The Radiuses of the Schwarzschild-de Sitter/AdS Black Holes

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A simple approach to calculate an approximate value for the black hole radius in space-time with the Schwarzschild-de Sitter (SdS) metric is presented. Equating the escape velocity to the speed of light leads to the cubic equation; those roots provide the solution for two horizons of the SdS black hole. The same approach leads to an estimate for the radius of the black hole in the Schwarzschild-anti de Sitter (AdS) space-time. Obtained result for the SdS metrics, when approximated to the second-order term, differs from the known expression for the Schwarzschild radius by a numerical factor of 3/4. Though in the case of the Schwarzschild-AdS metrics, such method results in approximation, which coincides with the classical expression for Schwarzschild black hole. The presented material would suit for pedagogical purposes for undergraduate students specializing in General relativity and astrophysics.

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There is increased interest to black holes in the de Sitter metric in recent studies motivated by the fact that the Universe metric may correspond to the de Sitter metric [7] as an exact solution of the Einstein field equation of General relativity for empty matter Universe [4],[8],[9]. There are few tests suggested based on the redshift value and the geodesics to verify possible Solar system effects in Schwarzschild-de Sitter (SdS) space-time [6]. Several recent works are dedicated to the calculation of the parameters of the SdS black holes, such as total emissivity, the radiation spectra, and the temperature of Hawking radiation [11]. The purpose of this study is to evaluate the SdS black hole radius as this simple topic is missing is such literature, but would be useful for further fast reference.

The exact expression for the Schwarzschild black hole radius is well known

$$r_g = \frac{2Gm}{c^2} \tag{1}$$

Generally, it follows from the solution to Einstein field equations (without the cosmological constant) of the General relativity (GR). This radius defines the event horizon of a static Schwarzschild black hole. Just because the GR field equations lead to the Schwarzschild metric in case of a spherically symmetric gravitational field

$$ds^2 = -\left(1 - \frac{2Gm}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2Gm}{c^2 r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{2}$$

where $d\Omega^2 = sin^2 \theta \, d\varphi^2 + d\theta^2$ and then basically (1) follows from (2). The fact however is that the Schwarzschild radius for a black hole can also be derived via classical approach without knowledge of GR - using just the escape velocity value

$$v = \sqrt{\frac{2Gm}{r}} \tag{3}$$

and then equating it to the speed of light. Such approach may provide an intuitive way to obtain the solution because it obviously means that at such radius the light cannot escape from gravitation. (By the way the expression (1) first time was derived by Michel in 1783 and by Laplace in 1796).

It is important to note that in case of presence in the GR field equation of a positive cosmological constant, the Einstein's "greatest blunder", the solution is given then by the Schwarzschild-de Sitter (SdS) metrics [4],[8],[9] which reads

$$ds^2 = -\left(1 - \frac{2Gm}{c^2 r} - \frac{H^2 r^2}{c^2}\right) c^2 dt^2 + \left(1 - \frac{2Gm}{c^2 r} - \frac{H^2 r^2}{c^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{4}$$

where H is the Hubble constant. Or via the lambda-term [7]

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$$ds^2 = -\left(1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} r^2\right) c^2 dt^2 + \left(1 - \frac{2Gm}{c^2 r} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5}$$

The obtaining the escape velocity in General relativity seems as not obvious task, even for the Schwarzschild metric. One may refer to reviews given in [1][12]. For an arbitrary spherically symmetric case the metric is given by

$$ds^2 = -g_{00} c^2 dt^2 + g_{11} dr^2 + g_{22} d\Omega^2 \tag{6}$$

where $g_{22}=r^2$ and the escape velocity is to be derived from the geodesic equation

$$\frac{d^2 x^k}{ds^2} + \Gamma^k_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0 \tag{7}$$

Using the transform that allows writing t instead of s and expressing the Christoffel symbols via metric tensor components [12] one may obtain

$$\frac{d^2 r}{dt^2} + c^2 \frac{g_{11}'}{2g_{11}} - \left(\frac{g_{00}'}{g_{00}} - \frac{g_{11}'}{2g_{11}}\right) \left(\frac{dr}{dt}\right)^2 = 0 \tag{8}$$

where (‘) means differentiation *d/dr*. The resulting escape velocity may be confusing, and in fact it coincidences with usual escape velocity only in Newtonian approximation or in weak gravitational field. But the most important point that the resulting value from (8) is the escape velocity measured by a *distant observer.* Another approach mentioned in [1], referring to Hartley [5], is to use the energy of a test particle in spherically symmetric gravitational field (in the way it is done when deriving the gravitational redshift). In such case the only g_{00} component of the metrical tensor shall be used

$$g_{00} (v_0)^2 = c^2 \tag{9}$$

where v_0 is the temporal component of the escape four-velocity. Basically, this relation expresses the equality of relativistic kinetic energy of a test particle with its potential energy in gravitational field. Importantly it gives the escape velocity as measured by *stationary observer* located in the gravitational field. The (9) simply is

$$\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{g_{00}}} \tag{10}$$

Then, in case of the Schwarzschild-de Sitter metric (4) the escape velocity is given by*

$$v(r) = \left(\frac{2Gm}{r} + H^2 r^2\right)^{\frac{1}{2}} \tag{11}$$

In order to find the black hole radius in the Schwarzschild-de Sitter space-time one has to equate (11) with speed of light *v(r)=c* as

$$\frac{2Gm}{r} + H^2 r^2 = c^2 \tag{12}$$

The solution is not trivial as it was in case of the Schwarzschild metric. But this is form of the cubic equation can be easily reduced to depressed cubic equation form as

$$r^3 - \frac{c^2}{H^2} r + \frac{2Gm}{H^2} = 0 \tag{13}$$

We will use Vita's method to solve it [10]. Denoting the coefficients as

$$p = -\frac{c^2}{H^2} \qquad q = \frac{2Gm}{H^2} \tag{14}$$

We see that p has an order of the cosmological distance (the Hubble length) squared and q has dimension of the volume, those value is also relatively big but smaller than p. It is easy to check that the discriminant is negative:

$$\Delta = 4p^3 + 27q^2 = -\frac{4c^6}{H^6} + 27 \frac{4G^2 m^2}{H^4} < 0 \tag{15}$$

therefore, in this case we may expect to have three real roots. The solutions are given by

$$r_k = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\varphi}{3} + \frac{2k\pi}{3}\right) \tag{16}$$

Where *k=0, 1, 2* and

* the same estimate for the SdS metric is given in [6], see expression (41).

$$\varphi = \text{acos}\left(\frac{3q}{2p}\sqrt{\frac{-3}{p}}\right) \tag{17}$$

Substituting the coefficients (14)

$$r_k = \frac{2}{\sqrt{3}}\frac{c}{H}\cos\left(\frac{\varphi}{3} + \frac{2k\pi}{3}\right) \tag{18}$$

$$\varphi = \text{acos}\left(-\frac{3\sqrt{3}GmH}{c^3}\right) \tag{19}$$

The argument of *acos* is an extremely small number, and then we may assume that $\phi \approx \pi/2$ obtaining three approximate roots of (13) as

$$r_0 = \frac{c}{H} \qquad r_1 = -\frac{c}{H} \qquad r_2 = 0 \tag{20}$$

The second root (r1) with a minus sign can be withdrawn as non-physical. The first and the third ones provide the solutions for the distance where the velocity (11) becomes equal to c. The first root (r0) corresponds to a maximal distance, which is the cosmological horizon. The last one corresponds to the gravitational radius being almost zero. So, we see that the Schwarzschild-de Sitter black hole has two event horizons (r0 and r2) instead of the simple one as we had in case of a black hole in the Schwarzschild metric.

Coming back to estimation of the SdS black hole radius. As it was shown that it should be a root with minimal radius, specifically r_2(*k=2*). So, let's now evaluate it without such a rough estimation for φ as it was done before. Rewriting the cosine in (18) as a sum of its arguments

$$r_k = \frac{2c}{\sqrt{3}H}\left(\cos\left(\frac{\varphi}{3}\right)\cos\left(\frac{2k\pi}{3}\right) - \sin\left(\frac{\varphi}{3}\right)\sin\left(\frac{2k\pi}{3}\right)\right) \tag{21}$$

And for *k=2*

$$r_2 = \frac{2c}{\sqrt{3}H}\left(\cos\left(\frac{\varphi}{3}\right)\cos\left(\frac{4\pi}{3}\right) - \sin\left(\frac{\varphi}{3}\right)\sin\left(\frac{4\pi}{3}\right)\right) \tag{22}$$

Using the known values for

$$\cos\left(\frac{4\pi}{3}\right) = -\frac{1}{2} \qquad \sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2} \tag{23}$$

Substitution of them into (22) leads to

$$r_2 = \frac{c}{\sqrt{3}H}\left(-\cos\left(\frac{\varphi}{3}\right) + \sqrt{3}\sin\left(\frac{\varphi}{3}\right)\right) \tag{24}$$

and then approximating only the cosine as before (assuming $\phi \approx \pi/2$)

$$r_2 = \frac{c}{H}\left(-\frac{1}{2} + sin\left(\frac{\varphi}{3}\right)\right) \tag{25}$$

Now one may use the Taylor series expansion for small argument values for *acos* in (19) such as

$$\text{acos}(x) = \frac{\pi}{2} - x - \frac{x^3}{6} - \cdots \tag{26}$$

Then (19) becomes

$$\varphi \approx \frac{\pi}{2} + \frac{3\sqrt{3}GmH}{c^3} \tag{27}$$

And therefore

$$\frac{\varphi}{3} \approx \frac{\pi}{6} + \frac{\sqrt{3}GmH}{c^3} \tag{28}$$

But before substitution of this value into (25), one needs to express *sin(ϕ/3)* as sum of its arguments given by (28) as

$$\sin\left(\frac{\varphi}{3}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\sqrt{3}GmH}{c^3}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\sqrt{3}GmH}{c^3}\right) \tag{29}$$

Using the numerical values (23) again

$$\sin\left(\frac{\varphi}{3}\right) = \frac{1}{2}\cos\left(\frac{\sqrt{3}GmH}{c^3}\right) + \frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}GmH}{c^3}\right) \tag{30}$$

Then the use of the Taylor series expansion for *sin* and *cos* for small arguments

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots$$
$$\sin(x) = x - \frac{x^3}{6} + \cdots \quad (31)$$

And taking into account only the first terms of the approximations then (30) becomes

$$\sin\left(\frac{\varphi}{3}\right) = \frac{1}{2} + \left(\frac{3GmH}{2c^3}\right) \quad (32)$$

Substitution into (25) leads to the solution for the radius of a black hole in the Schwarzschild-de Sitter metric

$$r_2 = \frac{3Gm}{2c^2} \quad (33)$$

The expression is not exact as all above approximations by the second order term were used. However, it shows that the radius of the SdS black hole is 4/3 times smaller than the Schwarzschild radius.

The case of the Schwarzschild-AdS metrics

The Schwarzschild-Anti-de Sitter metrics is given by the same form as (4) and (5) but with negative cosmological constant. So the only difference is the signs in our equation (13) which looks now as

$$r^3 + \frac{c^2}{H^2} r - \frac{2Gm}{H^2} = 0 \quad (34)$$

Denoting the coefficients as

$$p = \frac{c^2}{H^2} \qquad q = -\frac{2Gm}{H^2} \quad (35)$$

Obviously, contrary to the previous case, the discriminant is positive:

$$\Delta = 4p^3 + 27q^2 = \frac{4c^6}{H^6} + 27\frac{4G^2m^2}{H^4} > 0$$

therefore in this case the only one real root given by hyperbolic sinus is expected as

$$r = -2\sqrt{\frac{p}{3}} \sinh\left(\frac{\varphi}{3}\right) \quad (36)$$

Where

$$\varphi = \operatorname{asinh}\left(\frac{3q}{2p}\sqrt{\frac{3}{p}}\right) \quad (37)$$

Substituting the coefficients (34)

$$r = -\frac{2}{\sqrt{3}}\frac{c}{H} \sinh\left(\frac{\varphi}{3}\right) \quad (38)$$

$$\varphi = \operatorname{asinh}\left(-\frac{3\sqrt{3}GmH}{c^3}\right) \quad (39)$$

This case it is much easier because the Taylor series are given

$$\operatorname{asinh}(x) = x - \frac{x^3}{6} + \cdots \qquad \sinh(x) = x + \frac{x^2}{6} + \cdots \quad (40)$$

Therefore

$$\varphi = -\frac{3\sqrt{3}GmH}{c^3} \quad (41)$$

And finally,

$$r = \frac{2}{\sqrt{3}}\frac{c}{H}\frac{\sqrt{3}GmH}{c^3} = \frac{2Gm}{c^2} \quad (42)$$

So, an approximation to the second order term leads to the same expression for the black hole radius as one could have it in the Schwarzschild space-time. The reader can check the radiuses of two remaining imaginary roots using the same procedure if one believes that an imaginary radius and distance could have any physical significance.

Testing the second order term

The second-order term in the above approximations should be examined for the consistency of the obtained solutions. Denoting the second-order term as $r^{(2)}$ in case of the Schwarzschild-AdS metrics using (40) one would write

$$\text{asinh}(a) = -\frac{3\sqrt{3}GmH}{c^3} + \frac{1}{6}\left(\frac{3\sqrt{3}GmH}{c^3}\right)^3 +.. \tag{43}$$

and therefore

$$r^{(2)} = \frac{1}{6}\frac{2}{\sqrt{3}}\frac{c}{H}\left(\frac{\sqrt{3}GmH}{c^3}\right)^3 + \cdots = 3\left(\frac{H}{c}\right)^2\left(\frac{Gm}{c^2}\right)^3 = \frac{3}{8}\frac{r_g^3}{R_H^2} \tag{44}$$

where $R_H$is the Hubble radius or the de Sitter horizon distance. Obviously, the value for $r^{(2)}$ is negligible small comparing to the Schwarzschild radius r_g. Therefore, the second-order term can be safely neglected, and it shows that the solutions (33) and (42) are given with satisfactory precision.

Conclusion

It was shown that the radiuses of black holes in the case of the SdS and the Schwarzschild-AdS metric could be evaluated with reasonable accuracy. Importantly, the obtained results do not depend on the value of the cosmological constant, contrary to, for example, the parameters for the Hawking radiation of the SdS black hole obtained in [11]. The most existing recent observations of the Black hole in M87 galaxy unfortunately still do not provide the precise value of the black hole mass yet. Though it would be highly challenging to check the exact theoretical parameters of the observed black holes, especially for the consistency with the de Sitter Universe and a possible correspondence with the obtained the SdS radius (33).

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