Stability and interaction in flatline games

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Abstract

Starting from a given one-shot game played by a finite population of agents living in flatline, a circular or constrained grid structured by the classical definitions of neighborhood, we define transformation rules for cellular automata, which are determined by the best-reply behavior in standard two-person symmetric matrix games.

A meaningful concept of solution for the underlying population games will necessarily include robustness against any possible unilateral deviation undertaken by a single player. By excluding the invisible hand of mutation we obtain a purely deterministic population model. The resulting process of cellular transformation is then analyzed for chicken and stag-hunt type cellular games and finally compared with the outcomes of more prominent evolutionary models. Special emphasis is given to an exhaustive combinatorial description of the different basins of attraction corresponding to stable stationary states.

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1. Introduction

In recent years the literature on game theory has experienced a considerable shift towards the study of interactive (learning) behavior. Nash equilibrium—the standard solution concept for strategic form games—had always been defined as a self-stabilizing behavioral expectation. However, neither the scholastic theories on equilibrium refinements nor the sisyphean tasks of equilibrium selection have been able to explain how players know which Nash equilibrium is to be played if the game under consideration has multiple equally reasonable equilibria. But even in the case of a unique plausible equilibrium, the mode of behavior leading to it was in no way clear.
This deficient situation changed completely with the rise of evolutionary dynamics—a new area of research in game theory. Unlike classical games, evolutionary models describe the adaptation process by which a population of agents changes its distribution of types, i.e. of strategies, during a repeated game scenario. Working with a dynamical system means to apply a consistent tool to the analysis of out of equilibrium behavior, to develop an adequate approach for dealing with transition towards stable configurations (related to the equilibria of the corresponding game) and for solving the problem of equilibrium selection.

For excellent surveys on history and literature of adaptive dynamics in population games, see Blume [1], Samuelson [2] and Young [3].

Significant contributions focusing on a broad range of evolutionary learning procedures in agent populations have been written by Kandori et al. [4], Young [3, 5], Blume [1, 6], Ellison [7], Eshel et al. [8], Dawid and Mehlmann [9] and Shaked et al. [10]. Blume [1] and Ellison [7] devote their attention to models of local strategic interaction. Agents are locally imitating a best-reply learning behavior, disrupted by mutations and embedded in a Markov process.

Our approach displays the following differences with the cited literature on local interaction:

(i) We consider a large but denumerable population of agents situated on a linear grid. Agents may be described as bounded rational players which are of two types (strategies). Each agent earns his payoff by interacting only with his nearest neighbors within his interaction neighborhood. The evolutionary setup is described in terms of a one-dimensional cellular automaton which has cells in two possible states: 0 or 1 (say, dead or alive, off or on, white or black, playing the first or the second strategy). Starting from an initial generation called the seed, i.e. the first row of cells with a predetermined pattern of strategies, line after line the next offspring generations are obtained by the simple transformation rule mimicking a myopic best-reply mode of learning. Models where agents are represented as a population of cells in cellular automata, but learn (in contrast to our model) through imitation of successful strategies, have been successfully used by Kirchkamp [11]. Novak et al. [12] and Page et al. [13]. Kirchkamp [11] and Novak et al. [12] present simulation results on the spatial evolution of cooperative behavior imposed by an iterated prisoner’s dilemma. Page et al. [13] perform numerical simulations for one- and two-dimensional spatial ultimate games.

(ii) In order to perfectly understand the unperturbed influence of local myopic rules on the global behavior of a population of economic actors, we deliberately exclude mutational trembles and concentrate on a pure deterministic process of cellular transformation. Indeed, on a (odd length) playing grid shaped by classical game patterns of coordination, we need no amount of noise at all to ensure that the process of cellular transformation selects only (the “risk dominant”) one among several equilibria. There is, however, no problem to introduce noise into cellular automata rules—for example, by reversing the value of a randomly chosen cell $i$ at each time step with probability $z_i$. Wolfram [14] gives a precise analysis of the local properties of cellular automata with noise increasing from zero. As shown by Gach et al. [15], a cellular automaton ultimately visits every possible configuration, when $z_i \neq 0$ for all $i$.

(iii) We furthermore distinguish robust and unstable stationary states of the cellular dynamics defined for the games of chicken and stag-hunt. The basins of attraction corresponding to these equilibria are thoroughly identified by analyzing the iterative rules of cellular transformation. This approach is entirely based on mathematical reasoning; simulation is only used for illustration purposes. As a side effect the total number of steps needed to reach stability from an arbitrary starting point can be explicitly determined from the size and (positional) configuration of homogeneous strategic blocks in an arbitrary seed.
By describing the adaptive process of a population game in discrete computational terms, we definitely offer an artificial paradigm for dealing with the natural mechanisms of social behavior. Instead of exclusively relying on numbers, equations and continuous time adaptation, we represent population states and local transformation rules both by an elementary collection of lists. Nevertheless, our approach should not be classified as experimental mathematics or as a computational technique of implementing behavior in a dynamic system.

It is quite more than that. Wolfram argues in [16] that even systems of quite different provenance can be viewed as computations of equivalent sophistication. By taking a complex structured process of group or individual behavior apart into its constituent pieces we will lastly always deal with a sort of modelling at the cellular level.

Research in discrete dynamical networks of the kind described in this paper has mainly been based on computer experiments using advanced software such as Wuensche’s complex “Discrete Dynamics Lab”, see [17] Our aim is to include mathematical intuition directly into the exploration of the disequilibrium process and to provide a solid characterization of the basins of attraction by analyzing strategic details of configuration in the initial population.

The paper is structured as follows: the local interaction model is described in Section 2. Section 3 contains the main theoretical results on the basins of attraction for chicken type game situations. Section 4 discusses the equilibrium selection problem for stag-hunt type coordination games.

2. From bimatrix games to cellular automata

Let $A$ be the row player’s payoff matrix in a $2 \times 2$ symmetric normal form game and denote by 0 and 1 the two pure strategies available to both players, i.e.:

$$
\begin{pmatrix}
0 & 1 \\
0 & a \\
1 & c & d
\end{pmatrix}
$$

We will assume that this one-shot game is repeatedly played by a finite population of agents living in flatline, a circular or constrained grid structured by the simplest definition of neighborhood. An individual agent $i$—named after his relative location (position $i$) in flatline’s coordinate system—is matched at each discrete point in time, say, $t = 1, \ldots, \tau, \ldots$, with all his neighbors situated in position $i-j$ for $j = \pm 1, \ldots, \pm r$. For the case of a constrained grid the number of neighbors available to a certain agent may depend, however, as well on his distance from the grid’s boundary.

At time $t = 1$ each agent will start the game by selecting an available pure strategy. At each discrete point $t \geq 2$ in time he will act by choosing a best reply strategy to the distribution of strategies played by his opponents at time $t - 1$. Since we consider only (symmetric) $2 \times 2$ games, we may furthermore restrict the best reply to the choice of a pure strategy. This means that even if our agent is indifferent between his two pure strategies, he will not change his current (best reply) strategy. Let $\sum_{i}^{t-1}$ be the pure strategy played by agent $i$ at time $t - 1$ and denote by $p^{t-1}$ the fraction of agents in the neighborhood of $i$ using 0 as their strategy at time $t - 1$. Then the pure strategic choice for agent $i$ at time $t \geq 2$ can be
given by the following myopic best reply rule:

\[
\sum_i t' = \begin{cases} 
0 & \text{if } (a - c)p_i^{t-1} > (d - b)(1 - p_i^{t-1}), \\
0 & \text{if } (a - c)p_i^{t-1} = (d - b)(1 - p_i^{t-1}) \text{ and } \sum_i t'^{t-1} = 0, \\
1 & \text{if } (a - c)p_i^{t-1} = (d - b)(1 - p_i^{t-1}) \text{ and } \sum_i t'^{t-1} = 1, \\
1 & \text{otherwise.}
\end{cases}
\] (1)

The basic framework of locally interactive learning behavior in repeated games described above is evidently analogous to the mathematical model of a cellular automaton. A one-dimensional cellular automaton with a finite total number \(m\) of cells is commonly defined at each discrete time point \(t\) by his state \(s = s_1s_2 \cdots s_m\), where \(s_i\), the \(i\)th component of \(s\), (a binary variable) denotes the value of cell \(i\), and by the local rule describing how the value of each cell will change in terms of a Boolean function of the cells within the neighborhood.

We may now interpret each agent playing strategy 0 (strategy 1) as a cell with value 0 (value 1). Following the notation proposed by John H. Conway for his special cellular automaton, the “Game of Life”, see Gardner [18] and Berlekamp et al. [19] we will speak of dead cells (cells with value 0) and living cells (cells with value 1). Additionally, a local (transformation) rule for our cellular automaton will be defined by the myopic best-reply rule (1).

**Definition 2.1.** By a cellular game corresponding to a given payoff matrix \(A\) we will understand a cellular automaton whose local (transformation) rule is equivalent to the myopic best-reply rule (1).

The following lemma deals with the structural influence of payoff in \(A\) on the local rule in the corresponding cellular game. These results hold as well for cellular games defined on rectangular shaped game boards structured by the Moore or von Neumann definitions of neighborhood (see [Fig. 1]).

**Lemma 2.1.** (1) In cellular games generated by a payoff matrix \(A\) with identical rows no cell will change its value.

(2) Let \(a \neq c\) and \(b = d\). Then any cell without dead neighbors will keep its value. If \(a > c\), any cell with dead neighbors will die or remain dead. If \(a < c\), any cell with dead neighbors will survive or gain new life.
(3) Let \( a = c \) and \( b \neq d \). Then any cell without living neighbors will keep its value. If \( b > d \), any cell with living neighbors will die or remain dead. If \( b < d \), any cell with living neighbors will survive or gain new life.

(4) Let \( a > c \) and \( (a - c)(d - b) < 0 \). Then any cell will die or remain dead.

(5) Let \( a < c \) and \( (a - c)(d - b) < 0 \). Then any cell will survive or gain new life.

(6) Let \( a > c \) and \( (a - c)(d - b) > 0 \). Then

\[
\begin{align*}
&\text{(i) any dead cell will remain dead iff the proportion of living to dead neighbors in its neighborhood does not exceed } (a - c)/(d - b) ; \\
&\text{(ii) any living cell will survive iff the proportion of dead to living neighbors in its neighborhood does not exceed } (d - b)/(a - c); \\
&\text{(7) Let } a < c \text{ and } (a - c)(d - b) > 0. \text{ Then}
\end{align*}
\]

\[
\begin{align*}
&\text{(i) any dead cell will remain dead iff the proportion of dead to living neighbors in its neighborhood does not exceed } (d - b)/(a - c); \\
&\text{(ii) any living cell will survive iff the proportion of living to dead neighbors in its neighborhood does not exceed } (a - c)/(d - b).
\end{align*}
\]

Proof. Follows from the myopic best-reply rule (1). \( \square \)

Let us now consider a (one-dimensional) cellular game variant of the symmetric game of chicken. The payoff matrix is given by

\[
\begin{bmatrix}
0 & 1 & 4 \\
1 & 2 & 3
\end{bmatrix}
\]

and we choose a simple neighborhood radius of \( r = 1 \). This restriction defines eight possible states of neighborhood (death or life for the left, center or right cell). For each state of neighborhood the local (transformation) rule is uniquely determined by the corresponding condition in Lemma 2.1, 7, (i)–(ii). This very rule for updating the grid holds as well for the equivalent payoff matrix

\[
\begin{bmatrix}
0 & 1 & a & b \\
1 & c & d
\end{bmatrix}
\]

with \( c - a = b - d = v \) for an arbitrary strictly positive real value \( v > 0 \), and can now be stated as follows:

<table>
<thead>
<tr>
<th>neighborhood</th>
<th>111</th>
<th>110</th>
<th>101</th>
<th>100</th>
<th>011</th>
<th>010</th>
<th>001</th>
<th>000</th>
</tr>
</thead>
<tbody>
<tr>
<td>output bit</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

By using the common terminology for denoting the rule of a (one-dimensional) cellular automaton, we may identify this cellular game of chicken by translating the binary sequence of output bits 01001101 into the hexadecimal number 4D.

If the cellular game is not defined on a circular grid but on a simple finite constrained grid, the following additional rule (with corresponding hexadecimal number 53) has to be defined for 8 boundary situations
Fig. 2. The one-dimensional cellular game of chicken starting from a near center seed at position 116. The x-axis points (from left to the right) to the 230 positions on the circular cellular grid; the y-axis indicates (from top to the bottom) the first 109 generations.

(the first four for the cell situated at the left margin, the remaining for the right margin cell):

```
boundary neighborhood  11 10 01 00 11 10 01 00
output bit              0 1 0 1 0 0 1 1
```

In Fig. 2 the first 108 iterations for the circular cellular game of chicken have been depicted by using Brian S. Macherone’s “Cellabration” [http://classes.yale.edu/math190a/Fractals/MacSoftware/cellabration.sit.hqx], a computer program supporting the text [20]. The initial generation consists of a single living cell (painted all in black) located in position 116 on a grid of length 230. Due to the transformation rule this cell will never die and will only admit dead neighbors. The cellular game converges against a fixed-point of the transformation rule 4D, where the even (odd) grid positions are populated by dead (living) cells, respectively. The final state corresponds evidently to the unique symmetric equilibrium of the game of chicken, since a cell randomly drawn from this alternating sequence of living and dead cells will be alive with probability 1/2.

3. Chicken on line

**Definition 3.1.** By a circular (constrained) cellular game we will understand a cellular game with a finite total number $m$ of cells situated on a circular (constrained) grid. Let $s = s_1 s_2 \cdots s_m$ be a state of a circular cellular game. The state $\hat{s}$ such that $\hat{s}_j := s_{m+1-j}$ for $j = 1, \ldots, m$ will be called a reflection of $s$. Any state $\sigma$ whose components $\sigma_i; i = 1, \ldots, m$ satisfy the equation

$$
\sigma_i = \begin{cases} 
  s_{i+j} & \text{if } i + j \leq m, \\
  s_{i+j-m} & \text{if } i + j > m,
\end{cases}
$$

for an arbitrary natural number $0 \leq j \leq m - 1$, will be called equivalent to $s$.

It can easily be shown that Eq. (2) defines an equivalence relation between states of a circular cellular game. Two states are equivalent if each can be obtained from the other by simply turning the circular grid’s wheel, say, anticlockwise by a given number $j$ of cells. By choosing a representative state $s$ and by
moving the circular grid’s wheel by \( j = 1, \ldots, m - 1 \) cells we obtain all states of a circular cellular game which are equivalent to \( s \).

Let us place after each move a cut through the grid right between position \( m \) and position 1. This cut generates for each state equivalent to \( s \) a state of the corresponding constrained cellular game. Note, however, that states of a constrained cellular game which correspond to equivalent states of a circular cellular game are structurally different due to the limited neighborhood at both ends of the constrained grid.

**Definition 3.2.** Let \( \langle CG \rangle^o_m \) be a circular cellular game of chicken with a finite total number \( m \) of cells and denote by \( \langle CG \rangle^{\circ \triangleleft}_m \) the corresponding constrained game of chicken. By a stationary state \( s^o \) of \( \langle CG \rangle^o_m \) we will understand a fixed-point of the local transformation rule 4D. A stationary state \( s^{\circ \triangleleft} \) of \( \langle CG \rangle^{\circ \triangleleft}_m \) is defined as fixed-point of the local transformation rules 4D and 53.

Any state \( \sigma \) which is equivalent to \( s^o \) or to the reflection of \( s^o \) will be as well a stationary state of \( \langle CG \rangle^o_m \).

**Proposition 3.1.** (1) No stationary state will contain more than two consecutive cells of equal value.

(2) Let \( s^o \) be a stationary state of \( \langle CG \rangle^o_m \). If neither the first two nor the last two cells of \( s^o \) are of equal value, then this state is as well a stationary state of \( \langle CG \rangle^{\circ \triangleleft}_m \).

(3) If the values of the first two cells in \( s^o \) coincide, move the circular grid’s wheel anticlockwise by one cell to generate a stationary state \( s^{\circ \triangleleft} \) of \( \langle CG \rangle^{\circ \triangleleft}_m \) satisfying:

\[
 s^{\circ \triangleleft}_i = \begin{cases} 
 s^o_{i+1} & \text{if } i = 1, \ldots, m - 1, \\
 s^o_1 & \text{if } i = m. 
\end{cases}
\]

(4) If the values of the last two cells in \( s^o \) coincide, move the circular grid’s wheel clockwise by one cell (or anticlockwise by \( m - 1 \) cells) to generate a stationary state \( s^{\circ \triangleleft} \) of \( \langle CG \rangle^{\circ \triangleleft}_m \) satisfying:

\[
 s^{\circ \triangleleft}_i = \begin{cases} 
 s^o_{i-1} & \text{if } i = 2, \ldots, m \\
 s^o_m & \text{if } i = 1. 
\end{cases}
\]

**Proof.** (1) Assume that a stationary state contains a group with at least three consecutive cells of equal value. Then — according to the general transformation rule 4D — any cell cornered by neighbors belonging to the same group will change its value. This is in contradiction to the postulated stationarity.

(2) The complementary rule for cells situated at the boundary implies that no state, where the first two or the last two cells are of equal value, can qualify as a stationary state of \( \langle CG \rangle^{\circ \triangleleft}_m \).

(3) Let \( s^o \) be a stationary state of \( \langle CG \rangle^o_m \) and assume that the first two cells have the same value. Since \( s^o \) is located on a circular grid the general transformation rule 4D implies that the last cell (in position \( m \)) will not have its value in common with the first two cells. To adapt our stationary state to a finite constrained grid we will have to cut through the circular grid so that neither the first two nor the last two positions of the new grid are occupied by cells of equal value. So the adequate position for cutting through is right between the first and the second cell on the circular grid.

(4) Use similar arguments as in step 3. The adequate position for cutting through the circular grid is right between the cells \( m - 1 \) and \( m \). \( \Box \)
Note that stationarity has to be adapted to become a meaningful concept of solution for the underlying population game. Our suggestion is to include robustness against any possible unilateral deviation undertaken by a single player, i.e. a single cell. This property may be interpreted as a minimal variant of evolutionary stability, i.e. a state possessing this stability property will easily withstand the pressure of mutation exercised by the smallest possible proportion of the whole population of cells.

**Definition 3.3.** By a stable state we understand a stationary state \( s \) of \( \langle CG \rangle_m \) (or of \( \langle CG \rangle_m^{-\circ} \)) which will be restored (by the process of cellular transformation) from each unilateral deviation \( s_i, i = 1, \ldots, m \),

\[
i s_j = \begin{cases} 
  s_j & \text{if } j \neq i. \\
  1 - s_i & \text{if } j = i.
\end{cases}
\]

This means, that we will have to exclude the two simplest situations from our consideration. If our circular or constrained cellular game consists of only two cells, there exist only four possible states. None of the two possible stationary states will satisfy the stability property, due to the fact that the only possible unilateral deviations will generate either two living or two dead cells. Such states are, however, caught in a common cycle. A similar argument holds as well in the case of three cells. We may always define a unilateral deviation transforming the original stationary state into a state with equal value cells.

All cellular games considered in this section will, therefore, contain more than three cells, i.e. \( m > 3 \).

**Proposition 3.2.** (1) No stable state will contain two consecutive cells of equal value.

(2) A circular or constrained cellular game of chicken with an even number \( m \) of cells admits only two stable states, \( 01s \) and \( 10s \), satisfying each:

\[
01s_i = \begin{cases} 
  0 & \text{for } i \text{ odd}, \\
  1 & \text{for } i \text{ even}.
\end{cases} \quad 10s_i = \begin{cases} 
  1 & \text{for } i \text{ odd}, \\
  0 & \text{for } i \text{ even}.
\end{cases}
\]

(3) A constrained cellular game of chicken with an odd number \( m \) of cells admits only \( 01s \) and \( 10s \) as stable states.

(4) A circular cellular game of chicken with an odd number \( m \) of cells admits no stable state.

**Proof.** (1) Assume that the stable state \( s \) contains (at least) a pair of consecutive cells of equal value. Denote this pair’s location on the circular grid by the positions 2 and 3. We can always turn the circular grid’s wheel by the necessary number of cells to satisfy this assumption. Choose cell 4 as deviating neighbor. A deviating cell on position 4 implies that our pair is now included in an even greater group of equal value cells consisting either of three, or of four or lastly of five members. Other group sizes are not admissible due to the stationarity of \( s \). After a single iteration in the first two of the above three cases and after two steps in the third case we arrive at stationary states where the pair of cells situated on positions 2 and 3 have different values (the value of cell 2 remaining always unchanged). But this simply means that, in contradiction to the postulated stability, the state \( s \) can not be restored from the specified unilateral deviation.

(2) Let the number of cells in our grid be even. Since no stable state will contain two consecutive cells of equal value, only two candidates for stability are left, the state \( 01s \) and its reflection \( 10s \). Both candidates are evidently stable. Start with an arbitrary unilateral deviation. In the circular cellular game the deviating cell is always cornered by two cells of equal value. Both these corner cells will have as well a neighbor of opposite value and therefore no reason to change their own values. The deviator is,
A. Mehlmann / Computers & Operations Research ( ) – 9

Fig. 3. Restoring the two stable states in the circular game of chicken with 4 cells.

Fig. 4. Attaining the stable states in the constrained game of chicken with 4 cells.

Fig. 5. The one-dimensional cellular game of chicken starting in “Cellabration” from a random seed (96.52% initially alive; after 37 iterations an unstable stationary state of 50% alive is reached). The x-axis points (from left to the right) to the 230 positions on the circular cellular grid; the y-axis indicates (from top to the bottom) the first 109 generations.

However, forced to return to his former state. A single iteration step will restore the candidates \(01s\) and \(10s\). In Fig. 3 the hexadecimal numbers represent each a state in the circular cellular game of chicken on a grid of length 4. The two stable states “0101” and “1010” have only a finite number of predecessors in the process of cellular transformation. Each such state corresponds uniquelly to a unilateral deviation from stability performed by a certain predetermined cell.

This line of arguments extends in principle to the constrained cellular game of chicken. The only exception occurs if one of the neighbors cornering the deviator is the grid’s first or last cell. In that case both the deviator and the borderline neighbor will switch to the deviator’s original value and a second iteration is additionally needed to restore the stable state. From Fig. 4 we can easily deduce that the set of possible predecessors to a stable state contains in addition to any (state of) unilateral deviation also transient states which qualify as stationary in the corresponding circular game (see e.g. C and 3).

(3) For a constrained cellular game of chicken with an odd number of cells the two states \(01s\) and \(10s\) are in fact each its own reflection and the only stable states.

(4) In a circular cellular game of chicken with an odd number of cells any state will automatically contain at least two consecutive cells of equal value (Figs. 5–8). For the states \(01s\) and \(10s\) take the first and last cell, respectively. This completes the proof. □
Fig. 6. The one-dimensional cellular game of chicken starting in “Cellabration” from a random seed (98.26% initially alive; after 66 iterations a stable stationary state of 50% alive is reached). The x-axis points (from left to the right) to the 230 positions on the circular cellular grid; the y-axis indicates (from top to the bottom) the first 109 generations.

Fig. 7. The one-dimensional cellular stag-hunt game starting in “Cellabration” from a random seed (99% initially alive; after 108 iterations declining 2.6% level of living cells). The x-axis points (from left to the right) to the 230 positions on the circular cellular grid; the y-axis indicates (from top to the bottom) the first 109 generations.

Fig. 8. The one-dimensional cellular stag-hunt game starting in “Cellabration” from a random seed (99% initially alive; after 66 iterations cycling at 50% level). The x-axis points (from left to the right) to the 230 positions on the circular cellular grid; the y-axis indicates (from top to the bottom) the first 109 generations.
Let \( \mu \) denote a binary value, i.e. 0 or 1. Each given initial state containing at least two cells of different value will be transformed to a stationary state with an easily identifiable final configuration. We shall focus now our attention on initial states containing at least an even length block of equal valued cells cornered by cells of opposite value.

**Lemma 3.1.** The process of cellular transformation will replace each block of 2\( k \) cells of equal value 1 – \( \mu \), cornered by (at least) two cells of opposite value \( \mu \), after \( k - 1 \) iterations by the following unstable, but stationary, configuration, cornered by two cells of value \( \mu \):

1. For \( k \) even:

\[
(1 - \mu) \mu(1 - \mu) \cdots \mu(1 - \mu) \mu(1 - \mu) \cdots \mu(1 - \mu) \mu(1 - \mu),
\]

2. For \( k \) odd:

\[
(1 - \mu) \mu \cdots (1 - \mu) \mu(1 - \mu)(1 - \mu) \mu(1 - \mu) \cdots \mu(1 - \mu).
\]

**Proof.** For \( k = 1 \), a block consisting of 2\( k \) equal valued cells coincides with a stationary configuration of type (4).

Let us assume that the statements of Lemma 3.1 are as well valid for a block of 2\( n \) cells. Consider now a block of 2\( (n + 1) \) cells of equal value 1 – \( \mu \) cornered by (at least) two cells of opposite value \( \mu \). By applying the process of cellular transformation on this block we arrive after one step at the following configuration:

\[
(1 - \mu) \mu \cdots \mu(1 - \mu).
\]

But according to our assumption the block

\[
\mu \cdots \mu
\]

cornered by two cells of opposite value 1 – \( \mu \) will be transformed after \( n - 1 \) steps to one of the following two configurations

1. For \( n \) even:

\[
\mu \mu(1 - \mu) \cdots (1 - \mu) \mu(1 - \mu) \mu(1 - \mu)(1 - \mu) \mu(1 - \mu) \cdots (1 - \mu) \mu
\]
(2) for \( n \) odd:
\[
\mu \left( \frac{1 - \mu}{2} \right)^{n-1} \mu \left( \frac{1 - \mu}{2} \right)^{n-1} \mu \left( \frac{1 - \mu}{2} \right)^{n-1} \mu \left( \frac{1 - \mu}{2} \right).
\]

(These configurations may be derived from those stated in Lemma 3.1 by simply replacing \( \mu \) by \( 1 - \mu \) and vice versa.)

By adjoining the two corner cells, whose values \( 1 - \mu \) will not be changed by the process of cellular transformation, we arrive as well for \( k = n + 1 \) at the stationary configurations (3) and (4).

Note that each stationary configuration described in Lemma 3.1 is unstable, since it contains two consecutive cells of equal value. Any initial state containing at least an even length block of equal valued cells cornered by cells of opposite value will therefore not be transformed to a stable stationary state.

The basins of attraction for stable stationary states in circular or constrained games of chicken can now easily be described.

**Proposition 3.3.** The process of cellular transformation defined by the rule 4D in a circular game of chicken, \((CG)_m^0\), with an even number \( m > 2 \) of cells, converges to one of the two unique stable states \( 01s, 10s \), if, and only if, the initial starting point is equivalent to a state

\[
m\widetilde{\sigma} = \mu \cdots \mu \left( \frac{1 - \mu}{2} \right) \cdots \mu \left( \frac{1 - \mu}{2} \right) \mu \cdots \mu \left( \frac{1 - \mu}{2} \right) \mu \left( \frac{1 - \mu}{2} \right)
\]

with \( r \) even, \( m_i \) odd for \( i = 1, \ldots, r \) and \( \sum_{i=1}^{r} m_i = m \).

**Proof.** Any initial state which is not equivalent to a state given in (5) will either contain only cells of equal value or will be equivalent to a state containing at least an even length block of equal valued cells cornered by (at least) two cells of opposite value. In both situations the process of cellular transformation will not converge to one of the two unique stable states \( 01s, 10s \). Therefore the assumption that an initial state is equivalent to \( m\widetilde{\sigma} \) is necessary for the convergence of the cellular transformation process to a stable state. The following lemma shows that this assumption is also sufficient.

To be precise, the state \( m\widetilde{\sigma} \) contains an even sequence of odd length blocks consisting each of equal valued cells, whereas no pair of adjacent blocks will have the same cell values in common. Note, that for each admissible choice of \( r, m_1 \cdots m_r \) we obtain a different set of states, say \( \{m\widetilde{\sigma}^j_0\} \), which are equivalent to a corresponding \( m\widetilde{\sigma} \). Each \( m\widetilde{\sigma}^j_0 \) can be derived from \( m\widetilde{\sigma} \) (or from its reflection) by simply turning the circular grid’s wheel, say anticlockwise, by a given number of cells.

**Lemma 3.2.** The stable stationary state:
\[
\left( \frac{1 - \mu}{2} \right)^{\frac{m}{2}} \left( \frac{1 - \mu}{2} \right)^{\frac{m}{2}} \left( \frac{1 - \mu}{2} \right)^{\frac{m}{2}} \left( \frac{1 - \mu}{2} \right)^{\frac{m}{2}}
\]
will be achieved in a circular game of chicken, \((CG)_m^0\), with an even number \(m > 2\) of cells, after \(m^* := \max_{i=1, \ldots, r} \left\lceil \frac{m_i-1}{2} \right\rceil\) iterations, if the initial state \(m\bar{\sigma}_j^0\) is derived from \(m\bar{\sigma}\) by turning the circular grid’s wheel anticlockwise either

(1) to an odd position in a block consisting of \((1 - \mu)\)-valued cells, or
(2) to an even position in a block consisting of \(\mu\)-valued cells.

**Proof.** The process of cellular transformation will replace the \(j\)th block of \(m\bar{\sigma}\) after \((m - 1)/2\) steps by the following stable configuration:

(1) for \(j\) even:

\[
\begin{array}{c}
(1 - \mu) \mu \cdots (1 - \mu) \mu (1 - \mu), \\
2 \quad 2 \\
\end{array}
\]

\[
\frac{m_j - 1}{2}
\]

(2) for \(j\) odd:

\[
\begin{array}{c}
\mu (1 - \mu) \cdots \mu (1 - \mu) \mu , \\
2 \quad 2 \\
\end{array}
\]

\[
\frac{m_j - 1}{2}
\]

The proof is easily done by induction along similar lines to those given in the proof of Lemma 3.1. As an immediate consequence, \(m\bar{\sigma}\) will be transformed after \(m^* := \max_{i=1, \ldots, r} \left\lceil \frac{m_i-1}{2} \right\rceil\) steps to the stable stationary state (6). Starting from \(m\bar{\sigma}\) by turning the circular grid’s wheel either to an odd position in a block consisting of \((1 - \mu)\)-valued cells or to an even position in a block consisting of \(\mu\)-valued cells we derive the desired result. \(\Box\)

In Lemma 3.1 and Proposition 3.3 we have demonstrated, that in a circular game of chicken no state containing at least an even length block of equal valued cells will belong to the basins of attraction of the two unique stable states \(01^s\), \(10^s\). If the same game is, however, played on a constrained grid, there exist certain states containing even length blocks at one or both margins, which will as well be transformed to \(01^s\), or to \(10^s\).

By cutting through the circular grid right between position \(m\) and 1, we obtain from each state \(m\bar{\sigma}_j^0\) the corresponding state \(m\bar{\sigma}_j^{\alpha <\alpha}\) of a constrained cellular game of chicken with a finite even length number \(m > 2\) of cells. Let us for instance assume that \(m\bar{\sigma}_j^0\) has been obtained from \(m\bar{\sigma}\) by turning the circular grid’s wheel to an even position in a block consisting of \((1 - \mu)\) valued cells. Then the corresponding state \(m\bar{\sigma}_j^{\alpha <\alpha}\) will start with an even length block of \((1 - \mu)\)-valued cells and will include at the end of the grid an odd number of remaining, \((1 - \mu)\)-valued, block elements. The total number of blocks in \(m\bar{\sigma}_j^{\alpha <\alpha}\) is now \(r + 1\).
We may as well define by \( m\bar{\sigma}^{\text{>}} \) the following (type of) state in a constrained cellular game of chicken with a finite even length number \( m > 2 \) of cells:

\[
m\bar{\sigma}^{\text{>}} = \mu \cdot (1-\mu) \cdot \cdots \cdot (1-\mu) \cdot \mu \cdot (1-\mu) \cdot \cdots \cdot (1-\mu)
\]

with \( r \) even, \( \sum_{i=1}^{r} k_i = m \), \( k_i \) odd for \( i = 2, \ldots, r - 1 \), and \( k_i \) even for \( i = 1, i = r \). This state consists of \( r \) blocks of equal valued cells. The first and last block are the only blocks of even length but do not have the same (cell) value in common.

By adjoining a cell of arbitrary value to the left or to the right of \( m-1\bar{\sigma}^{\text{>}} \) we shall define states \( m\bar{\sigma}^{\text{>}} \) and \( m\bar{\sigma}^{\text{<}} \) of a constrained cellular game of chicken with a finite odd number \( m > 3 \) of cells.

The following proposition and lemma are presented without proof, to avoid repetition of argumentative steps and techniques used before.

**Proposition 3.4.** (1) The process of cellular transformation defined by the rules 4D and 53 in a constrained game of chicken, \( (CG)^{m<} \), with an even number \( m > 2 \) of cells, converges against one of the two unique stable states \( 01s, 10s \), if, and only if, the initial starting point is given by a state \( m\bar{\sigma}^{\text{>}} \) or \( m\bar{\sigma}^{\text{<}} \).

(2) The process of cellular transformation defined by the rules 4D and 53 in a constrained game of chicken, \( (CG)^{m<} \), with an odd number \( m > 3 \) of cells, converges against one of the two unique stable states \( 01s, 10s \), if, and only if, the initial starting point is given by a state \( m\bar{\sigma}^{\text{>}} \) or \( m\bar{\sigma}^{\text{<}} \), and does not contain (at one or both of his margins) an even length group of equal valued cells cornered by cells of opposite values.

**Lemma 3.3.** The stable stationary state:

\[
\frac{(1-\mu)\mu (1-\mu)\mu \cdot (1-\mu)\mu}{2} \cdot \frac{2}{2} \cdot \frac{2}{m}
\]

will be achieved in a constrained game of chicken, \( (CG)^{m<} \), with an even number \( m > 2 \) of cells.

1. After \( m^* : = \max\{m_0^0 - 1, m_k - m_0^0 - 1, \max_{i=1, \ldots, r} \lfloor \frac{m_i - 1}{2} \rfloor \} \) iterations, if the initial state \( m\bar{\sigma}^{\text{>}} \) is derived from \( m\bar{\sigma} \) by turning the circular grid’s wheel anticlockwise to the odd position \( m_k^0 \) in the \( k \)-th block consisting of \( (1-\mu) \)-valued cells.

2. After \( m^* : = \max\{m_1^0 - 1, m_l - m_1^0 - 1, \max_{i=1, \ldots, r} \lfloor \frac{m_i - 1}{2} \rfloor \} \) iterations, if the initial state \( m\bar{\sigma}^{\text{>}} \) is derived from \( m\bar{\sigma} \) by turning the circular grid’s wheel anticlockwise to the even position \( m_l^0 \) in the \( l \)-th block consisting of \( \mu \)-valued cells.

3. After \( m^* : = \max\{m_1 - 1, m_r - 1, \max_{i=2, \ldots, r-1} \lfloor \frac{m_i - 1}{2} \rfloor \} \) iterations, if the initial state is given by \( m\bar{\sigma}^{\text{<}} \).
4. Cellular games of coordination

The stag-hunt game

\[
\begin{array}{c|c|c}
0 & 1 \\
\hline
2 & 4 \\
1 & 0 \\
\end{array}
\]

has often been used as a benchmark for comparing the predictive power of different dynamic approaches in (evolutionary) game theory. Whereas the replicator dynamics was not able to persuasively answer the question of choosing between the strict Nash equilibria of this game, the KMR-model, see Kandori et al. [4] as well as Ellison’s local coordination model [7] put all their focus on the choice of the “risk dominant” equilibrium (0, 0).

Both models use a simple mix of noise (mutation) and myopic response of players on past behavior of their opponents. Ellison [7] considers the case of local interaction allowing for a large variety of matching processes within the agent population. The dynamics of his model for the special case of uniform matching in a neighborhood containing only two agents coincides with those imposed by the best-reply myopic rule (1).

Although a cellular game represents in fact a purely deterministic model, a similar behavior as in [4] and [7] can be observed for cellular games derived from a stag-hunt type game. It can e.g. be shown that for the cellular stag-hunt game played, say, on a circular grid, the state corresponding to the “risk dominant” equilibrium qualifies as the unique stable one. Due to the use of cellular automata, our analysis of different basins of attraction of stationary states is more detailed than Ellison’s. Moreover, in contrast to the findings in [4] and [7] we need no noise at all to assure that (on an odd length circular grid) any initial population of heterogeneous agents will quickly reach the stable state.

For the lowest radius of neighborhood \( r = 1 \), we get the same cellular game for each coordination game:

\[
\begin{array}{c|c|c}
0 & a & b \\
\hline
1 & c & d \\
\end{array}
\]

where \( a - c = v > d - b = w > 0 \) with real values \( v, w \). Note that for the special game

\[
\begin{array}{c|c|c}
0 & v & 0 \\
\hline
1 & 0 & w \\
\end{array}
\]

belonging as well to the above mentioned class of equivalent coordination games, the stable state will correspond to the payoff dominant equilibrium.

Let \((S H G)^O_m\) \((SHG)_m^{\gg<}\) be a circular (constrained) cellular stag-hunt game with a finite total number \( m \) of cells. The cellular stag-hunt transformation rule on circular grids may be easily derived from Lemma 2.1, 6, (i)–(ii) as (hexadecimal number A0):

<table>
<thead>
<tr>
<th>neighborhood</th>
<th>111</th>
<th>110</th>
<th>101</th>
<th>100</th>
<th>011</th>
<th>010</th>
<th>001</th>
<th>000</th>
</tr>
</thead>
<tbody>
<tr>
<td>output bit</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
On constrained grids we have additionally (hexadecimal number AC):

\[
\begin{array}{cccccccc}
\text{boundary neighborhood} & 11 & 10 & 01 & 00 & 11 & 10 & 01 & 00 \\
\text{output bit} & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
\end{array}
\]

**Proposition 4.1.** (1) Both the circular and the constrained cellular stag-hunt game admit only two stationary states 0s and 1s, satisfying:

\[0s_i = 1s_i - 1 = 0; \quad i = 1, \ldots, m,\]

and corresponding to the strict Nash equilibria of the (stag-hunt) matrix game.

(2) With the notable exception of the circular cellular stag-hunt game played on an odd length grid, both the circular and the constrained cellular stag-hunt game admit additionally a sort of stationary equivalence class consisting of the two states 01s and 10s, with:

\[01s_i = \begin{cases} 0 & \text{for } i \text{ odd}, \\ 1 & \text{for } i \text{ even} \end{cases}, \quad 10s_i = \begin{cases} 1 & \text{for } i \text{ odd}, \\ 0 & \text{for } i \text{ even}. \end{cases}\]

As soon as the process of cellular transformation leads to a state belonging to this class, it will never leave it by cycling forever between the class members.

(3) Both the circular and the constrained cellular stag-hunt game admit (for \(m > 2\) and \(m > 3\), respectively,) the state 0s, corresponding to the risk dominant, strict Nash equilibrium of the (stag-hunt) matrix game, as the unique stable state.

**Proof.** (1) Both 0s and 1s are evidently stationary states. Now let us assume that there exists an additional stationary state, say, \(\hat{s}\). This means that \(\hat{s}\) contains (at least) two consecutive cells with opposite values. If these two cells are situated at the boundary of a constrained grid, then a living corner cell will die and a dead corner cell will gain life. If the two cells are situated either in the interior of a bounded or at an arbitrary position on a circular grid, then from the cellular transformation rules it follows that at least one of the two cells will change its status. This contradicts the postulated stationarity of \(\hat{s}\).

(2) In a circular cellular stag-hunt game played on an odd length grid, say \(m\), both states 01s and 10s are absorbed by 0s (01s) in \((m - 1)/2\) transformation steps; 10s needs exactly one step more. In all other cases 01s is transformed to 10s and vice versa.

(3) Each deviation from 0s undertaken by a single cell in a circular or constrained cellular game leads to a transient state from which the original stationary situation is quickly restored (in one step) by the cellular transformation rules.

To prove the uniqueness of 0s as a stable state, let us concentrate on deviations from the second stationary state 1s. We will at first restrict our attention to the cellular game played on a circular grid consisting of \(m\) cells. Each state derived by an arbitrary deviant behavior is equivalent to 0 1 \(\ldots\) 1 \(m-1\).

Denote by \(k\) the greatest integer such that \(2k < m - 1\). After \(k\) transformations according to the rule A0, we obtain from 0 1 \(\ldots\) 1 the state:

\[01 \ldots 01 \underbrace{0 \ldots 1}_{m-1} \underbrace{01 \ldots 01}_{k} \underbrace{01 \ldots 01}_{m-2k-1} \underbrace{01 \ldots 01}_{k},\]
for an even integer $k$, and

\[
\begin{array}{c}
  10\ldots101\ldots1010\ldots10, \\
  k+1 \quad m-2k-1 \quad k-1
\end{array}
\]

otherwise.

If our circular grid contains an even number of cells, then $m - 2k - 1 = 1$ and the middle group consisting of consecutive 1’s degenerates to a single 1. For $k$ even, we arrive at $01s$; $10s$ is the transformed state for $k$ odd. But each of these two states is the result of applying the cellular transformation rule $A0$ on the other one. So the one sided deviation from $1s$ leads, for the case of an even number of cells, to a final cycling between the two states: $01s$ and $10s$.

If, on the contrary, our circular grid contains an odd number of cells, then $m - 2k - 1 = 0$ and the middle group consisting of consecutive 1’s vanishes, leaving place for a block of two 0’s cornered by 1’s. But from the cellular rule $A0$, it immediately follows, that each state containing a block of two or more zeros will be transformed (after finitely many iterations) into the stable state $0s$. So the one sided deviation from $1s$ converges, for the case of an odd number of cells, to $0s$.

This means that a cellular stag-hunt game played on a circular grid admits only $0s$ as a stable state. □

The dynamics of cellular transformation concentrates on a reduced best-reply reaction of a finite number of agents at a local level of neighborhood. Neither the invisible hand of mutation nor the qualitative power of a global difference or differential equations system is needed to chose between equilibria.

Denote by $m\hat{\sigma}_j^0$, a state of the circular stag-hunt game with a finite even length number $m > 2$ of cells, which is equivalent to

\[
m\hat{\sigma} = 01\ldots101\ldots101\ldots1
\]

with $1 \leq r \leq m/2$, $m_i$ odd for $i = 1, \ldots, r$ and $\sum_{i=1}^r m_i = m - r$.

**Proposition 4.2.** (1) Starting from an arbitrary nonstationary state of a circular cellular game, $(SHG)_m^0$, with an odd number $m > 2$ of cells, the transformation rule $A0$ leads after finitely many iterations to the unique stable state $0s$.

(2) Starting from an arbitrary nonstationary state of a circular cellular game, $(SHG)_m^0$, with an even number $m > 2$ of cells, the process of cellular transformation defined by the rule $A0$ stops either after finitely many iterations in the unique stable state $0s$, or, if, and only if, the initial nonstationary state is given by $m\hat{\sigma}_j^0$, it will end by cycling forever between the two members of the equivalence class $\{01s, 10s\}$.

**Proof.** Without loss of generality, we will regard only nonstationary states which do not contain a block of two or more zeros (note that the block could consist as well only of the last and first cell on the grid). States containing blocks of two or more zeros are automatically transformed by the transformation rule $A0$ (after finitely many steps) into the stable state $0s$. 


Each nonstationary state under consideration will therefore contain only isolated cells of value 0 separating blocks of 1-valued cells and can be written as equivalent to

\[
0 \underbrace{1 \ldots 1} \underbrace{0 \ldots 0} \ldots 0 \underbrace{1 \ldots 1},
\]

with \(1 \leq r \leq m/2\), and \(\sum_{i=1}^{r} m_i = m - r\).

By generalizing the block analysis documented in the proof of Proposition 4.1 we may show, that if at least one of the \(m_i\)'s is an even number, then the nonstationary state will be as well transformed after finitely many steps into the stable state \(\sigma_0\).

1. Let us now consider a circular cellular game with an odd number \(m > 2\) of cells. For an even (odd) number \(r\) of isolated 0’s, \(m - r\) would then necessarily be odd (even). But if we assume that all \(m_i\)'s are odd, we get \(\sum_{i=1}^{r} m_i\) even (odd) for \(r\) even (odd). The equation \(\sum_{i=1}^{r} m_i = m - r\) will therefore only hold, if there exists a subset \(\{m_{j_1}, \ldots, m_{j_k}\} \subset \{m_1, \ldots, m_r\}\) with \(k \geq 1\) odd, such that each \(m_{j_l}, l = 1, \ldots, k\), is an even number. This implies, that in a cellular game played on an odd length circular grid, any nonstationary state will be transformed into the stable state \(\sigma_0\).

2. In a circular cellular game with an even number \(m > 2\) of cells, there certainly exist nonstationary states, say \(m^{\sigma_0}_j\), which are equivalent to:

\[
0 \underbrace{1 \ldots 1} \underbrace{0 \ldots 0} \ldots 0 \underbrace{1 \ldots 1},
\]

with \(1 \leq r \leq m/2\), \(m_i\) odd for \(i = 1, \ldots, r\) and \(\sum_{i=1}^{r} m_i = m - r\). By generalizing the single block transformations documented in the proof of Proposition 4.1 we may show, that each such nonstationary state will reach a member of the equivalence class \(\{01^s, 10^s\}\) after exactly \(k^* = \max_{i=1, \ldots, r} (m_i - 1)/2\) steps. For \(k^*\) even (odd) this member is given by \(01^s\) (\(10^s\)).

We may now use a similar approach as in Section 3, to describe the behavior of our cellular game in a play on a constrained grid.

By cutting through the circular grid right between position \(m\) and 1, we obtain from each such state \(m^{\sigma_0}_j\) the corresponding state \(m^{\sigma_{\sigma_0,j}^c}\) of a constrained cellular stag-hunt game with a finite even length number \(m > 2\) of cells.

Define the state \(m^{\sigma_{\sigma_0,j}^c}\) of a constrained cellular stag-hunt game with a finite odd number \(m > 3\) of cells by adjoining a cell of arbitrary value to the left or to the right of \(m^{\sigma_{\sigma_0,j}^c}\).

The following results are given without proof.

**Proposition 4.3.** (1) Starting from an arbitrary nonstationary state of a constrained cellular game, \((SHG)_{m}^{\sigma_{\sigma_0,j}^c}\), with an even number \(m > 3\) of cells, the process of cellular transformation defined by the rules \(A0\) and \(AC\) stops either after finitely many iterations in the unique stable state \(\sigma_0\), or, if, and only if, the initial nonstationary state is given by \(m^{\sigma_{\sigma_0,j}^c}\), it will end by cycling forever between the two members of the equivalence class \(\{01^s, 10^s\}\).

(2) Starting from an arbitrary nonstationary state of a constrained cellular game, \((SHG)_{m}^{\sigma_{\sigma_0,j}^c}\), with an odd number \(m > 3\) of cells, the process of cellular transformation defined by the rules \(A0\) and \(AC\) stops either after finitely many iterations in the unique stable state \(\sigma_0\), or, if, and only if, the initial nonstationary state is given by \(m^{\sigma_{\sigma_0,j}^c}\) and does not contain (at one or both of his margins) an even numbered group...
of consecutive 1’s cornered by single 0’s, it will end by cycling forever between the two members of the equivalence class $\{01^s, 10^s\}$.

5. Conclusion

In this paper we demonstrated how a game played repeatedly and locally by bounded rational, myopic agents can be described—in an equivalent manner—by a cellular automaton. We devoted our attention primarily to structural properties of the dynamic adaptive processes of cellular transformation for the special cases of chicken and stag-hunt type games. We did, moreover, illustrate how the dynamic paths that such an adaptive local interaction process will follow can be adequately predicted starting from an arbitrary initial strategic configuration.

By using cellular automata as a evolutionary tool we have, moreover, been able to persuasively solve the equilibrium selection problem in a class of $2 \times 2$ coordination games. By the very rules of the purely deterministic dynamic process of cellular transformation evolving on an odd length circular grid, the unique stable stationary state, which corresponds to the “risk dominant” equilibrium, has been shown to attract any initial seed containing at least two cells of different value.

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References