Intransitivity of Winningness of Chess Positions and Its Consequences for Game Theory

Alexander Poddiakov
Department of Psychology, National Research University Higher School of Economics
E-mail: apoddiakov@gmail.com

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Abstract
In contrast to assumptions in some approaches to chess and checkers declaring transitivity of positions' winningness, examples of some positions' intransitivity are shown. Namely, position A of White is preferable (it should be chosen if choice is possible) to position B of Black, position B of Black is preferable to position C of White, position C of White is preferable to position D of Black, but position D of Black is preferable to position A of White. Intransitivity of chess positions is considered to be a consequence of complexity of the chess environment—in contrast with simpler games with transitive positions only. The space of relations between winningness of chess positions is non-Euclidean. The Zermelo-von Neumann theorem is complemented by statements about possibility *vs.* impossibility of building pure winning strategies based on the assumption of transitivity of positions. Questions about the possibility of intransitive positions in other positional games are raised.

Key words: chess, checkers, intransitivity of winningness of positions, game theory

1. Introduction

Intransitivity of winningness in mathematical objects became widely known beginning from Martin Gardner's mathematical games columns in *Scientific American* (Gardner, 1970, 1974). This has to do, for example, with dice with such figures on their faces that in pair throws die A shows a greater number than die B, die B more often shows a greater number than die C, but die C more often shows a greater number than die A. Accordingly, in order to win, die A has to be chosen in the A-B pair, die B in the B-C pair, and die C in the A-C pair. This accords with the principle of rock-paper-scissors game, and contrasts with the transitive relation of dominance "if A≻B and B≻C then A≻C" seeming universal ("≻" means "is preferable to", "more favorable" etc.).[1]

Intransitive sets of playing cards, roulettes, lotteries, etc. may work on the same principle. By now many paradoxical examples of intransitive relation "stochastically greater than" have been invented and many studies in this area have been conducted (Akin, 2019; Bednay, Bozóki, 2013; Bozóki, 2014; Conrey et al., 2016; Grime, 2017; Hązła et al., 2020; Hulko, Whitmeyer, 2019; Lebedev, 2019; Pegg Jr., 2005; Poddiakov, Lebedev, 2021; Polymath, 2017; Trybuła, 1961; Van Deventer, 1992 etc.).

Such intransitive objects are considered in advanced game theory of Moulin (2008)—e.g., Gale's roulette.

"a) Each wheel has an equal probability to stop on any of its numbers. Player 1 chooses a wheel and spins it. Player 2 chooses one of the 2 remaining wheels (while the wheel chosen by 1 is still spinning), and spins it. The winner is the player whose wheel stops on the higher score. He gets $1 from the loser. Numbers on wheel #1: 2,4,9; on wheel #2: 3,5,7; on wheel #3: 1,6,8. Find the value and optimal strategies of this game.

b) Variant: the winner with a score of *s* gets $*s* from the loser" (Ibid., p. 17).

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[1] A note on terminology: the terms "intransitive" and "non-transitive" (e.g., "intransitive dice" and "non-transitive dice") are used for such sets as synonyms in the math literature in spite of

Butler and Pogrebna (2018) have used bespoke-designed intransitive lotteries to study intransitive preferences of participants in a behavioral economics experiment and shown that the participants realize here predictably intransitive choices (see also [Butler, Blavatskyy, 2020]).

The paradox of stochastic intransitivity can lead to paradoxical results in some cases of application of Wilcoxon–Mann–Whitney test in statistics (Butler, Blavatskyy, 2020; Korneev, Krichevets, 2011; Thangavelu, Brunner, 2007).

How does existence of intransitive objects affect the transition from the classical game theory to the advanced one? No answer to this question can be comprehensive, because a list of intransitive objects and systems and their types is getting ever longer.

For example, mechanically interacting geometrical objects in deterministic intransitive relations “to rotate faster than”. “to be stronger than”. “to lift and not be lifted” etc. are possible (Poddiakov, 2018, 2019; Van Deventer, 2019, n.d.).[2]

Intransitive gears are an example of intransitive machines (Fig. 1). This is a Condorcet-like composition reproducing, in mechanical and geometrical form, the structure of elements in the Condorcet paradox (Poddiakov, 2018). A player more competent in mechanics can always beat a less competent one by getting him/ her to be the first to choose an object and then choosing a construction which will rotate faster in the pair. Much more complicated constructions (e.g., meta-intransitive levers with nested intransitive loops of intransitive levers) are possible too (Poddiakov, Lebedev, 2021).

Thus, the possibility of the mechanical constructions in rock-paper-scissors relations can create new links between mathematical logic, theory of machines and mechanisms, and game theory.

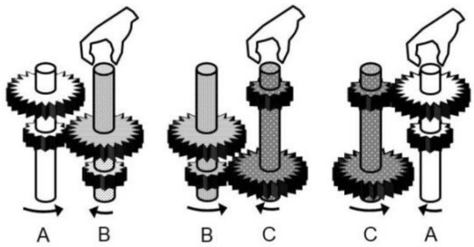

Figure 1. Intransitive gears
Gear A rotates faster than B in pair A-B, B rotates faster than C in pair B-C, and C rotates faster than A in pair A-C.

It has to be stressed that the existence of stochastic intransitive objects does not challenge the basic propositions of the probability theory, and the existence of mechanical intransitive objects does not violate laws of physics. The existence of such objects is a logical, albeit a counter-intuitive, consequence of these provisions and laws.

Yet what about deterministic positional games like chess and checkers? Dice are characterized by numbers on their faces, and some dice have such numbers that the dice are intransitive. Chess and checkers are characterized by positions of White and Black. Can these positions be intransitive?

The second note on terminology is about term “position”. The term has two meanings. On the one hand, it means one player’s position (e.g., in comparison with the other player’s position). Some examples of using the term in this meaning are the following.

David Brodsky entitles his post *Winning Equal Positions: An Imperfect Job* and writes: “Both players’ positions are fairly solid. White’s rook on h1 isn’t doing anything for the moment, and his superfluous knights aren’t awe-inspiring, but he can improve his pieces”

[2] Cf. *Robot Darwinism*, a “rock-paper-scissors”-like phenomenon similar to real biological intransitive competition (Special.., 2018), in Battle Bots shows (Atherton, 2013).

(Brodsky, 2018). “You need to learn to find weaknesses in your opponent's positions” (Van Apeldoorn, 2014). “If the idea of balance was sought of as an old fashion scale, like the scales of justice, both players’ position would hang equally in relation to one another. However, if one player has better development the scale will tilt in his or her favor” (Patterson, 2014).

On the other hand, “position” means a whole including both players’ positions (positions of White and Black): “The position on the board looks very interesting” (it is not about one player’s position). Here one should quote the following statements concerning transitivity of positions in the second meaning.

“Current schemes for machine evaluation of a chess position are based on two assumptions: First, for any two positions, it is possible to decide which is more favorable and second, that this relation is transitive. In consequence of these assumptions, *a number that serves as a measure of worth can be assigned to each possible chess position*” (Atkinson, 1998, p. 38) (my italics – *A.P.*).

“The construction of a checkers endgame database is simply the computation of a transitive closure. Each position is a member of either the set of wins, losses or draws. Once computed, the classification of a database entry represents *perfect knowledge as to the theoretical value of that position*” (Lake et al., 1993, p. 3) (my italics – *A.P.*).

Account of perfect values based on playout values has been used to build a common evaluation function for playing Chinese Checkers with two or more players (Sturtevant, 2016).

A clarification is in order which is important for practical play but not for theory. Two above-cited works consider sets only of such pairs of positions transition between which according to the rules of the game is possible during playout. Based on this approach checkers has been solved (Schaeffer et al., 2007). Without questioning the result achieved (“checkers is solved”) in a set of positions, arising of which is possible in playout games, we can tackle the question of whether it is possible to calculate perfect theoretical values of each player’s position in the whole set of all positions.

The main questions of the article: are transitive ordering of all the positions and account of theoretical, perfect value of both player’s position (in any “absolute rating” of the positions in the first meaning) possible in chess and checkers? Are transitive ordering of all the positions (in the second meaning—as wholes) and account of theoretical, perfect value of each position possible?

The answer is: it is impossible.

To ground this claim let us use the methodological approach of building “anomalies” (“pathologies”, “monsters”--according to Lakatos [1976]). In mathematics, the term “pathological” is used to refer to examples “specifically cooked up to violate certain almost universally valid properties. Pathological problems often provide interesting examples of counterintuitive behavior” (Weisstein, n.d.). “The idea of mathematical pathologies” refers “to examples that are specifically designed to violate properties that are perceived as valid” (Haavold, Sriraman, 2021).

We hope that the examples presented below are “hopeful monsters”, in Lakatosian terms, i.e. holding out a promise of some new ideas in game theory.

2. Examples of intransitive chess and checkers positions

Let us consider specially constructed chess positions chains of which cannot arise in a game unfolding according to classical rules, but which are not forbidden by these rules.

Let us consider the following four positions: position A of White, position B of Black, position C of White, and position D of Black (Fig. 2) (Poddiakov, 2016, p. 48).

Position A of White | Position B of Black | Position C of White | Position D of Black

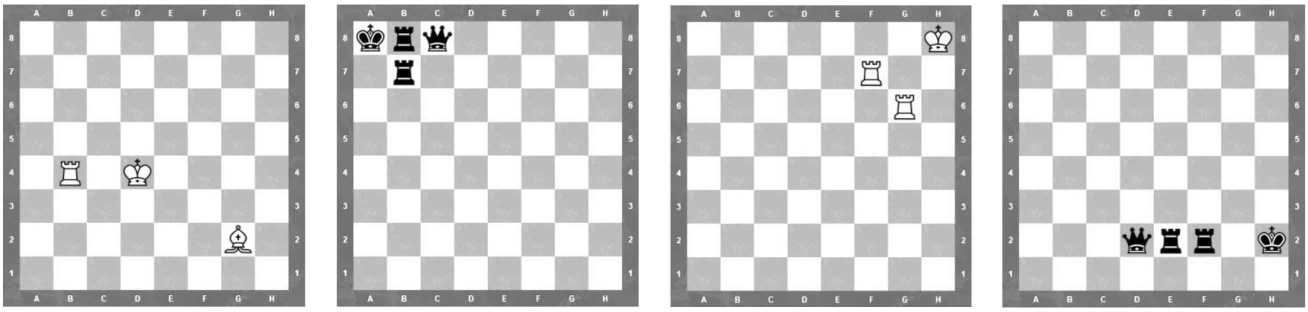

Figure 2. Four chess positions to be superposed for demonstration of intransitivity of their winningness

Now let us check their pairwise superpositions. Let us lay positions

- A and B
- B and C
- C and D
- A and D

on a chess board (Fig. 3).

White start in all the compositions in accordance with the rules of chess problem composing.

$A \succ B$ | $B \succ C$ | $C \succ D$ | $D \succ A$

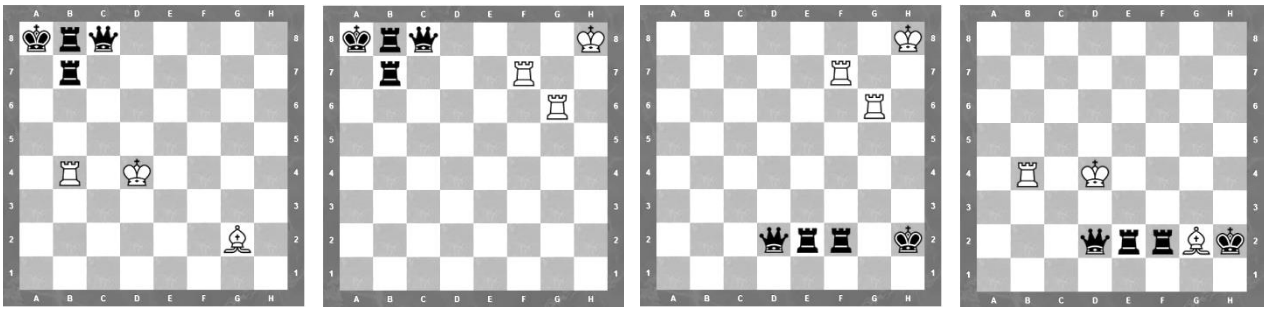

Figure 3. The chess positions superposed for demonstration of intransitivity of their winningness

One can see that:

- position A of White is preferable (it should be chosen if choice is possible) to position B of Black;
- position B of Black is preferable to position C of White;
- position C of White is preferable to position D of Black;
- but position D of Black is preferable to position A of White

(like in rock-paper-scissors game).

Let us now analyze these compositions in terms of "positions" in the second meaning (positions as wholes including both players' positions). Let us denote the first composition (including position A of White and position B of Black) as AB, the second one—BC, the third—CD, and the fourth—AC. The conclusion is the same:

- position AB is preferable (it should be chosen if choice is possible) to position BC;
- position BC is preferable to position CD;
- position CD is preferable to position AD;
- but position AD is preferable to position AB.

Alexander Filatov (2017) has designed a minimalist and symmetrical intransitive chess positions (a lesser number of pieces is impossible) (Fig. 4).

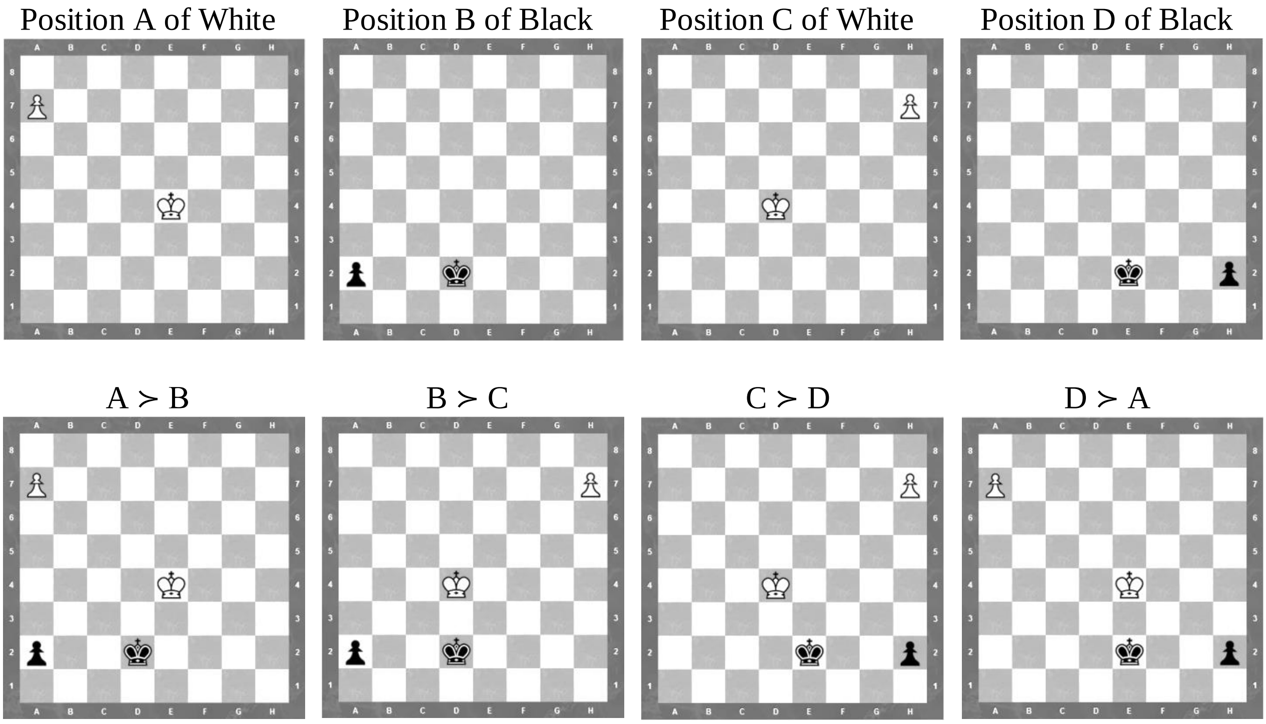

Figure 4. Filatov's minimalist and symmetrical intransitive chess composition.

Various chess problems can be designed based on intransitivity. Grigory Popov, a chess composer, has designed the following problem (Fig. 5). For a possible solution see (Popov, 2021).

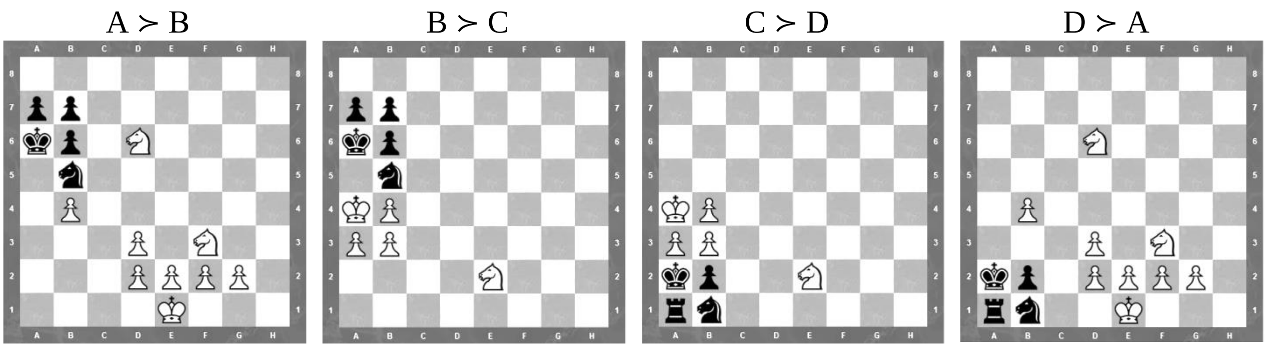

Figure 5. Popov's chess problem based on intransitivity.

3. How frequent are intransitive cycles of chess positions?

Alexander Filatov (2017), based on my example presented above, has shown that the number of intransitive positions in chess is astronomically large, and that intransitive chess chains can be of astronomical length (so it is futile to try to build the longest one).

What is the share of intransitive positions in the total body of positions?

No answer has been given, but we propose an approach to solving the question.

Since there exist the Nalimov tablebases, which make it possible to calculate the outcome of any endgame for 3-7 pieces, we can do the following.

1. Consider 4-7-piece endgames varying the length of position chains (4, 6, 8 positions). As it is difficult to go over all of them, let us fall back on the Monte-Carlo method to generate chains: we randomly generate position A of White (two pieces—the King and some other piece), position B of Black of two pieces, position C of White of two pieces and position D of Black of two pieces (for a chain of length 4). We see if an intransitive cycle of winningness is formed in this chain. Repeat the same for 4 new generated positions. Do it multiple times. Count the number of intransitive cases among all the cases considered. A similar procedure has been used to evaluate the share of intransitive chains for N-sided dice (Conrey et al., 2016).

Counterintuitively, but by analogy with the results of (Ibid.), with a large number of positions (A, B, C, D, E, F; A, B, C, D, E, F, G, H; etc.) the probability of encountering an intransitive chain does not radically diminish, remaining at least a value of the same order as the probability of drawing a transitive sequence and may even larger.

2. Compare the results for endgames with varying numbers of pieces and unvarying length of chains. Again, counterintuitively, the probability of “drawing” an intransitive chain may grow with the growth of the number of pieces in chains of equal length.

Intransitive checkers positions are also possible. See an example below (Fig. 6).

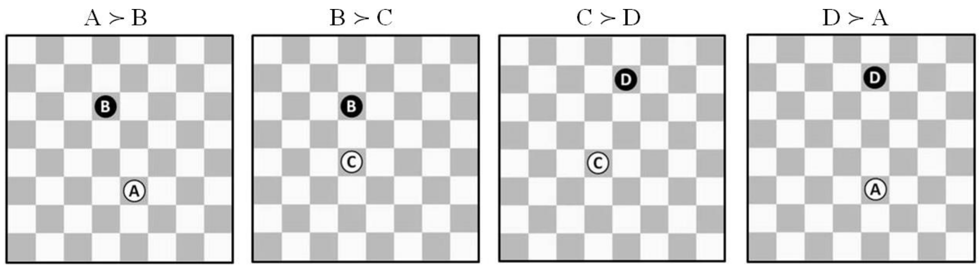

Figure 6. Zhurakhovsky’ intransitive checkers positions, in context of Poddiakov’s intransitive chess positions. A ≻ B, B ≻ C, C ≻ D, D ≻ A.

Using the methods analogous to those used by Filatov, it is apparently possible to show that the number of intransitive chains in checkers is also very large.

The intransitivity of positions shown above hardly matters for practical play. Let us repeat that intransitive positions exist in the space of all positions (and not in an ordinary played game) and are only revealed when superposed on one board artificially. But intransitivity of positions is important for theoretical assessment of (im)possibility of transitive ordering of all positions and for calculating the perfect value of each of them.

4. Intransitivity of chess positions is a consequence of complexity of chess environment

Intransitivity of chess positions is a previously unknown property of chess. No one initially sought to create conditions for the building of intransitive chess positions (including chains of astronomical length). This non-evident property was an unplanned result of a complicated environment created by humans.

I think that intransitive positions become impossible in simpler environments—e.g., on small chess boards (3×3? 4×4?). If so, it is interesting to determine the minimum size of the board on which intransitivity gets already possible.

And what about shapes of boards? Are intransitive chess positions possible on toroidal and cylinric boards?

Oleg Yarygin’s hypothesis (informal communication): there exists the number of moves between chess positions that must be exceeded before intransitivity becomes possible. It would be interesting to determine the limit beyond which intransitivity becomes possible and find out whether it can be a measure of the game’s complexity.

Intransitivity of positions may exist in other complex positional games, for example, in Chinese Checkers, Go, Reversi etc. Perhaps, the number and ratio of intransitive positions among all positions, and their diversity can serve as measures of complexity of a positional game.

Having shown that intransitive positions are possible in complicated games, let us now look at a simpler game with a *transitive order* of positions' winningness and the possibility of calculating their perfect values. We would need this for further comparison.

5. The Magicians: A simpler game with transitive positions only

I have designed The Magicians to study children's thinking and problem solving. In some aspects, The Magicians are similar to Reversi (see info about this game in [MacGuire, 2011]) but is much simpler than it.

The material of the game are similar cards. Each card has the picture of a good magician (a smiling rosy face) on one side and a bad magician (an angry blue face) on the other side.

Rules of the game. The cards are arranged in two horizontal rows.

Figure 7. Composition of the cards in Problem "Three good magicians against 2 bad ones".

By shifting the cards bad magicians can be turned into good ones and vice versa. The aim of the game is to turn all bad ones into good ones. The move consists in swapping one card from the upper row for one card in the lower row.

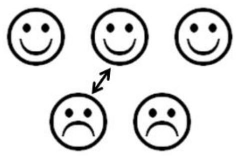

Figure 8. A possible move (swap) in Problem "Three good magicians against 2 bad ones".

Swapping within the row is forbidden. Shifting a magician within a row without a swap is forbidden.

If, as a result of the swap a magician swapped finds oneself between two wrong ones (i.e. a bad one between two good ones or a good one between two bad ones) he too becomes a "wrong one") (the card is turned with the other side up). "You've got surrounded and the same as surroundings" (a mnemonic rule for children).

In all other variants of neighbourhood no transformation occurs. It has to be stressed that only magicians who are new to a position are liable to be transformed. The rest do not change even if they get hostile neighbors as a result of the swap.

Examples of some problems are given below.

Figure 8. Problem "Five good magicians against 5 bad ones".

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Figure 9. Problem “Two good magicians against 5 bad ones” (preliminary swaps without the magicians transformations are necessary here, and it is difficult for most children).

I have devised a computer version of the game for which I have written an algorithm involving an active virtual enemy (the chief bad magician who plays for the bad ones).

This algorithm takes into account only: the position of friendly magicians in a row (the closer the magician is to the middle of the row the higher his value) and the working pairs. A working pair is two friendly magicians separated by one cell in which they convert foes under favorable circumstances, like on a conveyor belt. Each such pair adds to the score.

It is a simple game: the assessment of the position of “friends” does not include the position of the “strangers”, *all the positions (down to symmetrical ones) have their unique values, and there is no intransitivity*.

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Thus, there exist two types of deterministic positional zero-sum games.

1. Games without positions intransitive in terms of winningness. All the positions can be ordered transitively and perfect values can be built for all positions (like in The Magicians).

2. Games with positions of the sides intransitive in terms of winningness (chess, checkers). Positions of each side in such games do not lend themselves to absolute evaluation and absolute rating (there can be no perfect values for all the positions).

6. Conclusion

The novelty of this paper is in revealing unexpected intransitivity of positions’ winningness (in both meanings of term “position”) in some strategy deterministic games. The intransitivity is a consequence, a by-product of complexity of the games. The complexity of the chess (checkers) environment makes intransitive positions possible—in contrast to simpler games.

Peter C. Fishburn, the author of a theory of decision making without axiom of transitivity, wrote that rejection of intransitivity is analogous with “rejection of non-Euclidean geometry in physics”, which “would have kept the familiar and simpler Newtonian mechanics in place, but that was not to be” (1991, p. 117).

In turn, we have shown that for chess and checkers any Euclidean metric of positions’ winningness across their whole set is impossible because some positions form intransitive cycles. The space of mutual relations between winningness of chess positions is non-Euclidean. There are no perfect values (numbers in any absolute rating) of all the positions in Euclidean space.

Let us consider it in more detail. Perfect values of positions in case of their intransitive cycling should be equal to one another (each position beats another position and is beaten by the third one—like mutual beating of intransitive dice). The Euclidian distance between them should be equal to zero (there should be no any preferences between them).

Yet if A beats B then the value of A should be higher than the value of B (the distance between the positions does exist), and so on, and if the last member of the sequence beats the first one then the value of this last member should be higher than the value of the first one (the distance between the positions does exist).

Thus, the existence of intransitive positions means that a number that serves as a measure of worth (preference, winningness) cannot be assigned to each possible chess (checkers) position.

This contradicts statements like “a number that serves as a measure of worth can be assigned to each possible chess position.” Perfect values of positions in chess and checkers are impossible.

The number of such positions in chess is huge, but one example, like in checkers, suffices to prove that perfect values of positions are impossible.

What is called the Zermelo-von Neumann theorem consists in that all positional chess-like games with perfect information have solutions in pure strategies (Csákány, 2002; Kolchin, n.d.).

For my part, the following should be added.

For some of these positional games a solution is possible on the basis of the assumption on the transitivity of the winningness of positions in the whole set of positions.

But for some games such a blanket solution is impossible (because they have intransitive winningness positions). Other solutions are needed.

The results presented in the article can be seen as a contribution to advanced game theory.

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