Nonoscillation and positivity of solutions to first order state-dependent differential equations with impulses in variable moments

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Abstract

The state-dependent delay differential equation

$$\dot{x}(t) + \sum_{i=1}^{m} p_i(t)x(t - (H_i x)(t)) = f(t), \quad t \in [0, \infty),$$

with state-dependent impulses is under consideration.

Sufficient conditions for positivity of solutions to the Cauchy and periodic problems as well as conditions for positivity of solutions to the problem with a condition on the right end of the interval [0, \omega] are given. Sufficient conditions for nonoscillation of solutions to the homogeneous equation (f = 0, \varphi = 0) on the half-line are formulated.

MSC: 34K45; 34K05

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1. Introduction

The delay differential equations with impulses in fixed points independent of solutions, such as equations of the type

\[ \dot{x}(t) + \sum_{i=1}^{m} p_i(t)x(t - \tau_i(t)) = f(t), \quad t \in [0, \infty), \]  
\[ x(\xi) = \varphi(\xi), \quad \xi < 0, \]  
\[ x(t_j) = \beta_j x(t_j - 0), \quad j = 1, 2, \ldots, \]  

were studied by many researchers (see, e.g., [2,6–8]). Some higher order equations of such type were treated in [10,11]. In [1] ordinary differential equations without delays ($\tau_i = 0, i = 1, 2, \ldots, m$), but with impulses in variable time instants, were considered. The impulses there appeared at the points of intersection of the trajectory of the solution $x$ with some prescribed curves defined by continuous functions $\zeta_j : \mathbb{R} \to \mathbb{R}, j = 1, 2, \ldots$. This means that the points $t_j(x)$, where the impulses occur are the roots of the following equations:

\[ x(t) = \zeta_j(x(t)), \quad j = 1, 2, \ldots. \]  

Therefore conditions (1.3) become of the form

\[ x(t_j(x)) = \beta(t_j(x))x(t_j(x) - 0), \quad j = 1, 2, \ldots. \]

By a solution of (1.1), (1.2) due to the last condition, we understand a locally piecewise absolutely continuous function $x : [0, \infty) \to \mathbb{R}$, satisfying (1.7) and (1.1), (1.2) almost everywhere. Moreover, all the points of discontinuity $t_j$ of this function are among the roots of (1.4).

Several works (see the bibliography in [1]) deal with the question of existence of solutions to (1.1), (1.2) with impulses depending on solutions. The constraints imposed on the equations in those works are intended for excluding “throbbing” (i.e., the infinite number of intersections of the solution trajectory with the same curve $\zeta_j$). In other words the obtained conditions yield that (1.4) has only a finite number of roots for each $j = 1, 2, \ldots$.

During the last decade equations with impulses at variable time instances were considered in a number of papers (see, for example, [3–5]). The main object in those papers was an ordinary differential equation, and the well-known monotone techniques were used. In the case of an ordinary differential equation, a trajectory of a solution between two adjacent points of impulses satisfies this equation. The impulses actually only impose solution jumps from one trajectory of a solution to the nonimpulsive equation to the trajectory of another solution to the nonimpulsive equation. For equations with deviating argument the situation, however, is much more complicated. In the present paper we consider the equation

\[ \dot{x}(t) + \sum_{i=1}^{m} p_i(t)x(t - (Hix)(t)) = f(t), \quad t \in [0, \infty), \]  
\[ x(\xi) = \varphi(\xi), \quad \xi < 0, \]

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\[ \dot{x}(t) + \sum_{i=1}^{m} p_i(t)x(t - (Hix)(t)) = f(t), \quad t \in [0, \infty), \]  
\[ x(\xi) = \varphi(\xi), \quad \xi < 0, \]
with the impulses
\[ x(t_j(x)) = \beta(t_j(x))x(t_j(x) - 0), \quad j = 1, 2, \ldots \] (1.7)

The case (1.1), (1.2), (1.7) for \( m = 1 \) has been studied by the authors in [12]. Let us point out that the case when the delay depends on the state \( x \) is of importance in applications. For example, such equations arise in problems related to the echo effect. We assume that \( p_i: [0, \infty) \to \mathbb{R}, \ i = 1, \ldots, m, \) and \( f: [0, \infty) \to \mathbb{R} \) are locally summable, \( \varphi: (-\infty, 0) \to \mathbb{R} \) is measurable and bounded. The operators \( H_i, i = 1, 2, \ldots, m, \) act from the space \( \text{WAC} \) of piecewise absolutely continuous on every finite segment \([0, a], a \in (0, \infty)\) (locally piecewise absolutely continuous) functions \( x: [0, \infty) \to \mathbb{R} \) into the space of measurable bounded functions. Everywhere below we assume that

\[ (\forall x \in \text{WAC}) \quad 0 \leq (H_i x)(t) \leq \tau_i(t), \quad t \in [0, \infty), \ i = 1, \ldots, m, \] (1.8)

where \( \tau_i: [0, \infty) \to [0, \infty), \ i = 1, \ldots, m, \) are measurable and bounded.

**Remark 1.1.** Further we will assume that for any measurable and essentially bounded function \( z: (-\infty, \infty) \to \mathbb{R} \) and for any function \( x \in \text{WAC} \) the superpositions \( z(t - (H_i x)(t)), \ i = 1, 2, \ldots, m, t \in [0, \infty), \) are also measurable and essentially bounded functions. Conditions under which this assumption holds where studied in detail in [13].

**2. Impulsive conditions**

First let us reduce the study of the state-dependent equation (1.5), (1.6) with state-dependent impulses (1.7) to the investigation of the same state-dependent equation, but with impulses in fixed points.

Let \( \chi: [0, \infty) \times \mathbb{R} \to \mathbb{R} \) be a continuous in the union of the arguments function. Denote by \( \Gamma_1, \Gamma_2 \) Volterra (by Tikhonov) operators defined on the space \( \text{WAC} \). (An operator is called Volterra according to Tikhonov if any two functions coinciding on an interval \([0, t] \) have equal images on \([0, t], t \in [0, \infty)\)). Let \( \Gamma_1, \Gamma_2 \) be acting into the space of measurable and bounded functions \( y: [0, \infty) \to [0, \infty) \).

Pick a sequence of points \( a_1 < a_2 < \cdots, a_j \in (0, \infty), j = 1, 2, \ldots, \lim_{j \to \infty} a_j = \infty. \) Define a point \( t_1 \in [a_1, a_2) \) and a number \( \beta_1 \in [0, \infty) \) as follows

\[ t_1 = \max\{a_1, \inf\{t \in [a_1, a_2): (\Gamma_1 x)(t) = \chi(t, x(t))\}\}, \] (2.1)

where \( x: [0, a_2) \to \mathbb{R} \) satisfies (1.5), (1.6) almost everywhere on \([0, a_2)\);

\[ \beta_1 = \begin{cases} (\Gamma_2 x)(t_1), & \text{if } \{t \in [a_1, a_2): (\Gamma_1 x)(t) = \chi(t, x(t))\} \neq \emptyset, \\ 1, & \text{if } \{t \in [a_1, a_2): (\Gamma_1 x)(t) = \chi(t, x(t))\} = \emptyset. \end{cases} \] (2.2)

Let us define the points \( t_j \in [a_j, a_{j+1}) \) and the numbers \( \beta_j, j = 2, 3, \ldots, \) by the following equalities

\[ t_j = \max\{a_j, \inf\{t \in [a_j, a_{j+1}): (\Gamma_1 x)(t) = \chi(t, x(t))\}\}, \] (2.3)
where \( x : [0, a_{j+1}) \to \mathbb{R} \) satisfies (1.5), (1.6) almost everywhere;

\[
x(t_i(x)) = \beta(t_i(x))x(t_i(x) - 0), \quad i = 1, 2, \ldots, j - 1;
\]

\[
\beta_j = \begin{cases} (\Gamma_2x)(t_j), & \text{if } t \in [a_j, a_{j+1}): (\Gamma_1x)(t) = \chi(t, x(t)) \neq \emptyset, \\ 1, & \text{if } t \in [a_j, a_{j+1}): (\Gamma_1x)(t) = \chi(t, x(t)) = \emptyset. \end{cases} \tag{2.4}
\]

In what follows we will study the impulsive equation (1.5)–(1.7) with \( t_j, \beta_j, j = 1, 2, \ldots, \) defined by (2.1)–(2.4).

3. Main results

Let us formulate the main results of the paper.

Define

\[
\tau(t) = \max_{1 \leq i \leq m} \tau_i(t), \quad \gamma_j = \inf_{x \in \text{WAC}} \inf_{t \in [a_j, a_{j+1})} (\Gamma_2x)(t),
\]

\[
B(t) = \begin{cases} \prod_{j \in G_t} \gamma_j, & \text{if } G_t = \{j: [a_j, a_{j+1}) \cap [t - \tau(t), t) \neq \emptyset, \\ 1, & \text{if } G_t = \emptyset. \end{cases}
\]

**Theorem 3.1.** Let \( p_i(t) \geq 0, \ t \in [0, \infty), \ i = 1, \ldots, m, \ \text{mes}\{t \in [0, \omega]: \ p_i(t) > 0, \ t - \tau_i(t) > 0\} > 0, \ 0 < \gamma_j \leq 1, \ j = 1, 2, \ldots, \) and the following inequality is valid:

\[
\int_0^t \sum_{i=1}^m p_i(s) ds \leq \frac{1 + \ln B(t)}{e}, \quad t > 0, \quad h(t) = \max\{t - \tau(t), 0\}, \tag{3.1}
\]

then the following statements hold:

(1) A nontrivial solution \( x \) to the equation

\[
\dot{x}(t) + \sum_{i=1}^m p_i(t)x(t - (H_i x)(t)) = 0, \quad t \in [0, \infty), \tag{3.2}
\]

\[
x(\xi) = 0, \quad \xi < 0, \tag{3.3}
\]

with condition (1.7)

\[
x(t_j(x)) = \beta(t_j(x))x(t_j(x) - 0),
\]

does not have zeros on \([0, \infty)\). Here \( t_j, \beta_j, j = 1, 2, \ldots, \) are defined by formulae (2.1)–(2.4);

(2) The solution to the periodic problem (1.5)–(1.7) for \( t \in [0, \omega) \) and (3.4), where

\[
x(0) = x(\omega), \tag{3.4}
\]

is nonnegative (nonpositive) for any \( \omega \in (0, \infty) \) if \( f(t) \geq 0, \ t \in [0, \infty), \ \varphi(s) \leq 0, \ s \in (-\infty, 0), \ \varphi(t) \leq 0, \ t \in [0, \infty), \ \varphi(s) \geq 0, \ s \in (-\infty, 0))\);
(3) The solution to the problem (1.5)–(1.7) for \( t \in [0, \omega] \) and (3.5), where

\[
x(\omega) = 0,
\]

is nonnegative (nonpositive) for any \( \omega \in (0, \infty) \) if \( f(t) \leq 0, \ t \in [0, \infty) \), \( \varphi(s) \geq 0, \ s \in (-\infty, 0) \) \( f(t) \geq 0, \ t \in [0, \infty) \), \( \varphi(s) \leq 0, \ s \in (-\infty, 0) \);

(4) The solution to the Cauchy problem (1.5)–(1.7) and (3.6), where

\[
x(0) = \alpha, \quad \alpha > 0,
\]

is positive on \( [0, \infty) \) for \( f(t) \geq 0, \ t \in [0, \infty) \), \( \varphi(s) \leq 0, \ s \in (-\infty, 0) \).

**Proof.** The proof of the theorem is based on some results obtained in [7] and on the so-called “absorption method” introduced in [9]. The essence of this method could be formulated as follows: In the case when the “external” equation is linear and the only reason for nonlinearity is the state-dependent time lag, we analyze the behavior of equation’s solutions through the investigation of the corresponding linear equations, i.e., we “immerse” the solutions of a nonlinear equation into the set of solutions to the corresponding linear equation.

**Step 1.** Consider the following linear equation with impulses

\[
(Ly)(t) \equiv \dot{y}(t) + \sum_{i=1}^{m} p_i(t)y(t - \Theta_i(t)) = \psi(t), \quad t \in [0, \infty),
\]

\[
y(\xi) = 0, \quad \xi < 0,
\]

\[
y(s_j) = \delta_j y(s_j - 0).
\]

Here \( \psi : [0, \infty) \to \mathbb{R} \) is locally summable; \( \Theta_i : [0, \infty) \to [0, \infty), \ i = 1, \ldots, m \), are measurable and bounded. Moreover, \( \Theta_i(t) \leq \tau_i(t), \ i = 1, \ldots, m, \ t \in [0, \infty) \).

The difference between (3.7)–(3.9) and (1.5)–(1.7) is that

1. the delay does not depend on a solution;
2. the instants of impulses \( s_j \in [a_j, a_{j+1}) \), \( j = 1, 2, \ldots \), and the numbers \( \delta_j, \ j = 1, 2, \ldots \), do not depend on solution to (3.7)–(3.9) (i.e., given a priori).

The general solution to (3.7)–(3.9) (see [7]) has representation

\[
y(t) = C(t, 0)y(0) + \int_{0}^{t} C(t, s)\psi(s) \, ds,
\]

where \( C(t, s), 0 \leq s \leq t < \infty, \) is called the Cauchy function of this equation. If the boundary problem (3.7)–(3.10), where

\[
y(\omega) = 0,
\]

(3.10)
$\omega$ is a real positive number, is uniquely solvable, then its solution can be represented in the form

$$y(t) = \int_0^\omega G(t,s) \psi(s) \, ds,$$

where $G(t,s), t,s \in [0, \omega]$, is called the Green function of this problem.

If the periodic problem (3.7)–(3.9), (3.11), where

$$y(0) = y(\omega), \quad (3.11)$$

$\omega$ is a real positive number, is uniquely solvable, then its solution can be represented as follows:

$$y(t) = \int_0^\omega P(t,s) \psi(s) \, ds,$$

where $P(t,s), t,s \in [0, \omega]$, is the Green function of the periodic problem.

It is known [7] that for each fixed $s \in [0, \infty)$, $C(\cdot, s)$ is a solution to the “$s$-truncated” equation (3.7), (3.12), (3.9), where

$$y(\xi) = 0, \quad \xi < s, \quad (3.12)$$

with the initial condition

$$C(s,s) = 1;$$

$$G(t,s) = C(t,s) - \frac{C(\omega,s)C(t,0)}{C(\omega,0)}, \quad (3.13)$$

$$P(t,s) = C(t,s) + \frac{C(\omega,s)C(t,0)}{1 - C(\omega,0)}, \quad (3.14)$$

where $C(t,s) = 0$ for $t < s$. Let $C$ be the space of continuous on each interval $[s_j, s_{j+1}], j = 0, 1, 2, \ldots$ ($s_0 = 0$) functions, satisfying condition (3.9). Define an operator $K : C \to C$ by the equality

$$(Ky)(t) = -\int_0^\omega G_0(t,s) \sum_{i=1}^m p_i(s) y(s - \Theta_i(s)) \, ds, \quad (3.15)$$

where $y(\xi) = 0$ for $\xi < 0$. Here $G_0(t,s), t,s \in [0, \omega]$, is the Green function of the problem

$$\dot{y}(t) = \psi(t), \quad t \in [0, \omega], \quad y(s_j) = \delta_j y(s_j - 0), \quad j = 1, 2, \ldots, n, \quad y(\omega) = 0. \quad (3.16)$$

Note that $G_0(t,s) = 0$ for $0 \leq s \leq t \leq \omega$ and $G_0(t,s) < 0$ for $0 \leq t < s \leq \omega$. Using (3.13) one can find, for example, the Green function of problem (3.16) in the case $m = 3$. Namely,
\[ G_0(t,s) = \begin{cases} 
0, & 0 \leq s \leq t \leq \omega, \\
-1, & s_{i-1} \leq t < s < s_i \ (i = 1, 2, 3, \ s_0 = 0), \ s_3 \leq t < s \leq \omega, \\
-\frac{1}{s_1}, & t \in [0, s_1), \ s \in [s_1, s_2), \\
-\frac{1}{s_2}, & t \in [s_1, s_2), \ s \in [s_2, s_3), \\
-\frac{1}{s_3}, & t \in [s_2, s_3), \ s \in [s_3, \omega), \\
-\frac{1}{s_1s_2}, & t \in [0, s_1), \ s \in [s_2, s_3), \\
-\frac{1}{s_2s_3}, & t \in [s_1, s_2), \ s \in [s_3, \omega), \\
-\frac{1}{s_1s_2s_3}, & t \in [0, s_1), \ s \in [s_3, \omega). 
\end{cases} \]

**Lemma 3.2.** [7] Let \( p_i(t) \geq 0, \ t \in [0, \infty), \ i = 1, \ldots, m \), and assume that there exists \( i \) such that

\[ \text{mes}\{t \in [0, \omega]: \ p_i(t) > 0, \ t - \tau_i(t) > 0\} > 0, \ 0 < \delta_j \leq 1, \ j = 1, 2, \ldots. \]

The following assertions are equivalent:

1. The Cauchy function \( C(t,s) \) of (3.7)–(3.9) is positive for \( 0 \leq s \leq t < \infty \).
2. A nontrivial solution to the homogeneous equation (3.7)–(3.9) (\( \psi(t) = 0, \ t \in [0, \omega] \)) has no zeros on \([0, \omega]\).
3. The spectral radius of operator \( K \) is less than one.
4. Problem (3.7)–(3.10) is uniquely solvable for every summable \( \psi \), and its Green function \( G(t,s) \) is negative for \( 0 \leq t < s \leq \omega \) and nonpositive for \( 0 \leq s \leq t \leq \omega \).
5. Periodic problem (3.7)–(3.9), (3.11) is uniquely solvable, and its Green function \( P(t,s) \) is positive for \( t, s \in [0, \omega] \).
6. There exists a nonnegative absolutely continuous on each interval \([s_j, s_{j+1})\) function \( v \) satisfying condition (3.9) such that

\[ (\mathcal{L}v)(t) \leq 0, \ \ v(\omega) - \int_0^\omega (\mathcal{L}v)(s)\, ds > 0, \ t \in [0, \omega]. \]

**Step 2.** Let the conditions of the theorem be satisfied and a function \( x: [0, \infty) \to \mathbb{R} \) be the solution to the problem (1.5)–(1.7), (3.6). Then \( x \) is also a solution to the linear equation (3.7)–(3.9), where

\[ \Theta_i(t) = (H_i x)(t), \ t \in [0, \infty), \ i = 1, 2, \ldots, m, \ s_j = t_j, \ \delta_j = \beta_j, \ j = 1, 2, \ldots, \]

\[ \psi(t) = f(t) - \sum_{i=1}^m p_i(t)\tilde{\psi}(t - (H_i x)(t)), \quad \tilde{\psi}(t) = \begin{cases} \varphi(t), & t < 0, \\
0, & t \geq 0, \end{cases} \]

and \( t_j, \beta_j, \ j = 1, 2, \ldots, \) are defined by (2.1)–(2.4).

Define function \( v: [0, \infty) \to \mathbb{R} \) as follows:

\[ v(t) = \xi(t), \ t \in [0, s_1), \quad v(t) = \delta_1 \cdots \delta_k \xi(t), \ t \in [s_k, s_{k+1}), \ k = 1, 2, \ldots, \]
where
\[ \xi(t) = \exp\left(-e^t \int_0^t \sum_{i=1}^m p_i(s) \, ds \right). \]

This function satisfies condition (6) of Lemma 3.2. By virtue of this lemma we get the positivity of the Cauchy function \( C(t, s), 0 \leq s \leq t < \infty \). Then the positivity of \( x(t), t \in [0, \infty) \), follows from the solution’s representation formula.

Analogously, if \( x \) is a solution to (1.5)–(1.7), (3.5), then it is a solution to (3.7)–(3.10), where \( \theta_j, s_j, \delta_j, j = 1, 2, \ldots, \) are defined by (3.17), and \( t_j, \beta_j, j = 1, 2, \ldots, \) are computed according to (2.1)–(2.4), \( \psi \) is defined by (3.18). The function \( v : [0, \infty) \to \mathbb{R} \), constructed above, satisfies condition (6) of Lemma 3.2. Therefore, by virtue of the lemma the Green function \( G(t, s), t, s \in [0, \omega) \) of problem (3.7)–(3.10) is nonpositive, moreover, \( G(t, s) < 0 \) for \( 0 \leq t < s < \omega \). The solution’s representation formula of (3.7)–(3.10) implies the nonnegativity (nonpositivity) of \( x(t) \) for \( t \in [0, \omega) \) for any \( f(t) \leq 0, t \in [0, \omega] \), and \( \varphi(s) \geq 0, s \in (-\infty, 0) \) \( f(t) \geq 0, t \in [0, \omega], \varphi(s) \leq 0, s \in (-\infty, 0)) \). Exactly the same scheme works to prove the statements (1) and (2) of Theorem 3.1.

**Remark 3.3.** Let us point out that in case of nonimpulsive equation we have \( B(t) \equiv 1 \) and inequality (3.1) becomes unimprovable for nonoscillation of a nontrivial solution to the homogeneous equation (3.2), (3.3).

**Theorem 3.4.** Let \( \gamma_j \leq 1, p_i(t) \geq 0, t \in [0, \infty), i = 1, \ldots, m, \) and the following inequality holds
\[ \int_0^\omega \sum_{i=1}^m p_i(s) \, ds < \gamma_1 \gamma_2 \cdots \gamma_n, \]
where \( n \) is the number of points \( a_j \) in the interval \( (0, \omega) \).

Then a nontrivial solution \( x \) to the homogeneous equation (3.2), (3.3), (1.7) does not have zeros on \( [0, \omega] \) and the statements (3), (4) of Theorem 3.1 remain true. The statement (2) remains true if in addition there exists \( i \) such that \( \text{mes}\{t \in [0, \omega] : p_i(t) > 0, \, t - t_j(t) > 0\} > 0 \).

**Proof.** The Green function \( G_0(t, s), t, s \in [0, \omega] \), for the boundary value problem (3.16) has been constructed in [7]. From expression (3.13) for the Green function \( G_0(t, s) \) of problem (3.16) one can conclude that the following estimate for \( G_0(t, s) \)
\[ 0 \leq -G_0(t, s) \leq \frac{1}{\delta_1 \delta_2 \cdots \delta_n}, \quad t, s \in [0, \omega], \]
is valid. Moreover, this estimate is invariant with respect to the choice of the points \( s_j \in [a_j, a_{j+1}], j = 1, 2, \ldots, n \). Using the representation formula for the solution
\[ y(t) = \int_0^\omega G(t, s) \psi(s) \, ds \]
to the boundary value problem (3.16), in the case when

$$\psi(t) = -\sum_{i=1}^{m} p_i(t) y(t - \theta_i(t)), \quad t \in [0, \omega],$$

where $y(\xi) = 0$ for $\xi < 0$, we obtain

$$y(t) = (Ky)(t), \quad (3.21)$$

where operator $K : \mathbb{C} \to \mathbb{C}$ is defined by (3.15). Inequality (3.20) and the conditions

$$p_i(t) \geq 0, \quad t \in [0, \omega], \quad i = 1, \ldots, m, \quad \delta_j \leq 1, \quad j = 1, \ldots, n,$

allow to estimate the norm of the operator $K$ as follows:

$$\|K\| \leq \frac{1}{\delta_1 \delta_2 \cdots \delta_n} \int_0^{\omega} \sum_{i=1}^{m} p_i(s) ds.$$  

Inequality (3.19) implies that the norm $\|K\|$ is less than one, consequently the spectral radius of the operator $K$ is less than one. By virtue of Lemma 3.2 we have that $C(t, s) > 0$, $0 \leq s \leq t \leq \omega$, $P(t, s) > 0$, $G(t, s) \leq 0$ for $t, s \in [0, \omega]$ and a nontrivial solution to homogeneous equation (3.7)–(3.9) ($\psi(t) = 0$, $t \in [0, \omega]$) has no zeros on $[0, \omega]$.

To every solution $x$ to equation (1.5)–(1.7) we can put into correspondence equation (3.7)–(3.9), where $\Theta_i$, $i = 1, \ldots, m$, $s_j$ and $\delta_j$, $j = 1, 2, \ldots, n$, are found according to (3.17), and $\psi$ is defined by (3.18). The fact that for each such equation the statements of Lemma 3.2 remain true completes the proof of the theorem. \qed

**Remark 3.5.** Note that the suggested approach allows obtaining analogous results for equations of arbitrary order using the method introduced in [10,11]. An analog of Lemma 3.2 for a more general impulsive equation can be found in [8].

**References**


