing Machines provide a suitable model within which we can explore the degrees of infinitary computational complexity on infinitely long inputs. Building upon the undecidability of Cantor’s Continuum Hypothesis from ZFC, we use the ITTM model to relate the cardinality of the sets of computing steps of $P$ and $EXP$ tasks in an ITTM to $\aleph_0$ and $2^{\aleph_0}$, respectively, and thus show the unprovability of existence of complexity classes $X$ such that $P \subset X \subset EXP$, under the ITTM setting.

Keywords. $P$, $NP$, $EXP$, $ZFC$, Continuum Hypothesis, Infinite Time Turing Machines

Subject classification. 68Q15 – Complexity Classes, 03E10 – Ordinal and cardinal numbers

1. Introduction

The “$P = NP$?” problem has been one of the hardest in the history of Mathematics and Computer Science remaining unanswered still. It is known that $P \subseteq NP \subseteq PH \subseteq PSPACE \subseteq EXP$ and also that $P \subset EXP$ (Hartmanis & Stearns (1965)), but it is still unknown whether $P \subset NP$ or $P = NP$ hold, although it is widely suspected that the former is true rather than the latter.

For Infinite Time Turing Machines (ITTMs) it is already known (Hamkins (2005)) that $P \neq NP$. In this paper we take a step further and establish a one-to-one relationship between the cardinality of the sets of “operations” (or “transitions”) taken by an ITTM when deciding, according to a $P$ and a $EXP$ languages, on whether accepting or not a countably infinite input and the transfinite numbers $\aleph_0$ and $2^{\aleph_0}$. With this correspondence established we show that proving the existence of some $X$ such that $P \subset X \subset EXP$, would imply disproving Cantor’s Continuum Hypothesis, which we know is impossible.
to be disproved from the Zermelo-Fraenkel axiomatization with the Axiom of Choice. By contrapositive reasoning we conclude the existence of such an $X$, where $P \subset X \subset \text{EXP}$, cannot be proved from ZFC with ITTMs.

We conclude the paper with a listing of the still open possibilities this result allows, and consider them for future work.

2. The unprovability of $P \subset X \subset \text{EXP}$

Theorem. $P \subset X \subset \text{EXP}$ cannot be proved from ZFC with ITTMs.

Proof. Let $T$ be a deterministic universal Infinite Time Turing machine, and the languages $P$, and $E$ be, respectively, Polynomial, and EXP-complete. Let also $\mathcal{X}$ be any language such that $P \subset \mathcal{X} \subset E$. Let $i_L$ be an input for $P, \mathcal{X}, E$ where $L$ is the length of the input. We now define the sets of individual operations (or transitions) executed by $T$ when running an algorithm over input $i_L$. Since $T$ may execute the same operation more than once, and since for our purposes we need to consider the cardinality of such sets, we need to distinguish any two occurrences of the same operation in the set. We do so by attaching an index to each operation, which corresponds to $T$’s sequential execution order of that transition. Thus

$S_P(i_L) = \{op_0^T(P, i_L), op_1^T(P, i_L), op_2^T(P, i_L), \ldots \}$

$S_X(i_L) = \{op_0^T(X, i_L), op_1^T(X, i_L), op_2^T(X, i_L), \ldots \}$

$S_E(i_L) = \{op_0^T(E, i_L), op_1^T(E, i_L), op_2^T(E, i_L), \ldots \}$

are, respectively, the sets of individual operations (or transitions) performed by $T$ when running corresponding $P, \mathcal{X}, E$ algorithms to decide on whether accepting or not $i_L$. Hence, the set $S_P(i_L)$ (respectively, $S_X(i_L), S_E(i_L)$) above can be seen as a representation of an execution trace of $T$ running a $P$ (respectively, $\mathcal{X}, E$) algorithm over input $i_L$.

By definition of Polynomial, and EXP-complete languages, we have

- $\#S_P(i_L)$ is $O(L^k)$ for some $k \in \mathbb{N}$
- $\#S_E(i_L)$ is $O(2^{L^k})$ for some $k \in \mathbb{N}$

where $L$ is the length of the input $i_L$.

If we now consider countably infinite inputs $i_{\aleph_0}$ we have

- $\#S_P(i_{\aleph_0})$ is $O(\aleph_0^k)$ for some $k \in \mathbb{N}$
\[
\circ \#S_x(i_{\aleph_0}) \text{ is } O(2^{\aleph_0}) \text{ for some } k \in \mathbb{N}
\]
but since \(\aleph_0^k = \aleph_0\) for every \(k \in \mathbb{N}\) we have
\[
\circ \#S_P(i_{\aleph_0}) = \aleph_0
\]
\[
\circ \#S_E(i_{\aleph_0}) = 2^{\aleph_0}
\]

The Continuum Hypothesis (CH), advanced by Georg Cantor in (Cantor (1877)), states \(\nexists \alpha \in \omega \text{ such that } \aleph_0 < \alpha < 2^{\aleph_0}\).

If there is any \(X\) such that \(P \subset X \subset \text{EXP}\) then it must be the case that \(\#S_P(i_{\aleph_0}) < \#S_X(i_{\aleph_0}) < \#S_E(i_{\aleph_0})\), i.e., \(\aleph_0 < \#S_X(i_{\aleph_0}) < 2^{\aleph_0}\), which would mean the Continuum Hypothesis is false. I.e., proving \(P \subset X \subset \text{EXP}\) implies disproving CH. Since Gödel showed (Gödel (1940)) the CH cannot be disproved from the Zermelo-Frankel axiomatization (Zermelo (1930)) — even if the Axiom of Choice (Zermelo (1904)) (ZFC) is adopted —, this means the existence of an \(X\) such that \(P \subset X \subset \text{EXP}\) cannot be proved from ZFC as well for ITTMs. □

### 3. Open possibilities

With this, for ITTMs, either \(P \subset X \subset \text{EXP}\) holds but cannot be proved from ZFC, or \(P \subset X \subset \text{EXP}\) does not hold at all. Since we know that \(P \subset \text{EXP}\), and \(P \subseteq \text{NP} \subseteq \text{PH} \subseteq \text{PSPACE} \subseteq \text{EXP}\), if \(P \subset X \subset \text{EXP}\) does not hold for any \(X\), then for every such \(X\) either \(P = X\) or \(X = \text{EXP}\) (but not both). If indeed \(P \subset X \subset \text{EXP}\) does hold that can only be provable from an axiomatization in some sense richer than ZFC.

### References


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