ALADINS: an ALgebraic splitting time ADaptive solver
for the Incompressible Navier-Stokes equations
Part 1: Basic settings and analysis

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Abstract

We address a time-adaptive solver specifically devised for the incompressible Navier-Stokes equations. One of the challenging issues in setting up a time-adaptive solver is the identification of a reliable \textit{a-posteriori} error estimator. Typical strategies are based on the combination of the solutions computed either with two different time steps or two schemes with different accuracy. In this paper we move from the \textit{pressure correction algebraic factorizations} formerly proposed by Saleri, Veneziani (2005). These schemes feature an intrinsic hierarchical nature, such that an accurate solution for the pressure is obtained by computing an intermediate low-order guess. The difference between the two estimates provide a natural \textit{a-posteriori} estimator. In this first part we address the properties of this approach, and its possible variants, including the \textit{pressure incremental} formulation. Numerical results refer to 2D test cases. The second part will cover implementation details and results in 3D. This work is dedicated to the memory of F. Saleri.

\textit{Keywords:} Incompressible Navier-Stokes, Time adaptivity, Inexact algebraic factorizations

1. Introduction

Accurate and effective methods for the numerical solution of incompressible fluid dynamics is an old but still important challenging problem, as more and more complex problems in engineering, biology, ecology, medicine,
Figure 1: Two problems demanding for time adaptivity: pressure dynamics of the oil in a brake (left - courtesy of Brembo, Italy) and flow rate in a carotid artery (right).

sport are tackled with computational methods. Many different options are available for improving the efficiency of an incompressible fluid simulation, ranging from parallelization to the identification of appropriate preconditioners [1, 2]. In some cases, the practical problems at hand involve the sequence of fast and slow transients, only partially (or not at all) predictable a priori. This is the case, for instance, of the pressure dynamics in the oil surrounding the piston of a brake (see Fig. 1, left), that experiences a peak in corresponding of a sudden braking action; or the blood flow in arteries (see Fig. 1, right), featuring the sequence of the systolic phase, when the aortic valve is open and the velocity is high, and the diastolic phase where the valve is closed and the velocity falls. The duration of each phase is patient-specific and could change in time. In these cases, a time adaptive implementation can reduce the computational effort by reducing the number of time steps required in the intervals of slow transients. Time adaptivity requires basically two ingredients: (a) a reliable and effective a posteriori error estimator for testing the adequacy of the current time step; (b) an adapting rule for selecting a proper time step on the basis of the estimated error. The latter is usually set up on an empirical basis, relying upon the features of the estimator and some “safety coefficients”. This has the role of finding the trade-off between the selection of the largest possible time step and the need of limiting the number of its variations. The former is a critical issue. Residual estimators check quantities that are prevented to vanish - as they would do at the continuous level - by the numerical scheme. More in general, estimators are computed
either by combining different schemes with a different order of accuracy, or using the same scheme with two different time steps (see e.g. [3]). Both these strategies can result in an increment of the computational cost, since the numerical solution is computed in two different ways. This has probably limited the extensive adoption of time adaptive schemes for the Incompressible Navier-Stokes (INS) equations, which are intrinsically fairly expensive at the numerical level. A few recent papers address time adaptivity for this problem. In [4] a stabilized space-time Galerkin method is presented for solving free-surface problems with a time adaptivity based on the residual of the continuity equation. In [5] adaptivity is based on the combination of two-steps linear implicit schemes with different order. In the recent papers [6, 7], a smart combination of the trapezoid rule and the Adams-Bashforth 2 schemes is used for estimating the local truncation error and eventually computing the proper time step. Adaptivity is applied to the coupled momentum-mass system rather than to segregated schemes, since the latter are expected to require smaller time steps. In the present work, we introduce however a time-adaptive method specifically devised for the algebraic splitting schemes we have previously proposed for the INS equations. Splitting schemes introduce in general a “consistency” error. In [8] we proposed high order splitting methods upon introducing a pressure correction step in the basic inexact LU factorization formerly proposed in [9, 10]. Performances of the method with spectral element space discretization have been presented in [11], while the theoretical properties (with fixed time step) have been investigated by Ger- vasio [12] and the first author [13]. The inexact LU factorization approach has been used also for preconditioning the INS equations [14, 15].

The pressure corrected schemes feature a hierarchical structure, such that the final pressure is the result of a sequence of low order approximations. For this reason, the comparison between two different estimates is a natural error estimator for the problem, where “natural” means that no additional computation is required. In this paper (Part 1 out of 2) we present the basic features of the time-adaptive solver implemented upon this idea. After recalling the pressure corrected schemes (Sect. 2) and its basic features, we present the error estimator (Sect. 3). We illustrate the method for the automatic selection of the time step and present some preliminary results in 2D. Even if the error estimates obtained in this way are in general reliable, this approach intrinsically suffers from some stability constraints induced by the approximate splitting that reflect in the automatic selection of the time step. The latter can be actually limited even when large time step could be selected.
with a monolithic (unsplit) solver. For this reason, a variant of the original idea, based on using the pressure corrected scheme as preconditioners and the comparison with the unsplit solver for the error estimation, is presented in Sect. 4. Even though this second approach is in general computationally more expensive, it does not suffer from the stability drawback induced by the splitting. Finally, we extend the analysis of the pressure corrected schemes of [12] to the case of incremental formulations (Sect. 5 and e.g. [16, 17]).

Part 2 of this work will cover implementation details and numerical results in 3D.

2. Yosida pressure corrected schemes

Let us introduce the basic notations and definitions used throughout the paper. Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^d \) where \( d = 2, 3 \). Velocity and kinetic pressure of the fluid are denoted by \( u(x, t) \) and \( p(x, t) \) (\( x \in \Omega, \ t > t_0 \)) respectively. The INS equations read

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nu \nabla \cdot (\nabla u + \nabla u^T) + (u \cdot \nabla) u + \nabla p &= f \quad \text{in } \Omega \times (t_0, T) \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \times (t_0, T) \\
B u &= g \quad \text{on } \partial \Omega \times (t_0, T) \\
u(x, t_0) &= u_0 \quad \text{in } \Omega.
\end{align*}
\]

Here \( \nu > 0 \) is the kinetic viscosity, \( f \) an external force field, and \( B \) a boundary (trace) operator, including Dirichlet, Neumann or Robin conditions on different portions of the boundary \( \partial \Omega \). Time and space discretization and linearization of the Navier–Stokes equations result in a sequence of large, sparse linear systems of the form

\[
\begin{bmatrix}
C & B^T \\
B & O
\end{bmatrix}
\begin{bmatrix}
U^{n+1} \\
P^{n+1}
\end{bmatrix}
= \begin{bmatrix}
f_1^{n+1} \\
f_2^{n+1}
\end{bmatrix}, \tag{5}
\]

where now \( U^{n+1}, P^{n+1} \) are finite-dimensional counterparts of velocity and pressure respectively evaluated at the instant \( t^{n+1} \) and \( f_1^{n+1} \) and \( f_2^{n+1} \) collect the discretization of the forcing term \( f \), terms coming from the time discretization, and possibly the linearization of the convective term and the non-homogeneous boundary terms. The (1,1) block of the matrix has the form

\[
C = \frac{\alpha}{\Delta t} M + A \tag{6}
\]
where M is the block-diagonal velocity mass-matrix, A collects the discretization/linearization of the convective term and the discretization of the viscous term. The coefficient $\alpha$ depends on the time advancing method adopted. In particular, we will refer to Backward Difference Formulas (BDF) and Picard-like linearizations. The matrix B in (5) is the discrete (negative) divergence, and $B^T$ the discrete gradient. The nonsymmetric system (5) is often referred to as a generalized saddle point problem; see [2]. We shall denote the coefficient matrix in (5) by $A$. If we denote with $N_u$ the number of velocity degrees of freedom and with $N_p$ the number of pressure degrees of freedom, the matrix $A$ is $N \times N$ with $N = N_u + N_p$. We assume that an inf-sup stable finite element pair is being used for $(u, p)$, so that no stabilization is needed [1]. Since matrix $A$ in (5) features bad spectral properties reflecting the saddle point nature of the INS system, several methods have been proposed for its effective numerical solution. Some of these resort to a splitting of the problem into a sequence of subproblems dealing with the computation of velocity and pressure in separate steps. A possible method in this class is the celebrated Chorin-Temam projection scheme [18]. Working directly on the discrete problem, splitting methods can be devised by exploiting the following “exact” block factorization of the matrix in (5)

$$\begin{bmatrix} C & B^T \\ B & 0 \end{bmatrix} = \begin{bmatrix} C & 0 \\ B & \Sigma \end{bmatrix} \begin{bmatrix} I_{N_u} & C^{-1}B^T \\ 0 & I_{N_p} \end{bmatrix},$$

(7)

where $\Sigma = -BC^{-1}B^T$ is the so-called pressure Schur complement. Moving from this factorization, the solution to the original problem can be calculated by solving a sequence of smaller problems, involving velocity and pressure separately. The bottleneck is the solution of linear systems involving $\Sigma$. A possible workaround is to approximate $\Sigma$. Since for $\Delta t$ small enough the dominating term in $C$ is the first one, a natural approach is to approximate

$$C^{-1} \approx \frac{\Delta t}{\alpha} M^{-1} \equiv H.$$

(8)

This can be regarded as the first order truncation of the Neumann expansion

$$C^{-1} = \frac{\Delta t}{\alpha} \sum_{j=0}^{\infty} \left( -\frac{\Delta t}{\alpha} M^{-1} A \right)^j M^{-1}.$$

According to (8), when $\Sigma$ is replaced by $S \equiv -BHB^T$, we obtain the so-called Yosida method. The accuracy of this method is driven by the approximation (8), that induces a (splitting) error.
2.1. Pressure corrections

The basic idea of pressure corrected schemes (see [14, 8, 11, 12, 13]) for reducing the splitting error is to replace the lower and upper block-triangular matrices in (7) with the following ones:

\[
\mathbf{A} = \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{B} & \Sigma \end{bmatrix} \left[ \begin{bmatrix} \mathbf{I}_{N_u} & \mathbf{C}^{-1} \mathbf{B}^T \\ \mathbf{O} & \mathbf{I}_{N_p} \end{bmatrix} \right] \cong \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{B} & \mathbf{S} \end{bmatrix} \left[ \begin{bmatrix} \mathbf{I}_n & \mathbf{C}^{-1} \mathbf{B}^T \\ \mathbf{O} & \mathbf{Q} \end{bmatrix} \right] = \hat{\mathbf{A}}. \tag{9}
\]

The matrix \( \mathbf{Q} \) is associated with the pressure correction step (see [8]) and the solution of the problem at each time step consists of the following linear systems

\[
\begin{align*}
1: \quad & \mathbf{C}\tilde{\mathbf{U}} = \mathbf{f}_1 \\
& \mathbf{S}\tilde{\mathbf{P}} = \mathbf{f}_2 - \mathbf{B}\tilde{\mathbf{U}}, \\
2: \quad & \mathbf{Q}\mathbf{P} = \tilde{\mathbf{P}} \\
& \mathbf{C}\mathbf{U} = \mathbf{f}_1 - \mathbf{B}^T\mathbf{P}. \tag{10}
\end{align*}
\]

When we replace \( \mathbf{A} \) with \( \hat{\mathbf{A}} \), we introduce a splitting error that is however confined to the continuity equation, as it is highlighted by the explicit form of \( \hat{\mathbf{A}} \):

\[
\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{C} & \mathbf{O} \\ \mathbf{B} & \mathbf{S} \end{bmatrix} \left[ \begin{bmatrix} \mathbf{I}_{N_u} & \mathbf{C}^{-1} \mathbf{B}^T \\ \mathbf{O} & \mathbf{Q} \end{bmatrix} \right] = \begin{bmatrix} \mathbf{C} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{S}\mathbf{Q} - \Sigma \end{bmatrix}. \tag{11}
\]

An exact factorization would be yielded by the choice \( \mathbf{Q} = \mathbf{S}^{-1} \Sigma \equiv \mathbf{Q}_{ex} \). This choice is however unfeasible, since it still requires the inversion of the Schur complement \( \Sigma \). The Yosida pressure-corrected schemes stem from the choice \( \mathbf{Q} = \mathbf{Q}_q \) where \( q \) is a parameter related to the time accuracy of the method, such that \( ||\Sigma - \mathbf{S}\mathbf{Q}_q|| = \mathcal{O}(\Delta tf(q)) \) where \( f(q) \) is an appropriate (linear) function of \( q \). Set \( \mathbf{D}_k = \mathbf{B}(-\mathbf{H})^k \mathbf{H} \mathbf{B}^T, \ k \geq 0 \), one can show that \( ||\mathbf{D}_k|| = \mathcal{O}(\Delta f^{k+1}) \). In particular, \( \mathbf{S} = -\mathbf{D}_0 \). Now, let us denote \( \mathbf{R} = \mathbf{S}^{-1} \sum_{k \geq 1} \mathbf{D}_k \), for which \( ||\mathbf{R}|| = \mathcal{O}(\Delta t) \). For \( \Delta t \) small enough such that the spectral radius \( \rho(\mathbf{R}) < 1 \), we have

\[
\Sigma = -\sum_{k \geq 0} \mathbf{D}_k \Rightarrow \Sigma^{-1} = \left(-\sum_{k \geq 0} \mathbf{D}_k\right)^{-1} = \sum_{k \geq 0} \mathbf{R}^k \mathbf{S}^{-1}. \tag{12}
\]

Consequently, we have \( \mathbf{Q}^{-1}_{ex} = \Sigma^{-1} \mathbf{S} = \sum_{k=0}^q \mathbf{R}^k + \sum_{k=q+1} \mathbf{R}^k = \mathbf{I} + \sum_{k=1}^q \mathbf{R}^k + \mathcal{O}(\Delta t^{q+1}) \). As pointed out in [13], we can denote for \( k \geq 1 \),

\[
\mathbf{R}^k = \hat{\mathbf{R}}_k + \text{h.o.t.}
\]
where \( \hat{R}_k \) is a matrix of order \( \Delta t^k \) and “h.o.t.” denotes terms featuring a higher order dependence on \( \Delta t \). For instance, we have \( \hat{R}_1 = S^{-1}D_1 \), \( \hat{R}_2 = S^{-1}D_2 + (S^{-1}D_1)^2 \), \( \hat{R}_3 = S^{-1}D_3 + (S^{-1}D_1S^{-1}D_2 + S^{-1}D_2S^{-1}D_1) + (S^{-1}D_1)^3 \).

A viable approximation of \( Q_{ex} \) is given therefore by

\[
Q_{ex}^{-1} \approx Q_q^{-1} = \sum_{k=0}^q \hat{R}_k.
\] (13)

The correction steps then read (we omit the time index for the sake of notation) \( \hat{Q}_q P = \tilde{P} \), and give \( P = \sum_{k=0}^q \hat{R}_k \tilde{P} \). Denoting by \( z_k \) the vectors such that \( \hat{R}_k^{-1} z_k = \tilde{P} \), we have \( P = \sum_{k=0}^q z_k \). Solution to the sequence of systems in \( \hat{R}_k^{-1} \) can be rearranged in a hierarchical way. Indeed, let us consider the case \( q = 3 \). Set \( z_0 = \tilde{P} \), then

\[
\begin{align*}
z_1 &= S^{-1}D_1 \tilde{P} \quad \Rightarrow \quad Sz_1 = D_1z_0 \\
z_2 &= (S^{-1}D_2 + (S^{-1}D_1)^2) \tilde{P} \quad \Rightarrow \quad Sz_2 = D_2z_0 + D_1z_1 \\
z_3 &= (S^{-1}D_3 + S^{-1}D_2S^{-1}D_1 + S^{-1}D_1S^{-1}D_2 + (S^{-1}D_1)^2) \tilde{P} \quad \Rightarrow \quad Sz_3 = D_3z_0 + D_2z_1 + D_1z_2.
\end{align*}
\] (14)

More accuracy is obtained by solving more systems which, however, all entail the same matrix \( S \). A partial analysis of these pressure correction methods has been carried out in [12, 13]. In particular, it is possible to prove that for any \( q \in \mathbb{N} \) the matrices \( Q_p \) are non-singular for \( \Delta t \) small enough and that the consistency error is such that \( ||\Sigma - SQ_p|| = O(\Delta t^{q+2}) \), for all \( q \) [13]. This is confirmed by the numerical results obtained on a unit square (2D lid-driven cavity) reported in Fig. 2.

We recall here the accuracy result proven in [12]. At each time step \( t^n \) we denote \( \mathbf{E}_u^k \) and \( \mathbf{E}_p^k \) the vectors of the nodal values of the splitting errors (i.e. the difference between the pressure corrected solution and the solution of the monolithic system). For a generic time step \( k \), denoting with \( K \) the stiffness matrix for the velocity, we set \( ||\mathbf{E}_u^k||_0^2 \equiv \mathbf{E}_u^k M \mathbf{E}_u^k \), \( ||\mathbf{E}_u^k||_1^2 \equiv \mathbf{E}_u^k K \mathbf{E}_u^k \) and \( ||\mathbf{E}_p^k||_0^2 \equiv \mathbf{E}_p^k M_p \mathbf{E}_p^k \) where \( M_p \) is the mass matrix for the pressure unknown.

**Theorem 1.** If there exists a positive constant \( c \) such that \( \sum_{n=0}^{Nt-1} \Delta t ||p^{n+1}||^2 \leq c \) and \( \Delta t \) is sufficiently small, then there exist three positive constants \( c_1, c_2 \) and \( \sigma \) dependent on the space discretization and independent of \( \Delta t \) such that
the pressure correction schemes for $q = 0, 1, 2$ applied to the Stokes problem satisfy

$$
||E_u^{N_T}||_0^2 + \sigma \sum_{n=n_0}^{N_T-1} \Delta t ||E_u^n||^2_1 \leq c_1 \Delta t^{2q+3}, \quad \sum_{n=n_0}^{N_T-1} \Delta t ||E_p^n||^2_1 \leq c_2 \Delta t^{2q+2}.
$$

(15)

A major drawback of this approach is that if $\Sigma - SQ_p$ is negative definite, then the region of absolute stability of the time advancing scheme is in general reduced (see [8]). In particular, it is possible to prove (for the case of Stokes problem) that $\Sigma - SQ_0$ is positive, $\Sigma - SQ_1$ is negative, while $\Sigma - SQ_2$ is conditionally positive (i.e. for $\Delta t$ small enough) - see [13]. In Fig. 4 we report the error obtained when solving in a 2D rectangular domain the pressure drop problem with a sinusoidal time dependence of the pressure difference between inlet and outlet sections (Womersley test case: the 3D analytical solution can be found in [19], the 2D one can be found e.g. in [20]) in the case of $q = 2$. For larger time steps, the solution blows up independently of the BDF scheme adopted, as a consequence of a loss of stability induced by the splitting. As we will see, this affects the selection of the time step in the...
Figure 3: Velocity (left) and pressure (right) errors for a pressure corrected scheme with \( q = 1 \). As expected, for a BDF formula of order 4 the splitting is limiting the velocity accuracy with an order between 3 and 4. Pressure exhibits a slightly higher convergence order than the one predicted by the theory.

3. Algebraic splitting-based estimator

The pressure corrected method resorts to an approximation of the pressure at each time step in the form (the time index is still dropped)

\[
P = \sum_{k=0}^{q} z_k
\]

From the convergence Theorem we infer that the addition of each term \( z_k \) increases the accuracy of the pressure. Even though the estimate refers to the \( L^2 \) norm in time, we heuristically “localize” it and assume that for \( q = 0, 1 \ldots \)

\[
O(\Delta t^{q+1}) = \|p_{ex} - P_q\| \leq \|P_q - P_{q+1}\| + \|p_{ex} - P_{q+1}\| = \|z_{q+1}\| + \text{h.o.t.}
\]

This inequality highlights the role of vectors \( z_k \) as possible error estimators to be used for time adaptivity. These vectors are a by-product of the hierarchical
structure of the scheme. We stress that they do not need any additional computation.

In Fig. 5 we report the quantification of the norm $L^2$ and $L^\infty$ of vectors $z_k$ for $k = 1, 2, 3$ in a 2D problem on a unit square. The expected behaviour is confirmed by the numerical evidence. We report also the estimators obtained by using combinations of vectors $z_k$, still featuring the expected dependence on the time step.

Let $\chi$ be a multiplicative factor of the current time step, so that $\Delta t_{\text{new}} = \chi \Delta t_{\text{old}}$. We look for $\chi$ such that for a prescribed tolerance $\varepsilon$ we have

$$\|p_{ex} - P_q\| \leq \varepsilon \chi \Delta t_{\text{old}}$$

The prospected error will be

$$O(\Delta t_{new}^{q+1}) = \|p_{ex} - P_q\| (\Delta t_{new}) \approx O(\chi^{q+1} \Delta t_{old}^{q+1}) \approx$$

$$\chi^{q+1} \|p_{ex} - P_q\| (\Delta t_{old}) \approx \chi^{q+1} \|z_{q+1}\| (\Delta t_{old}).$$

The computation of $\chi$ will require therefore that

$$\chi^{q+1} \|z_{q+1}\| \leq \varepsilon \chi \Delta t_{old} \Rightarrow \chi \lesssim \left( \frac{\varepsilon \Delta t_{old}}{\|z_{q+1}\|} \right)^{1/q}.$$  

(16)
Figure 5: Norm of the error estimator as a function of the time step. $L^2$ norm $\|z\|_{L^2}$ on the left, infinity norm $\|z\|_{L^\infty}$ on the right. On the top we consider a single pressure increment $z_q$, on the bottom we consider the sum of the pressure increments $\sum_{i=q}^3 z_i$. 

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In practice, the selection of $\chi$ will be based on the previous estimate in a limited range of values, so that

$$\chi = \max \left( \min \left( \frac{1}{\varepsilon} \Delta t_{\text{old}}, \chi_{\text{max}} \right), \chi_{\text{min}} \right).$$

Details on the implementation will be given in the second part of the present work.

**Remark 1** - For variable-time-step-BDF schemes, coefficients of the discretization can be computed following the so-called variable coefficient technique. Stability bounds for these time-advancing methods as function of the step-size ratios $\Delta t_{j+1}/\Delta t_j$ are available in literature (see e.g. [21, 22]).

In Fig. 6 we present a 2D test case, where a forcing pressure drop between inlet and outlet boundaries in a channel is applied with a sinusoidal waveform in the first part of the time interval. Then, the pressure drop is turned off. In the first quarter of the time interval adaptivity is not activated and the time step is fine enough for the dynamics forced by the pressure drop. Successively time step is selected according to the adaptive strategy. The results are obtained by comparing the solutions with $q$ and $q+1$ for $q = 1$ and a tolerance $\varepsilon = 0.01$. When the time step is accepted, even though the estimate refers to the error for $q = 1$, the “more accurate” solution is available and it is retained.

When we adopt a BDF of order 3, we find the expected behaviour on the selection of time step, and the final $\Delta t$ is selected to be large when the forcing pressure drop is off (Fig. 6, left). When we use the BDF of order 4, the reduced stability induced by the combination of the time advancing BDF4 and the splitting prevents the selection of large time steps (Fig. 6, right).

A less trivial example of adaptivity is presented in Fig. 7, where a 2D domain resembling a vascular bifurcation has been simulated with a physiological-like waveform in the flow rate. The waveform reported in the picture refers to three heartbeats and has been used for modulating the inflow velocity profile. Neumann conditions have been prescribed at the outlets. Adaptivity has been activated after the first heart beat. The initial time step has been tuned for a correct representation of the first part of each heart beat (systole), that features a fast transient. The adaptive scheme (BDF3 with $q = 1$) works in the expected way. At the beginning (during the second
systole) the time step is not changed significantly, then during the diastolic phase a larger $\Delta t$ is automatically selected. The overall number of time steps is reduced from 3334 (with no adaptivity) to 1127 per heart beat.

4. Adaptivity for preconditioned unsplit solvers

In real applications reduction of stability induced by the combination of high order schemes with the splitting can be a major issue, limiting the effectiveness of the adaptive procedure. As we have seen, this can prevent the automatic selection of large time steps. This depends on the size of the space mesh (as illustrated in Fig. 4). As a matter of fact, a few highly stretched elements or mesh singularities may drastically increase the spectral radius of the Neumann expansion matrix $HA$, leading to severe constraints on the choice of the time step. A possible workaround in this case is the adoption of inexact (pressure corrected) LU factorizations as preconditioners for system (14). Performances of these preconditioners have been investigated in [15] for the case $q = 0$ and [14] for the case $q = 1$. More in general, Fig. 8 pinpoints how the eigenvalues of the preconditioned Schur Complement $(SQ_q)^{-1} \Sigma$ for
Figure 7: Time adaptivity in a 2D bifurcation with an input time-dependent velocity modulated by a physiological waveform. Three heart beats with a physiological peak Reynolds of about 700. In the first beat, adaptivity is off and time step is selected for capturing the fast transients of the first part of the heart beat (systole). In the second and third heart beat, the adaptivity maintains the same time step during systole (see the zoomed box below) and selects larger steps during the subsequent phase (diastole). In this way, one third of time steps used by the non-adaptive computation are required.
the non symmetric Navier-Stokes problem are clustering around 1 for a lid driven cavity with $Re = 1000$ for $q = 0, 1, 2, 3$ and different values of $\Delta t$.

It has been proved in [15] that each preconditioned iterations provide a progressively more accurate estimate for the pressure. A possible error estimator can be therefore obtained by comparing the solution obtained by the pressure corrected scheme (corresponding to one preconditioned iteration) $P_q$ (typically with $q = 0, 1$) and the converged (unsplit) solution $P_{unspl}$.

$$\|p_{ex} - P_q\| \leq \|p_{ex} - P_{unspl}\| + \|P_{unspl} - P_q\|$$

Let us assume that the order of accuracy $r$ of the time discretization for the unsplit scheme is such that

$$O(\Delta t^r) = \|p_{ex} - P_{unspl}\| \ll \|P_{unspl} - P_q\| = O(\Delta t^{q+1})$$

being $r > q + 1$. We have then

$$\|p_{ex} - P_q\| \leq \|p_{ex} - P_{unspl}\| + \|P_{unspl} - P_q\| = \|P_{unspl} - P_q\| + h.o.t..$$
By following the same procedure as before, we select the time step $\Delta t_{\text{new}} = \chi \Delta t_{\text{old}}$ by choosing
\[
\chi = \max \left( \min \left( \left( \frac{\varepsilon \Delta t_{\text{old}}}{\| P_{\text{unspl}} - P_q \|} \right)^{1/q}, \chi_{\text{max}} \right), \chi_{\text{min}} \right).
\]
Since at the end of the time step, the solution $P_{\text{unspl}}$ is retained, the overall stability of the method is driven by the stability of the BDF scheme. Numerical performances of this approach (with incremental pressure corrected schemes) will be illustrated in Part 2.

5. Pressure corrected incremental Yosida schemes

A variant to the original splitting methods (both at the differential and algebraic level) often advocated is the so-called incremental formulation (see [23, 17, 16]). The basic idea is to rewrite the original unsplit form as follows.

Let us write at each time step
\[
P^n = \delta P^n + \sigma^n_P,
\]
where $\sigma^n_P$ is an extrapolation of $P^n$ based on the previous time steps, such that $P^n - \sigma^n_P = \mathcal{O}(\Delta t^s)$ with $\mathbb{N} \ni s \geq 1$. More explicitly, in the case of constant time step, for $s = 1$ we have $\sigma^n_P = P^{n-1}$, and for $s = 2$ we have $\sigma^n_P = 2P^{n-1} - P^{n-2}$. Then, we write
\[
\begin{bmatrix}
C & B^T \\
B & O
\end{bmatrix}
\begin{bmatrix}
U^n \\
P^n
\end{bmatrix}
= \begin{bmatrix}
f_1^n \\
f_2^n
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
C & B^T \\
B & O
\end{bmatrix}
\begin{bmatrix}
U \\
\delta P^n
\end{bmatrix}
= \begin{bmatrix}
f_1^n - B^T \sigma^n_P \\
f_2^n
\end{bmatrix}
\]
(17)

Previous works on splitting methods for the solution of Navier-Stokes equations argue that solving for incremental pressure has better convergence properties than those of the non-incremental one, and that it does not affect the overall stability of the method. In particular, in [24] the authors note that the incremental method gives stable velocity and pressure fields under minimal hypotheses, provided that the finite element discretization fulfills the inf-sup condition.

In [17], the authors suggest the incremental method as a valid and computationally convenient alternative to using high order matrix factorizations. However, they notice that incremental methods may accumulate round-off
errors, in the case of long time simulations. For this reason they provide an alternative formulation in which they correct the end of step pressure to take into account the accumulation errors.

Here we analyse the pressure-corrected incremental scheme, i.e. the scheme where the idea of the pressure corrections is applied to the incremental formulation. Let us denote by $U_{q,s}, P_{q,s}$ the solution of the $q$-pressure corrected Yosida scheme, with an increment of order $s$. In order to investigate the splitting error introduced at each step, we postulate the localizing assumption, i.e. we assume that the solution at the time steps prior the current one correspond to the unsplit solution, i.e. $U_i^{q,s} = U_i$ and $P_i^{q,s} = P_i$ for $i = k - 1, k - 2, \ldots k - p$. In the following results, $C$ denotes a constant (not necessarily the same at each occurrence) independent of $\Delta t$.

We denote the local splitting errors by $E_k^{*,*}$ and $e_k^{*, *}$.

**Theorem 2.** If the pressure $p \in W^{s, \infty}(0, T; L^2(\Omega))$ for some $s \geq 1$ and $\Delta t$ is sufficiently small, then the local splitting error of a $q$-pressure corrected $s$-incremental Yosida scheme applied to the Stokes problem satisfies at each time step $k$

\[
\|E_k^{*,*}\|_0 \leq C \Delta t^{q+s+2}, \quad \|E_k^{*,*}\|_1 \leq C \Delta t^{q+s+3/2}, \quad \|e_k^{*,*}\|_0 \leq \Delta t^{q+s+1}. \tag{18}
\]

**Proof**

A BDF approximation of order $p$ for the Stokes system reads

\[
\begin{cases}
\frac{\alpha_0}{\Delta t} MU^k + \nu KU^k + B^T P^k = f^k_1 + \frac{1}{\Delta t} \sum_{i=1}^{p} \alpha_i MU^{k-i} \\
BU^k = f^k_2
\end{cases}, \tag{19}
\]

while the Yosida $s$-incremental pressure corrected splitting reads

\[
\begin{cases}
\frac{\alpha_0}{\Delta t} MU_{q,s}^k + \nu KU_{q,s}^k + B^T P_{q,s}^k = f^k_1 + \frac{1}{\Delta t} \sum_{i=1}^{p} \alpha_i MU_{q,s}^{k-i} \\
BU_{q,s} - (\Sigma - SQ_q)\delta P_{q,s}^k = f^k_2
\end{cases}. \tag{20}
\]

Thanks to the localizing assumption, by subtracting (20) from (19) we get

\[
\begin{cases}
\frac{\alpha_0}{\Delta t} ME^{k,*} + \nu K E^{k,*} + B^T e^{k,*} = 0 \\
BE^{k,*} - (\Sigma - SQ_q)e^{k,*} = -(\Sigma - SQ_q)(\delta^k P)
\end{cases}. \tag{21}
\]
With standard manipulation, we get
\[ \alpha_0 (\mathbf{M} \mathbf{E}^{k,*}, \mathbf{E}^{k,*}) + \nu \Delta t (\mathbf{K} \mathbf{E}^{k,*}, \mathbf{E}^{k,*}) + \Delta t ((\Sigma - \mathbf{S} \mathbf{Q}) \mathbf{e}^{k,*}, \mathbf{e}^{k,*}) = \Delta t ((\Sigma - \mathbf{S} \mathbf{Q}) \mathbf{\delta P}^{k}, \mathbf{e}^{k,*}). \] (22)

Thanks to Cauchy-Schwarz and Young inequality, we have
\[ \alpha_0 \| \mathbf{E}^{k,*} \|^2_0 + \nu \Delta t \| \mathbf{E}^{k,*} \|^2_1 \leq \frac{1}{2\varepsilon} \| (\Sigma - \mathbf{S} \mathbf{Q}) \mathbf{\delta P}^{k} \|^2 + \left( 2\varepsilon + \gamma \frac{\| (\Sigma - \mathbf{S} \mathbf{Q}) \|}{\Delta t} \right) \Delta t^2 \| \mathbf{e}^{k,*} \|^2, \] (23)
where \( \gamma = 0 \) if \( \Sigma - \mathbf{S} \mathbf{Q} \) is semidefinite positive, 1 otherwise. Notice that in [13] it has been proved that \( \Sigma - \mathbf{S} \mathbf{Q} \) is semidefinite positive for all even splitting order, provided that \( \Delta t \) is small enough.

Since \( \| (\Sigma - \mathbf{S} \mathbf{Q}) \| \leq c \Delta t^{q+1} \), we have
\[ \alpha_0 \| \mathbf{E}^{k,*} \|^2_0 + \nu \Delta t \| \mathbf{E}^{k,*} \|^2_1 \leq \frac{1}{2\varepsilon} \| (\Sigma - \mathbf{S} \mathbf{Q}) \mathbf{\delta P}^{k} \|^2 + \left( 2c + \gamma c \Delta t^{q+1} \right) \Delta t^2 \| \mathbf{e}^{k,*} \|^2. \] (24)

Since the finite element discretization is inf-sup compatible, we have
\[ \| \mathbf{e}^{k,*} \| \leq \frac{1}{\beta} \sup_{\mathbf{V}} \frac{|(\mathbf{B}^T \mathbf{e}^{k,*}, \mathbf{V})|}{\| \mathbf{V} \|_1} = \frac{1}{\beta} \sup_{\mathbf{V}} \frac{|(\alpha_0 \mathbf{M} \mathbf{E}^{k,*} + \nu \mathbf{K} \mathbf{E}^{k,*}, \mathbf{V})|}{\| \mathbf{V} \|_1} \leq \frac{1}{\beta} \left[ C_1 \alpha_0 \| \mathbf{E}^{k,*} \|_0 + \nu \| \mathbf{E}^{k,*} \|_1 \right], \] (25)
where \( \beta \) is independent of the time step. Therefore
\[ \Delta t^2 \| \mathbf{e}^{k,*} \|^2 \leq C_* \left[ \| \mathbf{E}^{k,*} \|^2_0 + \Delta t^2 \nu \| \mathbf{E}^{k,*} \|^2_1 \right], \] (26)
being \( C_* = 2 \max \left( \frac{C_2 \alpha_0^2}{2\varepsilon}, \frac{\nu}{\beta^2} \right) \).

Let \( \varepsilon = \frac{1}{4C_*} \) and substitute (26) in (24)
\[ \left( \frac{1}{2} - \gamma c \Delta t^{q+1} \right) \left( \alpha_0 \| \mathbf{E}^{k,*} \|^2_0 + \nu \Delta t \| \mathbf{E}^{k,*} \|^2_1 \right) \leq 2C_* c \Delta t^{q+1} \| \mathbf{\delta P}^{k} \|^2. \] (27)

Under the assumption of regularity for the pressure we have \( \| \mathbf{\delta P}^{k} \| \leq C \Delta t^{s} \), so that
\[ \left( \frac{1}{2} - \gamma c \Delta t^{q+1} \right) \left( \alpha_0 \| \mathbf{E}^{k,*} \|^2_0 + \nu \Delta t \| \mathbf{E}^{k,*} \|^2_1 \right) \leq K \Delta t^{q+4+2s}, \] (28)
where $K = 4C_\ast c^2 \left( \max_t \left\| \frac{\partial^s p}{\partial t^s} \right\| \right)^2$ is a constant independent of $\Delta t$. Should either $\gamma = 0$ or $\Delta t$ small enough, the local splitting error is such that

$$\|E^{k,*}\|_0 \leq C \Delta t^{q+s+2}, \quad \|E^{k,*}\|_1 \leq C \Delta t^{q+s+3/2}, \quad \|e^{k,*}\| \leq C \Delta t^{q+s+1}. \quad (29)$$

### Remark 2

When we remove the localizing assumption and introduce the error propagation, it is possible to prove that

$$\|E^k\|_0 \leq C \Delta t^{q+s+1},$$

as a consequence of the previous Theorem and of the classical propagation error analysis of linear multistep method (see for example [25] pages 235-249).

More complicated is the estimation of the pressure global splitting error $e^k$. From numerical evidence (see Part 2 of the present work), we conjecture that also the pressure splitting error reduces of one order the accuracy of the splitting, leading to the following estimate

$$\|e^k\| \leq C \Delta t^{q+s+1}.$$  

A rigorous proof is still missing.

### 5.1. Time adaptivity with incremental schemes

When resorting to incremental methods, we expect the estimators $z_{k,s}$ to increase their accuracy according to the order of the extrapolation. Consequently, incremental pressure methods lead to higher order error estimator with respect to the non incremental ones. For example, for the algebraic splitting-based estimator we have

$$O(\Delta t^{q+s+1}) = \|p_{\text{ex}} - P_{q,s}\| \leq \|P_{q+1,s} - P_{q,s}\| + \|p_{\text{ex}} - P_{q+1,s}\| = \|z_{q+1,s}\| + h.o.t.$$  

By the same procedure as in Sect. 3, we select the time step $\Delta t_{\text{new}} = \chi \Delta t_{\text{old}}$ by choosing

$$\chi = \max \left( \min \left( \left( \frac{e \Delta t_{\text{old}}}{\|z_{q+1,s}\|_{L^2}} \right)^{1/(q+s)} , \chi_{\max} \right), \chi_{\min} \right),$$

where $\chi_{\max}$ and $\chi_{\min}$ are chosen such that the time step belongs to a user defined range, $\Delta t \leq \Delta t \leq \Delta t$, and the stability bound for variable time step BDF, discussed in [21, 22], are satisfied.
Similar arguments can be used for the incremental formulation of the time adaptive method advocated in Sect. 4.

Numerical evidence of the effectiveness of the incremental scheme is given in the second part of the present work.

6. Conclusion

In this paper we have introduced an *a posteriori* error estimator for the error in time for a solver of the incompressible Navier-Stokes equations. The estimator stems from the comparison of different guesses of the pressure that are computed by the segregated pressure correction scheme of [8]. The error estimator is thus computed with no additional cost. The results in 2D presented here show that the error is actually estimated with good reliability. Overall accuracy can be improved with the incremental formulation of the method. However, the splitting behind the pressure-correction approach, in particular with high order corrections, can affect significantly the stability of the time advancing. This eventually prevents the time adaptive method from selecting large time steps. A different - still hierarchical - approach consists of using low order pressure corrections as preconditioner. The comparison of the iterates of the preconditioned scheme with the converged solution provides in this case the error estimator. The stability of the time discretization method is retained, even if the computational costs are increased.

Several issues still need to be addressed, in particular concerning the actual implementation and the performances of the method in 3D. These are the subject of Part 2.

References


