Calibration of the default probability model

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Received 1 November 2003; accepted 1 November 2004

Abstract

In this paper, we study the calibration problem for the Merton–Vasicek default probability model [Robert Merton, On the pricing of corporate debt: the risk structure of interest rate, Journal of Finance 29 (1974) 449–470]. We derive conditions that guarantee existence and uniqueness of the solution. Using analytical properties of the model, we propose a fast calibration procedure for the conditional default probability model in the integrated market and credit risk framework. Our solution allows one to avoid numerical integration problems as well as problems related to the numerical solution of the nonlinear equations.

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Keywords: Credit risk; Default probability; Merton–Vasicek model; Calibration

1. Introduction

Calibration of the default probability model in the integrated market and credit risk framework [7] is one of the problems that require intensive computations. Indeed, this problem should be solved separately for each obligor. The number of obligors contained in a corporate portfolio may be very large. Taking into account that the straightforward implementation of the calibration algorithm (algorithm NINS) contains numerical integration and numerical solution of a nonlinear equation [11], we conclude that an efficient solution to this problem becomes critical for an industrial implementation of this model. An alternative solution can be based on reduction of the calibration problem to computation of the bivariate normal integral [14]. The latter approach is faster than NINS algorithm [11] but also requires numerical solution of the nonlinear equation for the parameters of the model.

In this paper, we describe an efficient approach to the calibration of the default probability model based on analytical properties of the calibration equation in the integrated market and credit risk framework. In this paper, we consider the calibration problem for the one step Merton–Vasicek default model. This model is

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2 The integrated market and credit risk model is described in details in [7,6].
described, in particular, in [9,13,12,4–6,8]. This model is actively used in one-factor setting for estimation of asset correlations [2,3].

The calibration procedure studied in this paper is based on matching the first two moments of the conditional default probabilities. We find an approximate solution to the calibration problem that does not require numerical integration or numerical solution of the nonlinear equation. Our solution provides sufficiently small level of relative error acceptable for practical implementation. Our approach allows us to capture asymptotic behaviour of the parameters when the correlation of the obligor’s creditworthiness index and the credit driver is close to 0 as well as close to ±1. In the latter case application of maximum likelihood method usually leads to numerically unstable results.

This paper is organized as follows. In Section 2, the model for conditional default probability of the obligor is introduced. In Section 3.1, we study the moments of the conditional default probabilities. In Section 4, we construct three different approximations of the second moment of the default probability covering different areas of the parameters value. These approximations are obtained using the results of Section 3.1.

In Section 5, we propose a calibration procedure based on the approximations studied in Section 4. In Section 6, we analyze the accuracy of our solution and compare the computation time of the straightforward calibration procedure and that proposed in Section 5.

2. The model

In the integrated market and credit risk model, the default event is governed by a credit worthiness index, \( Y_t \). This index is described by the random process

\[
Y_t = \beta \cdot \xi_t + \alpha \cdot \epsilon_t, \quad t = 1, 2, \ldots, \quad \beta^2 + \alpha^2 = 1, \tag{1}
\]

where both \( \xi_t \) and \( \epsilon_t \) are independent canonical processes [6], i.e. the random variables \( \xi_t \) and \( \epsilon_t \) all have standard normal distribution \( \mathcal{N}(0,1) \), \( t = 1, 2, \ldots \).

The processes \( \xi_t \) and \( \epsilon_t \) are interpreted as follows: \( \xi_t \) represents a systematic component of the market risk (including credit drivers) and \( \epsilon_t \) represents an idiosyncratic component. The default time, \( \tau \), in this model is represented as a first hitting time of the process \( Y_t \)

\[
\tau = \inf_{t>0} \{ t : Y_t < b(t) \},
\]

where the boundary \( b(\cdot) \) is either a deterministic continuous function in the continuous-time case or a deterministic sequence in the discrete-time case.

Consider the default event \( (Y_1 < b) \) in a scenario in which the value of the systematic component is equal to \( x \). Then from (1) we have for \( t = 1 \), \( Y_1 = \beta \cdot x + \alpha \cdot \epsilon_1 \), and in the case of default \( \epsilon_1 < \frac{b-\beta x}{\alpha} \).

Denote \( p = \Pr \{ Y_1 < b \} \). Then \( p = \Phi(b) \), where \( \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du \) is the cumulative distribution function of a standard normal random variable. The conditional probability of default,

\[
P(\xi_1) = \Pr( Y_1 < b | \xi_1),
\]

therefore, is

\[
P(\xi_1) = \Phi(\frac{b - \beta \cdot \xi_1}{\alpha}). \tag{2}
\]

Since \( \xi_1 \) has a standard normal distribution, we can consider the distribution of the conditional default probability \( \eta = P(\xi_1) \). We have from (2)

\[
\Pr\{ \eta < v \} = \Pr\{ \Phi(\frac{b - \beta \cdot \xi_1}{\alpha}) < v \} = \Pr\{ \xi_1 > \frac{b - \alpha \cdot \Phi^{-1}(v)}{\beta} \}, \quad \text{where} \ 0 \leq v \leq 1. \tag{3}
\]

Denote the cumulative distribution function of \( \eta \) by \( F(v) \). Then from (3) we derive

\[
F(v) = \Phi(\frac{b - \alpha \cdot \Phi^{-1}(v)}{\beta}), \tag{4}
\]

where \( \Phi(\cdot) = 1 - \Phi(\cdot) \).

Let us denote by $f_\beta(v)$ the probability density function of $\eta$. Then from (4) we have

$$f_\beta(v) = \lambda \cdot \exp \left( -\frac{1}{2} \left( \gamma^2 - 2 \cdot \gamma \lambda \Phi^{-1}(v) + (\lambda^2 - 1) \cdot (\Phi^{-1}(v))^2 \right) \right),$$

(5)

where

$$\lambda = \frac{a}{b}, \quad \gamma = \frac{b}{b}.$$

It follows from (5) that for sufficiently small unconditional default probabilities $p < 0.5$ we have $\gamma < 0$ and then

$$\lim_{v \to 0} f_\beta(v) = \begin{cases} \infty, & \text{if } \lambda \leq 1, \\ 0, & \text{if } \lambda > 1 \end{cases}$$

and

$$\lim_{v \to 1} f_\beta(v) = \begin{cases} \infty, & \text{if } \lambda < 1, \\ 0, & \text{if } \lambda \geq 1. \end{cases}$$

If $p = 0.5$ then $\gamma = 0$. In this case we have $f_\beta(v) = 1$, if $\lambda = 1$; if $\lambda < 1$ then

$$\lim_{v \to 0} f_\beta(v) = +\infty, \quad \text{and} \quad \lim_{v \to 1} f_\beta(v) = +\infty.$$

If $\lambda > 1$ then

$$\lim_{v \to 0} f_\beta(v) = 0, \quad \text{and} \quad \lim_{v \to 1} f_\beta(v) = 0.$$

Fig. 1 illustrates the variety of shapes of the family $f_\beta$ of the default probability density consistent with the relations for $\lim_{v \to 0} f_\beta(v)$ and $\lim_{v \to 1} f_\beta(v)$.

![Fig. 1. Density of the default probability distribution.](image-url)
3. The calibration problem

The model described by Eq. (2) contains two unknown parameters, \( b \) and \( \beta \). Therefore, if we can find two independent equations linking these parameters to some statistical characteristics of \( g = P(x) \) we will be able to solve the calibration problem. In this Section, we apply the idea of matching the first two moments of the random variable \( g \).

Thus, our analysis will focus on the solution of the calibration equations

\[
\begin{align*}
\mathbb{E}\eta(b, \beta) &= p, \\
\mathbb{E}\eta^2(b, \beta) &= Q,
\end{align*}
\]

where \( Q = p^2 + \sigma^2 \) and \( \sigma \) is an empirical standard deviation of the conditional default probability.

Using analytical properties of the function \( Q(b, \beta) = \mathbb{E}\eta^2(b, \beta) \) we will prove that the calibration problems (6) and (7) has a unique solution

\[
b = b(p, Q), \quad \beta = \beta(p, Q),
\]

under some natural constraints on \( p \) and \( Q \). In Section 3.1, we prove the main analytical result – integral representation of the function \( Q(b, \beta) \). The proof is based on four lemmas that describe the properties of the second moment of the default probability. The derivation of these four statements is placed in Appendix A.

3.1. The second moment of the default probability

The first important property of the unconditional default probability is invariance of \( \mathbb{E}\eta \) with respect to the parameter \( \beta \).

**Lemma 1.** The expectation of the conditional default probability satisfies the relation

\[
\mathbb{E}\eta(b, \beta) = p,
\]

for all \( \beta \), \(-1 \leq \beta \leq 1\), where \( p = \Phi^{-1}(b) \) is the unconditional default probability.

**Proof.** The proof of (8) is based on the following statement.

**Proposition 2.** Let \( \xi \) be a random variable having a standard normal distribution, \( \xi \sim N(0, 1) \). Then the expectation of the random variable \( \Phi(a \cdot \xi + c) \) equals

\[
\mathbb{E}\Phi(a \cdot \xi + c) = \Phi\left(\frac{c}{\sqrt{1 + a^2}}\right).
\]

Proposition 2 is known. For the sake of completeness we give a short proof of (9) in the Appendix A. Let us now derive (8) from Proposition 2. The expectation of the conditional default probability, \( \mathbb{E}\eta(b, \beta) \), satisfies the relation

\[
\mathbb{E}\eta(b, \beta) = \mathbb{E}\Phi\left(\frac{b - \beta \cdot \xi}{\alpha}\right), \quad \alpha^2 + \beta^2 = 1, \quad \Phi(b) = p,
\]

where \( \xi \) is a random variable having a standard normal distribution. Then, applying (9), we obtain

\[
\mathbb{E}\eta(b, \beta) = \Phi\left(\frac{b}{\sqrt{1 + \frac{\beta^2}{\alpha^2}}}\right) = \Phi(b) = p,
\]

as was to be proved.
Lemma 1 implies that the equation
\[ \mathbb{E}\eta(b, \beta) = p, \]
has a unique solution for the parameter \( b, b = \Phi^{-1}(p) \). Therefore, to solve the calibration problem one has to find a solution of the equation \( \mathbb{E}\eta^2(b, \beta) = Q \) given \( b \), where \( Q \) is the empirical second moment of the conditional default probability.

The second moment, \( Q(b, \beta) = \mathbb{E}\eta^2(b, \beta), \) of the conditional default probability satisfies the relation
\[ Q(b, \beta) = \int_{-\infty}^{\infty} \Phi^2 \left( \frac{b - \beta x}{\sqrt{1 - \beta^2}} \right) d\Phi(x), \quad -\infty < b < \infty, \quad -1 < \beta < 1. \] (10)

Let us consider analytical properties of the function \( Q(b, \beta) \). Denote \( k(\beta) = \sqrt{\frac{1 - \beta^2}{1 + \beta^2}} \).

Theorem 3. The function \( Q(b, \beta) \) has the following integral representation
\[ Q(b, \beta) = \Phi(b) - \frac{1}{\pi} \int_0^{k(\beta)} \frac{\exp \left( -\frac{k^2(1+1)^2}{2} \right)}{t^2+1} dt. \] (11)

The partial derivative \( \frac{\partial Q(b, \beta)}{\partial \beta} \) equals
\[ \frac{\partial Q(b, \beta)}{\partial \beta} = \frac{1}{\pi} \frac{\beta}{\sqrt{1-\beta^2} \cdot \sqrt{1+\beta^2}} \cdot e^{-\frac{x^2}{1+\beta^2}}. \] (12)

Before sketching the proof of Theorem 3, we formulate an important corollary that states monotonicity of the function \( Q \).

Corollary 4. For fixed \( b \), the function \( Q(b, \beta) \) is a monotonically increasing function of \( \beta \) for \( \beta \geq 0 \).

The proof of Corollary 4 follows directly from (12).

Proof of Theorem 3. The proof is based on the following statements.

Lemma 5. The partial derivative of the function \( Q(b, \beta) \) satisfies the relation
\[ \frac{\partial Q(b, \beta)}{\partial b} = 2\phi(b)\Phi(b \cdot k(\beta)), \] (13)
where \( \phi(x) = \exp(-x^2/2)/\sqrt{2\pi} \) is the density function of the standard normal distribution.

Formula (13) implies that the function \( Q(b, \beta) \) can be represented in the form
\[ Q(b, \beta) = Q(0, \beta) + 2 \int_0^b \phi(u)\Phi(u \cdot k(\beta))du. \] (14)

Lemma 6. The function \( Q(b, \beta) \) satisfies the relations
\[ Q(b, -\beta) = Q(b, \beta), \] (15)
\[ Q(-b, \beta) = 1 - 2\Phi(b) + Q(b, \beta), \] (16)
\[ Q(b, 0) = \Phi^2(b), \] (17)
\[ Q(b, 1) = \Phi(b). \] (18)
Eq. (16) states that the solution \( \beta(p, Q) \) of the calibration problem with input data \((p, Q)\) satisfies the equality

\[
\beta(p, Q) = \beta(1 - p, Q + 1 - 2p).
\] (19)

Therefore, \( \beta(p, Q) \) is also a solution of the calibration problem with the input data \((1 - p, 1 - 2p + Q)\).

The next lemma allows us to find the function \( Q(b, \beta) \) when \( b = 0 \). This statement plays the key role in the construction of the approximation of the function \( Q(b, \beta) \), considered in Section 4.1.

**Lemma 7.** For \( b = 0 \), the function \( Q(b, \beta) \) satisfies the relation

\[
Q(0, \beta) = \frac{1}{\pi} \arctan \left( \frac{\sqrt{1 + \beta^2}}{1 - \beta^2} \right) = \frac{\pi - \arccos(\beta^2)}{2\pi}.
\] (20)

Let us now prove Theorem 3. Consider Eq. (14). The first term in the right hand side of (14) is given by formula (21). Denote

\[
G(b, c) = \sqrt{2/\pi} \int_0^b e^{-u^2/2} \Phi(u \cdot c) \, du, \quad c \in \mathbb{R}^1.
\]

Using this notation, Eq. (14) can be written as

\[
Q(b, \beta) = Q(0, \beta) + G(b, k(\beta)).
\] (21)

The function \( G(b, c) \) satisfies the relation

\[
G(b, c) = 2 \int_0^b \varphi(u) \Phi(u \cdot c) \, du = 2 \int_0^b \Phi(u \cdot c) \, d\varphi(u).
\]

The partial derivative \( \frac{\partial G(b, c)}{\partial c} \) is given by the integral

\[
\frac{\partial G(b, c)}{\partial c} = 2 \int_0^b \varphi(c \cdot u) \varphi(u) \, du = \frac{1}{\pi} \cdot \left( 1 - e^{-b^2(1+c^2)/2} \right) \frac{1}{c^2 + 1}.
\]

Since \( G(b, 0) = \Phi(b) - 1/2 \), we find

\[
G(b, c) = \Phi(b) - \frac{1}{2} + \frac{1}{\pi} \cdot \arctan c - \frac{1}{\pi} \int_0^c \frac{e^{-b^2(u^2+1)/2}}{u^2+1} \, du.
\] (22)

Then from (21), (22) and Lemma 7 we derive\(^4\) that

\[
Q(b, \beta) = \frac{1}{\pi} \arctan(1/k(\beta)) + \Phi(b) - \frac{1}{2} + \frac{1}{\pi} \cdot \arctan k(\beta) - \frac{1}{\pi} \int_0^{k(\beta)} \frac{e^{-b^2(u^2+1)/2}}{u^2+1} \, du
\]

\[
= p - \frac{1}{\pi} \int_0^{k(\beta)} \exp(-b^2 \cdot (u^2 + 1)/2) \, du.
\]

Finally, taking the partial derivative \( \frac{\partial Q(b, \beta)}{\partial \beta} \) in Eq. (11) we obtain formula (12). The proof of Theorem 3 is now complete. \( \square \)

It follows from Eq. (11) that

\[
Q(b, \beta) = \Phi^2(b) + \frac{1}{\pi} \int_0^\beta \frac{z}{\sqrt{1 - z^2} \cdot \sqrt{1 + z^2} \cdot e^{-\frac{1}{2}b^2 z^2}} \, dz.
\] (23)

Formula (23) can be used for efficient numerical integration of the second moment \( Q(b, \beta) \). In particular, formula (23) is very efficient for small \( \beta \). In turn, the integral representation in Theorem 3 is very efficient as \( \beta \approx 1 \). Indeed, in this case the upper limit in the integral in (11) is determined by \( k(\beta) \) and \( k(\beta) \approx 0 \).

\(^4\) We use the elementary identity \( \arctan w + \arctan(1/w) = \pi/2 \) in this derivation.
3.2. Solvability of the calibration problem

Suppose that the empirical mean, \( p \), of the default probabilities and the empirical variance, \( \sigma^2 \), satisfy the relations

\[
0 < p < 1, \quad 0 \leq \sigma^2 \leq p \cdot (1 - p).
\]

(24)

**Theorem 8.** The calibration problem (6), (7) has a unique nonnegative solution under the condition (24).

**Proof.** If the empirical variance satisfies (24), the empirical second moment, \( Q \), satisfies the inequalities

\[
p^3 \leq Q \leq p.
\]

Since the function \( Q(b, \beta) \) is a continuous, monotonically increasing function of \( \beta \) (see Corollary 4), \( Q(b, 0) = p^2 \) and \( Q(b, 1) = p \), there exists a unique solution of Eq. (7) satisfying the condition \( 0 \leq \beta \leq 1 \). If \( Q \not\in [p^2, p] \) the calibration problem has no solution. \( \square \)

4. Approximations of the second moment \( Q(b, \beta) \)

4.1. “Homotopy approximation” of \( Q(b, \beta) \)

Now we are ready to construct an approximation of the function \( Q(b, \beta) \). Consider the function

\[
Q^H(b, \beta) = (1 - \lambda(\beta)) \cdot \Phi(b) + \lambda(\beta) \cdot \Phi^2(b).
\]

(25)

where \( \lambda(\beta) = \frac{1}{2} \arccos(\beta^2) \).

The function \( Q^H(b, \beta) \) provides a continuous mapping, \( Q^H: \beta \rightarrow Q^H(b, \beta) \), of the unit interval into the subspace of the space of continuous functions such that

\[
Q^H(b, 0) = Q(b, 0) = \Phi^2(b),
\]

\[
Q^H(b, 1) = Q(b, 1) = \Phi(b),
\]

and for \( b = 0, Q^H(0, \beta) = Q(0, \beta) \). The mapping \( Q^H \) is a homotopy in the space of continuous functions. For this reason, it is natural to call \( Q^H(b, \beta) \) the homotopy approximation of the second moment.

From (25) it follows that the approximate solution to the calibration problem satisfies the relation

\[
\beta^H(p, Q) = \sqrt{\cos \left( \frac{\pi \cdot (p - Q)}{2 \cdot p - p^2} \right)}.
\]

(26)

We also note that the function \( \beta^H \) is a unique solution to the calibration problem for \( p = 1/2 (b = 0) \).

Approximation (26) provides an acceptable accuracy for all \( p \in [0.3, 0.7] \). In this case, the relative error of approximation of \( Q \) does not exceed 1%; the relative error of approximate solution of \( \beta \) is smaller than 3%. Even for \( p = 0.2 \) the homotopy approximation provides relative error of approximation \( \varepsilon = 5\% \) (see Fig. 2, subplot A). For small \( p \) this approximation has significant relative error (Fig. 2, subplot C). In this case we construct more complex approximation using asymptotics of \( Q(b, \beta) \) as \( \beta \rightarrow 0 \) and \( \beta \rightarrow 1 \).

4.2. Approximation for \( \beta \approx 0 \)

Eq. (23) implies that if \( \beta \) is close to 0, the function \( Q(b, \beta) \) can be approximated by the integral

\[
Q(b, \beta) = p^2 + \frac{1}{\pi} \int_0^\beta \exp \left( - \frac{b^2}{1 + x^2} \right) \frac{x}{\sqrt{1 - x^4}} \, dx \approx p^2 + \frac{e^{-b^2}}{\pi} \int_0^\beta \frac{x}{\sqrt{1 - x^4}} \, dx.
\]

Then we obtain
The latter relation implies the following approximation for the solution of the calibration problem:

\[ b(p, Q) \approx b_0(p, Q) = \sqrt{\sin(2\pi \cdot (Q - p^2) \cdot e^{k^2})}. \]  

(27)

Fig. 2, subplot B, displays the function \( b_0(p, Q) \). If \( Q \) is close to \( p^2 \) the function \( b_0(p, Q) \) provides a very good approximation of \( b(p, Q) \).

4.3. Approximation for \( b \approx 1 \)

If \( b \) is close to 1, then \( k(b) \) is close to 0. Then, from the integral representation (11) we derive

\[ Q(b, \beta) \approx p^2 + \frac{1}{2\pi} e^{-b^2} \arcsin b^2. \]

where \( z = \arctan(k(b)) \). The integral in (28) is equal

\[ \int_0^z \exp(-b^2 t^2/2) \, dt = \sqrt{2\pi} \cdot b^{-1}(\Phi(hz) - 1/2). \]

Then, from (28) we derive the asymptotic approximation for the solution of the calibration problem for \( b \approx 1 \):

\[ b(p, Q) \approx b_1(p, Q) = k(\tan(\phi^* / b)). \]

(29)
where
\[ \phi^* = \Phi^{-1}\left(\sqrt{\frac{\pi}{2}} \cdot |b| \cdot \exp\left(\frac{b^2}{2}\right) \cdot (p - Q) + 0.5\right) \quad \text{and} \quad k(x) = \sqrt{\frac{1 - x^2}{1 + x^2}}. \]

Fig. 2, subplot D, displays the function \( \beta_1(p, Q) \). If \( Q \) is close to \( p \) the function \( \beta_1(p, Q) \) provides a very good approximation of \( \beta(p, Q) \).

5. Calibration procedure

In this Section, we construct an approximate solution, \( \beta^*(p, Q) \), of the calibration problem using the approximations \( \beta_H(p, Q) \), \( \beta_0(p, Q) \) and \( \beta_1(p, Q) \). These approximations cover three different areas of values of the parameters \( p \) and \( Q \).

The function \( \beta^H \) provides a very accurate approximation of the function \( \beta(p, Q) \) as \( p \) is close\(^5\) to \( 1/2 \). If \( Q \approx p^2 \) (\( \beta \) is close to 0), the function \( \beta_0 \) approximates the solution of the calibration problem. If \( Q \approx p \) (\( \beta \) is close to 1) then the function \( \beta_1(p, Q) \) approximates the function \( \beta(p, Q) \). Thus, we have to find a sewing function that provides a smooth transition from one branch of the solution to another.

We construct the sewing function using a polynomial regression of the variables

\[ \beta_H = \beta^H(p, Q) = \sqrt{\cos\left(\frac{\pi}{2} \cdot \frac{p - Q}{p^2}\right)}, \]

\[ \beta_0 = \beta_0(p, Q) = \sqrt{\sin\left(2\pi \cdot (Q - p^2) \cdot e^{b^2}\right)}, \]

\[ \beta_1 = \beta_1(p, Q) = k(\tan(\phi^*/b)), \]

where
\[ \phi^* = \Phi^{-1}\left(\sqrt{\frac{\pi}{2}} \cdot |b| \cdot \exp\left(\frac{b^2}{2}\right) \cdot (p - Q) + 0.5\right) \quad \text{and} \quad k(x) = \sqrt{\frac{1 - x^2}{1 + x^2}}. \]

The regression coefficients are determined separately in each of the intervals, \([\hat{p}^{(i)}, \hat{p}^{(i+1)}]\), that form a cover of the unit interval,

\[ \bigcup_{i=1}^{N_i}[\hat{p}^{(i)}, \hat{p}^{(i+1)}] = [0, 1] \]

as a solution to the optimization problem

\[ \| \mathcal{P}_2(\beta_0, \beta_1, \beta_H) - \beta(p, Q) \|_{L^2} \rightarrow \min, \quad p' \in [\hat{p}^{(i)}, \hat{p}^{(i+1)}], \]

where \( \mathcal{P}_2(x, y, z) \) is a polynomial of degree 2 of the variables \( x, y \) and \( z \).

Given the empirical default probability, \( p \), and the empirical second moment, \( Q \), we compute \( b \) and \( \beta \) as follows:

Computation of the parameter \( b \): \( b = \Phi^{-1}(p) \).

Computation of \( \beta \):
- If \( p > 0.5 \) then \( p' = 1 - p \) and \( Q' = Q + 1 - 2p \)
- otherwise \( p' = p \) and \( Q' = Q \).
- If \( 0.3 \leq p' \leq 0.5 \) then \( \beta^* = \beta_H \)

\(^5\) More precisely, if \( p \in [0.3, 0.7] \) the error of approximation is smaller than 1%.

If $0.1 \leq p' < 0.3$, compute $\beta^l = a_1 \cdot \beta_0 + a_2 \cdot \beta_1 + a_3 \cdot \beta_H$ where $a_1 = 0.3882$, $a_2 = 0.2725$ and $a_3 = 0.3353$.

if $\beta^l > 0.7$, then $\beta^* = \beta_1$,

if $0.7 \geq \beta^l > 0.3$, then $\beta^* = \beta^l$,

else $\beta^* = \beta_0$.

If $0.05 \leq p' < 0.1$ then

if $\beta_1 > 0.7$, then $\beta^* = \beta_1$,

elseif $\beta_0 < 0.3$, then $\beta^* = \beta_0$,

else $\beta^* = \beta^l$.

If $0.01 \leq p' < 0.05$, compute

$$\beta^0 = a_1 \cdot \beta_0 + a_2 \cdot \beta_1 + a_3 \cdot \beta_H + a_4 \cdot \beta_0 \cdot \beta_1 + a_5 \cdot \beta_0 \cdot \beta_H + a_6 \cdot \beta_1 \cdot \beta_H + a_7 \cdot \beta_0^2 + a_8 \cdot \beta_1^2 + a_9 \cdot \beta_H^2$$

and the coefficients $a_i$ are given in the following table

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5285</td>
<td>-0.36768</td>
<td>1.7249</td>
</tr>
<tr>
<td>1.4945</td>
<td>-1.9464</td>
<td>1.7619</td>
</tr>
<tr>
<td>-0.0381</td>
<td>0.6292</td>
<td>-1.6296</td>
</tr>
</tbody>
</table>

if $\beta_1 < \beta^0$, then $\beta^* = \beta_1$,

elseif $\beta^0 > 0.2$, then $\beta^* = \beta^0$,

else $\beta^* = \beta_0$.

If $p' < 0.01$ we compute the following two quadratic functions

$$\beta^{QH} = a_1 \cdot \beta_0 + a_2 \cdot \beta_1 + a_3 \cdot \beta_H + a_4 \cdot \beta_0 \cdot \beta_1 + a_5 \cdot \beta_0 \cdot \beta_H + a_6 \cdot \beta_1 \cdot \beta_H + a_7 \cdot \beta_0^2 + a_8 \cdot \beta_1^2 + a_9 \cdot \beta_H^2$$

$$\beta^{QH} = c_1 \cdot \beta_1 + c_2 \cdot \beta_H + c_3 \cdot \beta_1 \cdot \beta_H + c_4 \cdot \beta_1^2 + c_5 \cdot \beta_H^2$$

where the coefficients $a_i$ are given by

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$a_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2778</td>
<td>-0.2341</td>
<td>1.4750</td>
</tr>
<tr>
<td>1.29100</td>
<td>-0.58969</td>
<td>1.1624</td>
</tr>
<tr>
<td>-0.3053</td>
<td>0.42448</td>
<td>-2.4124</td>
</tr>
</tbody>
</table>

and the coefficients $c_i$ are given by

<table>
<thead>
<tr>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.0025767</td>
<td>1.3554</td>
<td>-1.0888</td>
</tr>
<tr>
<td>1.0509</td>
<td>0.48947</td>
<td>1.0888</td>
</tr>
</tbody>
</table>

if $\beta^{QH} \geq 0.8$, then $\beta^* = \beta_1$,

elseif $Q' \geq (0.25 \cdot \exp(-b^2) + p^2)$, then $\beta^* = \beta^{QH}$,

elseif $\beta^{QH} \geq 0.157$, then $\beta^* = \beta^{QH}$,

else $\beta^* = \beta_0$.

6. Numerical results and performance

In this Section, we consider the approximate solution to the calibration problem for a small value of the probability of default, $p = 0.5%$. In Fig. 3 we display the main elements of the approximate solution: the
approximating functions $\beta_0(p, Q)$ and $\beta_1(p, Q)$. The function representing the homotopy approximations was already shown in Fig. 2.

The first and the second subplots display the approximations $\beta_0(p, Q)$ and $\beta_1(p, Q)$ working for $Q \approx p^2$ and $Q \approx \rho$ correspondingly. The third subplot represents the resulting approximation obtained by sewing the functions $\beta_0(p, Q)$ and $\beta_1(p, Q)$. The relative error of the approximate solution is shown in the last subplot. The maximal error is attained at the point $Q_c$ where the approximate solution switches from the branch corresponding to $\beta_0$ to the polynomial regression branch linking $\beta_0$ and $\beta_1$. In this case, the relative error of the approximation is smaller than 4%.

The calibration procedure was tested using exhaustive search over the set of parameters $p$ and $\beta$ in the area $10^{-6} \leq p \leq 0.5, 10^{-4} \leq \beta \leq 1 - 10^{-4}$. We create a grid in the space of parameters on which we study the error of the approximation. First, we computed the value of the second moment $Q_{p,\beta} = Q(p, \beta)$ considered as a function of the parameter $\beta$ given a value of $p$. The integral that determines the second moment is computed with the accuracy $10^{-10}$. After that we apply the calibration procedure to find an approximate value of $\beta^*(p, Q)$. The maximal relative error, $e(p)$, for a given $p$, $p > 0$ of the approximation is estimated as $e^* = \max_p e(p)$, where

$$e(p) = \max_{Q^2 \neq 0} \frac{|\beta(p, Q) - \beta^*(p, Q)|}{\beta(p, Q)}.$$

Our numerical experiment shows that the relative error of the approximation $e^* < 0.05$.

To analyze performance improvement, we measure the computation time of the calibration procedure proposed in Section 5 and the time of the calibration algorithm NINS based on numerical integration and numerical solution of the nonlinear equation

$$\hat{\sigma}^2(p, \beta) - \sigma^2 = 0,$$

(30)
where \( \tilde{\sigma}^2(p, \beta) = Q(b, \beta) - p^2 \) and \( \sigma \) is the empirical standard deviation of the default probability. The NINS algorithm uses a 30-node Gaussian quadrature (see [10]),

\[
Q(b, \beta) \approx \sum_{i=1}^{N} \omega_i \cdot \Phi^2\left(\frac{b - \beta x_i}{\alpha}\right), \quad N = 30,
\]

for computation of the variance \( \tilde{\sigma}^2(p, \beta) \). The corresponding summation weights, \( \omega_i \), and the abscissas, \( x_i \), are calculated by the Gauss–Hermite algorithm.

The NINS algorithm finds the positive root of Eq. (30) using the Brent method [1,10].

Then the performance ratio is defined as \( R = T_s/T_a \), where \( T_a \) is the time required for the solution of the calibration problem using our approximation and \( T_s \) is the computation time of the calibration procedure NINS. We measure the computation time of 100 calibrations corresponding to the values of the parameter \( \beta \) for a given value of the default probability \( p \). The results are shown in Fig. 4. One can see that for the range of values of the default probability \( p \in [0.005, 0.5] \) the performance ratio \( R > 100 \). The smallest value is achieved at a very small value of the empirical default probability \( p = 10^{-4} \). In this case \( R = 60 \). Thus, the calibration procedure suggested in this paper demonstrates significant performance advantage and acceptable accuracy.

7. Conclusion

We constructed an approximate solution of the calibration problem for the Merton–Vasicek model sewing together three approximations of the second moment of the conditional default probability. This solution allows one to compute the parameters of the model with relative error smaller than 5% even for very small values of the second moment (less than \( 10^{-7} \)). Our solution demonstrates significant performance advantage with respect to the calibration procedure NINS. Our calibration procedure is more than 100 times faster than the calibration algorithm NINS for \( p > 0.5\% \).

Acknowledgements

The authors thank the referees for the comments and suggestions that helped to improve the paper. We are also very grateful to David Saunders and Ian Iscoe for many useful discussions.
Appendix A. Proof of auxiliary results

Proof of Proposition 2

Let us compute the expectation $\mathbb{E}(a \cdot \xi + c)$, where $\xi \sim \mathcal{N}(0, 1)$ is a standard normal random variable. We have $\mathbb{E}(a \cdot \xi + c) = \int_{-\infty}^{\infty} \Phi(a \cdot x + c) \phi(x) dx$. The integral in this relation can be represented as the probability of the event $\{\xi < a\xi + c\}$, where the random variables $\xi$ and $\xi$ are independent and $\xi$ has a standard normal distribution. Then, from this probabilistic interpretation we obtain that

$$\mathbb{E}(a \cdot \xi + c) = \Pr\{\xi < a\xi + c\} = \Pr\{\xi - a\xi < c\}. \tag{A.1}$$

The random variable $\xi - a\xi$ has a normal distribution with the mean $0$ and the standard deviation $\sqrt{1 + a^2}$. Therefore,

$$\Pr\{\xi - a\xi < c\} = \Phi\left(\frac{c}{\sqrt{1 + a^2}}\right). \tag{A.2}$$

From (A.1) and (A.2) we obtain (9).

Proof of Lemma 5

Let us compute the partial derivative of the function $Q(b, \beta)$ with respect to the variable $b$. We have

$$\frac{\partial Q(b, \beta)}{\partial b} = \frac{\partial}{\partial b} \int_{-\infty}^{\infty} \Phi^2\left(\frac{b - \beta x}{\alpha}\right) \phi(x) \, dx,$$

where $\alpha = \sqrt{1 - \beta^2}$. It is not difficult to see that one can interchange the operators of differentiation and integration in the latter formula. Then we obtain

$$\frac{\partial Q(b, \beta)}{\partial b} = \frac{2}{\alpha} \int_{-\infty}^{\infty} \Phi\left(\frac{b - \beta x}{\alpha}\right) \varphi\left(\frac{b - \beta x}{\alpha}\right) \phi(x) \, dx$$

$$= \frac{1}{\pi \alpha} \int_{-\infty}^{\infty} \Phi\left(\frac{b - \beta x}{\alpha}\right) \exp\left(-\frac{1}{2} \left(\frac{b^2 + \beta^2 x^2 + 2b\beta\alpha^2}{\alpha^2}\right)\right) \, dx$$

$$= \frac{\sqrt{2}}{\pi} \cdot e^{-b^2/2} \int_{-\infty}^{\infty} \Phi\left(\frac{b - \beta x}{\alpha}\right) \frac{e^{-(x - b\beta)^2/(2\alpha^2)}}{\sqrt{2\pi\alpha}} \, dx. \tag{A.3}$$

Let $\xi \sim \mathcal{N}(b\beta, \alpha)$ be a random variable having a normal distribution with the expectation $b\beta$ and the standard deviation $\alpha$. Denote the distribution function of $\xi$ by $\Phi_\xi(x)$. Then formula (A.3) can be represented as follows:

$$\frac{\partial Q(b, \beta)}{\partial b} = \sqrt{2/\pi} \cdot e^{-b^2/2} \int_{-\infty}^{\infty} \Phi_\xi\left(\frac{b - \beta x}{\alpha}\right) \, d\Phi_\xi(x). \tag{A.4}$$

The integral in the latter equation satisfies the relation

$$\int_{-\infty}^{\infty} \Phi\left(\frac{b - \beta x}{\alpha}\right) \, d\Phi_\xi(x) = \mathbb{P}(a\eta + \beta\xi < b), \tag{A.5}$$

where $\eta \sim \mathcal{N}(0, 1)$. The random variable $\xi$ can be represented in the form $\xi = \beta b + \alpha \cdot \zeta$, where $\zeta$ has the standard normal distribution. Then from (A.5) we obtain using Proposition 2

$$\int_{-\infty}^{\infty} \Phi\left(\frac{b - \beta x}{\alpha}\right) \, d\Phi_\xi(x) = \mathbb{P}(a\eta + \beta(\beta b + \alpha \zeta) < b) = \mathbb{P}(\eta < b\alpha - \beta \zeta) = \Phi(b\alpha/\sqrt{1 + b^2}) = \Phi(bk(\beta)).$$

Finally from Eqs. (A.4), (A.5) and (A.6) we find

$$\frac{\partial Q(b, \beta)}{\partial b} = \sqrt{2/\pi} \cdot e^{-b^2/2} \Phi(b \cdot k(\beta)).$$
Proof of Lemma 6

Let us prove relation (15). We have

\[ Q(b, -\beta) = \int_{-\infty}^{\infty} \Phi^2 \left( \frac{b + \beta x}{\sqrt{1 - \beta^2}} \right) \cdot \varphi(x) \, dx. \]

Changing the integration variable \( x = -t \) and taking into account that the function \( \varphi \) is even, we obtain (15).

Formula (16) is proved as follows. For all real \( x \) the standard normal cumulative distribution function satisfies the identity \( \Phi(-x) = 1 - \Phi(x) \). Then we have

\[ Q(-b, \beta) = \int_{-\infty}^{\infty} \Phi^2 \left( \frac{-b - \beta x}{\sqrt{1 - \beta^2}} \right) \cdot \varphi(x) \, dx = \int_{-\infty}^{\infty} \left( 1 - \Phi \left( \frac{b + \beta x}{\sqrt{1 - \beta^2}} \right) \right)^2 \cdot \varphi(x) \, dx. \]

Then, using Proposition 2 and Eq. (15) we finally obtain (16).

Formula (17) follows directly from Eq. (10). To prove (18) notice that

\[ \lim_{b \to 1} \Phi((b - \beta x)/\sqrt{1 - \beta^2}) = \begin{cases} 1, & \text{if } x < b \\ 0, & \text{otherwise.} \end{cases} \]

Therefore

\[ \lim_{b \to 1} Q(b, \beta) = \int_{-\infty}^{b} d\Phi(x) = \Phi(b) = p. \]

Thus, Lemma 6 is proved.

Proof of Lemma 7

Consider the function \( Q(0, \beta) \). Let \( t = \beta/\sqrt{1 - \beta^2} \). Then the function \( Q \) can be written as

\[ Q(0, \beta) = \Psi(t) = \int_{-\infty}^{\infty} \Phi^2(-tx) \, d\Phi(x) = \int_{-\infty}^{\infty} \Phi^2(tx) \, d\Phi(x). \]

Let us compute the derivative of \( \Psi(t) \). We have

\[ \frac{d\Psi(t)}{dt} = 2 \int_{-\infty}^{\infty} \Phi(tx) \cdot \varphi(tx) \cdot x \cdot \varphi(x) \, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \Phi(tx) \cdot \exp \left( -\frac{x^2}{2} \cdot (t^2 + 1) \right) \, x \, dx, \]

and integrating by parts we obtain

\[ \frac{d\Psi}{dt} = \frac{1}{\pi} \cdot \frac{t}{t^2 + 1} \int_{-\infty}^{\infty} \exp \left( -\frac{x^2}{2} \cdot (t^2 + 1) \right) \cdot \varphi(tx) \, dx = \frac{1}{\pi} \cdot \frac{t}{t^2 + 1} \cdot \frac{1}{\sqrt{2t^2 + 1}}. \]

Therefore,

\[ \Psi(t) - \Psi(0) = \frac{1}{\pi} \int_{0}^{t} \frac{s}{s^2 + 1} \cdot \frac{1}{\sqrt{2s^2 + 1}} \, ds. \]

Changing the variable, \( u = \sqrt{2s^2 + 1} \), and taking into account that \( \Psi(0) = 1/4 \) we obtain

\[ \Psi(t) - \frac{1}{4} = \frac{1}{\pi} \int_{1}^{\sqrt{2t^2 + 1}} \frac{1}{u^2 + 1} \, du = \frac{1}{\pi} (\arctan \sqrt{2t^2 + 1} - \pi/4). \]
Thus, $\Psi(t) = \pi^{-1} \arctan \sqrt{2t^2 + 1}$, and we derive that

$$Q(0, \beta) = \frac{1}{\pi} \arctan \left( \sqrt{\frac{1 + \beta^2}{1 - \beta^2}} \right),$$

and after some algebraic transformations we finally obtain (20). Lemma 7 is thus proved.

References