A coupled SAFE-2.5D BEM approach for the dispersion analysis of damped leaky guided waves in embedded waveguides of arbitrary cross-section

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Abstract

The paper presents a Semi-Analytical Finite Element (SAFE) formulation coupled with a 2.5D Boundary Element Method (BEM) for the computation of the dispersion properties of viscoelastic waveguides with arbitrary cross-section and embedded in unbounded isotropic viscoelastic media. Attenuation of guided modes is described through the imaginary component of the axial wavenumbers, which accounts for material damping, introduced via linear viscoelastic constitutive relations, as well as energy loss due to radiation of bulk waves in the surrounding media. Energy radiation is accounted in the SAFE model by introducing an equivalent dynamic stiffness matrix for the surrounding medium, which is derived from a regularized 2.5D boundary element formulation.

The resulting dispersive wave equation is configured as a nonlinear eigenvalue problem in the complex axial wavenumber. The eigenvalue problem is reduced to a linear one inside a chosen contour in the complex plane of the axial wavenumber by using a contour integral method. Leaky and evanescent poles are obtained by choosing appropriately the phase of the wavenumbers normal to the interface in compliance with the nature of the waves in the surrounding medium. Finally, the obtained eigensolutions are post-processed to compute the energy velocity and the radiated displacement wavefield in the surrounding domain.

The reliability of the method is first validated on existing results for waveguides of circular cross section embedded in elastic and viscoelastic media. Next, the potential of the proposed numerical framework is shown by computing the dispersion properties for a square steel bar embedded in grout and for a H-shaped steel pile embedded in soil.

Key words: leaky guided waves, dispersion, viscoelasticity, nonlinear eigenvalue problem, contour integral method, SAFE, 2.5D BEM
1. INTRODUCTION

Application of Guided Ultrasonic Waves (GUW) in the field of Non-destructive Evaluation (NDE) is constantly increasing due to their capacity to provide long inspection ranges while screening the entire cross-section of the waveguide. As well known, knowledge of the dispersion properties of guided modes propagating in the structure of interest, phase velocity, energy velocity and attenuation, is indispensable for the design of both actuation and sensing systems, as well as to tune experimental set-up. To efficiently compute dispersion properties of guided waves in traction-free waveguides, analytical methods [47, 56, 70, 71, 80, 23] and finite element-based methods [37, 41, 42, 7, 24, 86, 55, 82] have been intensively investigated in the recent years.

In several circumstances, waveguides are embedded in solid media (tendons, foundation piles, buried pipes, embedded fibers) or immersed in fluids (pipelines).

In these cases, guided modes traveling with phase speed greater than the bulk speed of the surrounding media radiates energy into it. As a consequence, inspection ranges are generally reduced since the energy radiated in the surrounding media causes high attenuation rates of the guided modes (leaky modes).

The knowledge of dispersion properties of leaky modes is therefore fundamental in NDE testing of civil, mechanical and aerospace structures and mathematical tools able to describe waveguides with different geometric and mechanical characteristics are needed. In this context, several studies can be found in literature involving simple geometries, i.e. plate and cylindrical structures, in which analytical methods have been extensively applied.

A comprehensive study of matrix techniques for the computation of dispersion curves in free, embedded and immersed plates can be found in the work of Lowe [56]. The propagation of leaky Lamb waves in plates embedded in solids have been studied by Dayal and Kinra [29, 30] and Lowe [58]. In their works, Nayfeh and Chimenti [64] and Chimenti and Nayfeh [25] have analyzed the dispersive behaviour of waves in plates immersed in fluids, while Bernard et al. [14] focused on the energy velocity of waves propagating in absorbing and non-absorbing plates immersed in both vacuum and fluids. Dispersion properties of poroelastic plates immersed in fluids can be found in the work of Belloncle et al. [10, 11].

The propagation of non-leaky guided waves in elastic circular waveguides embedded in elastic media have been investigated by Parnes [68, 69] and Kleczewski and Parnes [46]. Dispersion relations for leaky modes in elastic circular rods embedded in isotropic elastic solids have been extracted by Thurston [85], Simmons et al. [81] and Viens et al. [88] using analytic dispersive equations. In their work, Nayfeh and Nagy [65] have applied the Transfer Matrix Method to investigate the propagation of axisymmetric waves in coaxial layered anisotropic fibers embedded in solids and immersed in fluids. General studies on wave propagation in transversely isotropic and homogeneous anisotropic circular rods immersed in fluids have been conducted by Dayal [28], Nagy [63], Berliner and Solecki [12, 13] and Ahmad [1].

The dispersion properties of leaky guided waves in both embedded and immersed cylindrical structures have been in depth analyzed by Pavlakovic [71] using the Global Matrix Method (GMM). This method has been used next to perform numerical analyses and support experimental investigations involving free pipes with defects [57], fluid-loaded pipes [4], buried pipes [54], embedded circular bars [73] and embedded tendons and bolts [8, 9]. An analytical method has been proposed by Laguerre et al. [49] to predict dispersion curves and to interpret the ultrasonic transient bounded-beam propagation in a solid cylindrical waveguide embedded in a solid medium.
Although very attractive for simple geometries, analytical approaches are generally unsuitable to extract dispersion properties for waveguides with irregular cross-section and, in these cases, one must resort to numerical methods.

Due to the capability to represent domains with different materials and arbitrary geometries while forming well posed polynomial eigenvalue problems, the SAFE method has been also extended in recent years to wave propagation problems involving unbounded domains. In their work, Castaings and Lowe [20] have used a SAFE mesh to discretize both the waveguide and the embedding medium. The material surrounding the waveguide was simulated by introducing a finite absorbing region of length proportional to the largest radial wavelength of the existing leaky waves. The method eliminates the well known problem of non-physical reflections which would arise using a finite mesh to model the unbounded surrounding domain. A similar method has been applied by Fan et al. [32] for the prediction of dispersion properties of torsional waves in waveguides of arbitrary cross-section immersed in fluids. However, this method may require very large meshes to properly model waves radiating in the surrounding media and guided modes with high rates of energy confined in the embedded cross-section need to be selected from a large set of eigensolutions.

A hybrid SAFE formulation has been proposed by Jia et al. [44] to study double layer hollow cylinders embedded in infinite media. In this work, the unbounded medium has been discretized by means of infinite elements, which overcomes the problem of energy reflection. However, the capability of infinite elements to correctly represent the physics of leaky waves is strongly related to the choice of the elements shape functions. Moreover, complicated geometries, such as H shaped beams, may result difficult to treat.

In their work, Lin et al. [51] have considered the presence of two isotropic elastic half spaces at the top and bottom interface of a SAFE-modeled layer by introducing appropriate analytical boundary conditions. The analytical boundary conditions have been adopted in order to satisfy the Snell’s law for radiated longitudinal and shear waves. Only solutions relative to evanescent wavefields in the surrounding medium have been considered in this work.

A further numerical technique that allows to model radiated waves without reflections has been proposed by Treyssède et al. [87] by coupling the SAFE method with the Perfectly Matched Layer (PML) method. Using the PML, leaky modes are defined through analytic extensions in terms of complex spatial coordinates. Although this method allows to preserve the original dimension of the problem as well as the nature of the dispersive wave equation, the radiation efficiency strictly depends on the choice of the complex-valued function used to represent geometric decay inside the PML domain.

A possible alternative is represented by the Boundary Element Method (BEM). Unlike FE-based formulations, the BEM allows to describe the unbounded surrounding domain by means of a boundary mesh only. Moreover, since the weight functions are represented by the fundamental solutions of the dynamic problem (Green’s function), no approximations are introduced in the definition of the radiated wavefield. Generally, FE-based formulations are preferred to BEM, since the latter presents numerical difficulties related to (i) singularities of fundamental solutions (ii) treatment of boundary corners and (iii) solution of the nonlinear eigenvalue problem resulting from the boundary element equations.

In recent years, different coupled FEM-BEM formulations have been proposed to investigate the wave propagation in waveguide-like structures. Such formulations are sometimes referred in literature as the wavenumber finite-boundary element method [78, 79], the waveguide finite-boundary element method [67] or the 2.5D finite-boundary element method [35, 27]. While most of these studies are focused on forced or induced vibrations problems, minor attention has been
dedicated to the study of dispersive characteristics of guided waves, especially when attenuation is involved.

Some exceptions are represented by the work of Gunawan and Hirose [40], where a BEM formulation has been applied in the extraction of dispersion properties of traction-free elastic waveguides, and by the works of Tadeu and Santos [84] and Zengxi et al. [90], which have adopted a 2.5D BEM for the computation of dispersion relations in fluid filled boreholes. However, attenuation information is not provided in these works. More recently, Nilsson et al. [67] have proposed a waveguide FEM-BEM formulation to study the radiation efficiency of open and embedded rails. In such work, dispersion relations for radiating modes have been obtained by considering complex wavenumbers, thus taking into account the amplitude decay due to attenuation. However, since the acoustic impedance mismatch between the rail and the air was very high, the authors have considered in their model only the influence of the rail on the fluid vibrations and not the one of the air on the rail (the model is not fully coupled).

In the present work, a SAFE formulation coupled with a regularized 2.5D BEM formulation is proposed to extract dispersion curves for viscoelastic waveguides of arbitrary cross-section embedded in viscoelastic isotropic materials. With respect to the above mentioned SAFE formulations that use absorbing regions, infinite elements and PMLs, the proposed SAFE-2.5D BEM formulation represents exactly the radiated wavefield from waveguides of arbitrary cross-section while preserving the dimension of the SAFE problem and without the need of special complex functions. The complex axial wavenumbers and the corresponding wavestructures are computed from a nonlinear eigenvalue problem solved via a contour integral method [2, 5, 15]. The complex poles associated to leaky and evanescent modes are obtained by choosing the arguments of the wavenumbers in the embedding medium consistently with the nature of the radiated waves and removing points of singularities and discontinuities from the complex plane of the axial wavenumber.

The method is first validated against available results obtained, for embedded circular bars, by means of alternative approaches [73, 20]. Next, dispersion curves are extracted for a viscoelastic square steel bar embedded in viscoelastic grout and for a viscoelastic HP200 steel pile embedded in a viscoelastic soil. To the best of authors knowledge, these cases are presented for the first time in literature. The proposed method can be useful to understand the physical behaviour of leaky guided waves as well as to design testing conditions in GUW-based inspections and experiments involving embedded beams or foundation piles.

2. Wave equation

The guided wave equation is derived for the system with z-invariant geometric and mechanical properties of Fig. 1. The wavenumber-frequency dependence is assumed in the form \( \exp[i(\kappa_z z - \omega t)] \), where \( i \) is the imaginary unit, \( t \) is time, \( \omega \) is the real angular frequency and \( \kappa_z = \text{Re}(\kappa_z) + i\text{Im}(\kappa_z) \) denotes the axial wavenumber, which real and imaginary parts represent the moduli of the propagation and attenuation vectors \( k_z^p \) and \( k_z^a \), respectively.

The longitudinal invariance allows to describe the three-dimensional wave propagation problem in the \( x - y \) plane, while the third dimension is accounted by contraction of any \( z \)-dependent scalar or vectorial field in the axial wavenumber domain through spatial Fourier transform. In particular, the waveguide cross-section of area \( \Omega_s \) is discretized using the SAFE method [7, 61] while the external medium of infinite extent \( \Omega_b \) is modeled via a 2.5D regularized boundary integral formulation [60, 35]. The in-plane position vector \( \mathbf{x} = [x, y]^T \) is used instead of
\( x = [x, y, z]^T \), assuming the cross section located at \( z = 0 \). For convenience of representation, the spatial coordinates \( x, y \) and \( z \) are freely interchanged with the subscripts 1, 2 and 3, respectively.

The SAFE and BEM meshes are defined with coincident nodes and matching shape functions at the coupling interface \( \partial \Omega_s = \partial \Omega_b \), where compatibility of displacements and equilibrium of tractions are enforced through the relationships

\[
\begin{align*}
\mathbf{u}(x, z, t)|_{\partial \Omega_b} &= \mathbf{u}(x, z, t)|_{\partial \Omega_s}, \\
\mathbf{t}(x, z, t)|_{\partial \Omega_b} &= -\mathbf{t}(x, z, t)|_{\partial \Omega_s},
\end{align*}
\]

(1)
denoting with \( \mathbf{u}(x, z, t) = [u_1, u_2, u_3]^T \) the displacements vector and \( \mathbf{t}(x, z, t) = [t_1, t_2, t_3]^T \) the tractions vector. The minus sign on the right hand side of the second expression in Eq. (1) accounts for the opposite sign of the outward normals of the SAFE and BEM regions at the boundary point \( x \), i.e. \( \mathbf{n}(x)|_{\partial \Omega_s} = -\mathbf{n}(x)|_{\partial \Omega_b} \) (see Fig. 1). The equilibrium equation for a waveguide embedded in an infinite surrounding media can be obtained by letting vanish the first variation of the Hamilton’s functional \( \mathcal{H}(\mathbf{u}) \) between two arbitrary time instants \( t_1 \) and \( t_2 \) [61] and applying the compatibility conditions in Eq. (1) for the boundary tractions, leading to

\[
\delta \mathcal{H}(\mathbf{u}, \delta \mathbf{u}) = \int_{t_1}^{t_2} \int_{\Omega_s} \left\{ -\int_{\Omega_s} \delta \mathbf{u}^T \rho(\mathbf{x}) \frac{\partial^2 \mathbf{u}}{\partial t^2} d\mathbf{x} d\mathbf{y} - \int_{\partial \Omega_s} (\delta \mathbf{e}(\mathbf{u}))^T \mathbf{s}(\mathbf{u}) d\mathbf{s} + \int_{\partial \Omega_t} \delta \mathbf{u}^T (\mathbf{t}_b - \mathbf{t}_t) ds \right\} dt = 0,
\]

(2)
where \( \rho(\mathbf{x}) \) is the material density at \( \mathbf{x} \in \Omega_s \), \( \mathbf{e}(\mathbf{u}) = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}]^T \) is the vector collecting the symmetric components of the linear strain tensor

\[
\mathbf{e}(\mathbf{u}) = \frac{1}{2} \left[ \nabla \mathbf{u} + (\nabla \mathbf{u})^T \right],
\]

(3)
while the entries of the vector \( \mathbf{s}(\mathbf{u}) = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}]^T \) represent the symmetric components of the Cauchy stress tensor

\[
\mathbf{\sigma}(\mathbf{u}) = \mathbf{C}(\mathbf{x}, t - \tau) : \mathbf{e}(\mathbf{u}),
\]

(4)
where \( \mathbf{C}(\mathbf{x}, t - \tau) \) is the fourth order tensor of relaxation functions for isotropic viscoelastic materials [26]. In Eq. (2), the first domain integral represents the first variation of the kinetic energy, while the second domain integral accounts for the stored elastic energy plus internally dissipated energy. Finally, the last boundary integral accounts for the virtual work done by the externally applied tractions \( \mathbf{t}_s \) on \( \partial \Omega_s \) and the virtual work done by the dynamic boundary tractions \( \mathbf{t}_b \) on \( \partial \Omega_t \) resulting from the interaction of the waveguide with the surrounding medium. Body forces are not considered in this work.

2.1. SAFE model of the embedded waveguide

The domain \( \Omega_s \) is discretized into a number \( N_s \) of quadratic semi-isoparametric finite elements of area \( \Omega_s^e \), with 3 degrees of freedom per node associated to the three displacement components \( u_j \). The displacement vector at point \( \mathbf{x}^e \in (\Omega_s^e \cup \partial \Omega_s^e) \) is approximated as
\[ \mathbf{u}^e (\xi, z, t) = \mathbf{N} (\xi) \mathbf{q}^e (z, t) , \]  

where \( \mathbf{N} (\xi) \) is the matrix collecting the quadratic shape functions for the parent element of area \( \Omega^\text{ref} \) in the natural reference system, which is described by the non-dimensional coordinates \( \xi = [\xi_1, \xi_2]^T \) [36], and \( \mathbf{q}^e (z, t) \) is the vector of nodal displacements. Eqs. (2) and (5) are reformulated in the wavenumber-frequency domain by means of the space-time Fourier transform \( \tilde{f} \{ f (z, t) \} (k, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f (z, t) \exp (-i (k z - \omega t)) \, dz \, dt \). Substituting Eq. (5) into the linear strain-displacement relations and taking the space-time Fourier transform allows to rewrite the vector \( \mathbf{e} (\mathbf{u}) \) in the \((k, \omega)\) domain as follows

\[ \tilde{\mathbf{e}}^e (\xi, k_z, \omega) = \left[ \mathbf{B}_{xy} (\xi) + i \kappa \mathbf{B}_x (\xi) \right] \tilde{\mathbf{q}}^e (k_z, \omega) , \]

where the compatibility operators \( \mathbf{B}_{xy} (\xi) \) and \( \mathbf{B}_x (\xi) \) are defined respectively as

\[ \mathbf{B}_{xy} (\xi) = \left[ \mathcal{L}_x \frac{\partial \mathbf{N} (\xi)}{\partial \xi_1} \frac{\partial \mathbf{N} (\xi)}{\partial y} + \mathcal{L}_y \frac{\partial \mathbf{N} (\xi)}{\partial \xi_1} \frac{\partial \mathbf{N} (\xi)}{\partial y} \right] , \quad \mathbf{B}_x (\xi) = \mathcal{L}_z \mathbf{N} (\xi) , \]

in which

\[ \mathcal{L}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad \mathcal{L}_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad \mathcal{L}_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \]

The Fourier-transformed constitutive relations in Eq. (4) are expressed in the \((k_z, \omega)\) domain as

\[ \tilde{\sigma}^e (\xi, k_z, \omega) = \tilde{\mathbf{C}}^e (\omega) : \tilde{\mathbf{e}}^e (\xi, k_z, \omega) , \]

where \( \tilde{\mathbf{C}}_{iklm}^e (\omega) = \tilde{\lambda}(\omega) \delta_{ij} \delta_{lm} + \tilde{\mu}(\omega) (\delta_{ik} \delta_{jm} + \delta_{im} \delta_{jk}) \) is the complex tensor of viscoelastic moduli for linear isotropic materials [26], in which \( \lambda \) and \( \mu \) denote the first and second complex Lamé constants, respectively.

Assuming the material attenuation vectors for both longitudinal and shear bulk waves perpendicular to their wavefronts (i.e., the material attenuation and propagation vectors are parallel), the complex Lamé constants can be expressed as [56]

\[ \tilde{\lambda}(\omega) = \rho \left[ \tilde{c}_L^2 (\omega) - 2 \tilde{c}_S^2 (\omega) \right] , \quad \tilde{\mu}(\omega) = \rho \tilde{c}_S^2 (\omega) , \quad \tilde{c}_L (\omega) = \frac{c_L}{1 + i \beta_L (\omega) / 2\pi} , \quad \tilde{c}_S (\omega) = \frac{c_S}{1 + i \beta_S (\omega) / 2\pi} , \]

where \( c_L \) and \( c_S \) are the phase velocities of longitudinal and shear bulk waves, respectively, while \( \beta_L (\omega) \) and \( \beta_S (\omega) \) represent their corresponding attenuation coefficients per wavelength.

Substituting Eqs. (5), (6) and (9) into Eq. (2), after some algebraic manipulations the following \( N \)-dimensional linear system of equations in the \((k_z, \omega)\) domain is obtained

\[ \left[ \kappa^2 \mathbf{K}_1 + i \kappa \mathbf{K}_2 + \mathbf{K}_1 - \omega^2 \mathbf{M} \right] \mathbf{q} (k_z, \omega) + \mathbf{F}_b (k_z, \omega) = \mathbf{F}_1 (k_z, \omega) . \]
where the different matrix operators, which result from the application of a finite element assembling procedure for all the $N_s$ elements of the mesh, are defined as

\[
\begin{align*}
K_3 &= \bigcup_{e=1}^{N_s} \int_{\Omega_e} (B_e(\xi))^T \bar{C}^e(\omega) B_e(\xi) J_p^e(\xi) \, d\xi_1 \, d\xi_2, \\
K_2 &= \bigcup_{e=1}^{N_s} \int_{\Omega_e} \left[ (B_{\eta_2}(\xi))^T \bar{C}^e(\omega) B_{\eta_2}(\xi) - (B_{\eta_1}(\xi))^T \bar{C}^e(\omega) B_{\eta_1}(\xi) \right] J_p^e(\xi) \, d\xi_1 \, d\xi_2, \\
K_1 &= \bigcup_{e=1}^{N_s} \int_{\Omega_e} (B_{\eta_3}(\xi))^T \bar{C}^e(\omega) B_{\eta_3}(\xi) J_p^e(\xi) \, d\xi_1 \, d\xi_2, \\
M &= \bigcup_{e=1}^{N_s} \int_{\Omega_e} (N(\xi))^T \rho^e(\mathbf{x}) N(\xi) J_p^e(\xi) \, d\xi_1 \, d\xi_2,
\end{align*}
\]

in which $J_p^e(\xi) = \det \left[ \frac{\partial \mathbf{x}(\xi)}{\partial \xi} \right]$ represents the Jacobian of the isoparametric mapping in the $x-y$ plane for the $e$th semi-isoparametric finite element. The vectors of nodal displacements on $\Omega_s \cup \partial \Omega_s$, $\mathbf{Q}(\kappa_1, \omega)$, and nodal forces on $\partial \Omega_s$, $\mathbf{F}_s(\kappa_1, \omega)$ and $\mathbf{F}_b(\kappa_2, \omega)$, are expressed as

\[
\begin{align*}
\mathbf{Q}(\kappa_1, \omega) &= \bigcup_{e=1}^{N_s} \mathbf{q}_e^e(\kappa_1, \omega), \\
\mathbf{F}_s(\kappa_1, \omega) &= \bigcup_{e=1}^{N_s} \int_{\partial \Omega_e} (N(\xi(\eta)))^T \bar{T}_e(\kappa_1, \omega) J_p^e(\xi(\eta)) \, d\eta, \\
\mathbf{F}_b(\kappa_2, \omega) &= \bigcup_{e=1}^{N_s} \int_{\partial \Omega_e} (N(\xi(\eta)))^T \bar{T}_b(\kappa_2, \omega) J_p^e(\xi(\eta)) \, d\eta,
\end{align*}
\]

where $\xi(\eta)$ is a coordinate transformation for the in-plane mapping of the edge of an element which nodes belong to $\partial \Omega_s$, $J_p^e(\xi(\eta)) = \left| \frac{\partial \mathbf{x}(\xi)}{\partial \xi} \right|$ the corresponding Jacobian and $N_b$ the total number of edges that discretize $\partial \Omega_s$.

It is worth noting that, while the matrix operators $K_1$, $K_2$ and $K_3$ can be either dependent or independent on the frequency with varying rheological models [7], the vector $\mathbf{F}_b(\kappa_2, \omega)$ always depends on wavenumber and frequency since it accounts for the acoustic mechanical and geometric properties of the external medium.

2.2. Regularized 2.5D BEM model of the surrounding medium

The surrounding medium of domain $\Omega_b$ is assumed to be isotropic and linear viscoelastic, with mechanical properties defined by mass density $\rho$ and complex bulk velocities $\hat{c}_L$ and $\hat{c}_S$.

The 2.5D boundary integral formulation is obtained from the application of the Maxwell-Betti reciprocity theorem between two different dynamic states [60]. The first state is represented by the unknown displacements $\ddot{\mathbf{u}}(\mathbf{x}, \kappa_1, \omega)$ and tractions $\mathbf{t}(\mathbf{x}, \kappa_2, \omega)$ at a receiver point $\mathbf{x} \in \partial \Omega_b$ (see Fig. 1). The second state is assumed as the state of fundamental solutions in the full space for the spatial- and time-harmonic problem, i.e. the dynamic Green functions in terms of displacements and tractions at $\mathbf{x}$ due to a harmonic line load $\mathbf{p}(\mathbf{x}', \omega) = \delta(\mathbf{x} - \mathbf{x}') \exp \left[ i(\kappa z' - \omega t) \right]$
with plane coordinates \( x' \in \Omega_b \), where \( \delta (\cdot) \) denotes the Dirac’s delta function. The classical procedure adopted to extend the boundary integral formulation to source points \( x' \) belonging to the boundary involves the limiting process \( x' \in \Omega_b \rightarrow x' \in \partial \Omega_b \) [18, 16]. As a result, the 2.5D Somigliana’s identity is obtained, in which boundary integrals are expressed in the Cauchy principal value sense [78, 40, 60].

In this work, numerical difficulties in treating Cauchy principal value integrals and boundary corners are overcome by using the so called rigid body motion technique [60, 35]. Following the work of Lu et al. [60], the regularized 2.5D boundary integral equation in the \((\kappa_z, \omega)\) domain for a source point \( x' \in \partial \Omega_b \) and in absence of body forces can be expressed as

\[
\tilde{u}(x', \kappa_z, \omega) = \int_{\partial \Omega_b} \left[ \tilde{U}^D(r, \kappa_z, \omega) - \tilde{U}^S(r) \right] \hat{f}(x, \kappa_z, \omega) \, ds(x) \\
+ \int_{\partial \Omega_b} \tilde{U}^S(r) \hat{f}(x, \kappa_z, \omega) \, ds(x) \\
- \int_{\partial \Omega_b} \left[ \tilde{T}^D(r, \kappa_z, \omega) \tilde{u}(x, \kappa_z, \omega) - \tilde{T}^S(r) \tilde{u}(x', \kappa_z, \omega) \right] ds(x), \quad (x, x') \in \partial \Omega_b,
\]

where \( r = |x - x'| \) is the source-receiver distance in the \( z = 0 \) plane (see Fig. 1). The fundamental dynamic solutions \( \tilde{U}^D_{ij}(r, \kappa_z, \omega) \) in Eq. (16) express the \( j \)th displacement component at \( x \) when the harmonic line load of plane coordinates \( x' \) is acting in the \( i \)th direction. For a homogeneous isotropic linear viscoelastic full space, the following relations have been proposed by Li et al. [50]:

\[
\tilde{U}^D_{ij}(r, \kappa_z, \omega) = \frac{i}{4\mu} \left[ H_0^{(1)}(\kappa_z r) \delta_{ij} + L_{ij} \left[ H_0^{(1)}(\kappa_z r) - H_0^{(1)}(\kappa_\omega r) \right] \right], \quad i, j = 1, 2, 3
\]

(17)

where \( \kappa_\omega = \pm (\kappa_z^2 - \kappa_\omega^2)^{1/2} \) and \( \kappa_\omega = \pm (\kappa_z^2 - \kappa_\omega^2)^{1/2} \) denote the wavenumbers normal to the interface, which depend on the axial wavenumber \( \kappa_z \) and the complex wavenumbers of the longitudinal and shear bulk waves, \( \kappa_L = \omega/\tilde{c}_L \) and \( \kappa_S = \omega/\tilde{c}_S \), respectively. In Eq. (17), \( H_0^{(1)}(\cdot) \) is the zero order Hankel function of first kind and

\[
L_{ij} = \frac{1}{\kappa_z^2} \left[ \delta_{ij} \frac{\partial^2}{\partial \lambda \partial \lambda} - i \kappa_L \left( \delta_{ij} \frac{\partial^2 \delta_{ij}}{\partial \lambda \partial \lambda} + \delta_{ij} \frac{\partial^3 \delta_{ij}}{\partial \lambda \partial \lambda} \right) - \kappa_S^2 \delta_{ij} \frac{\partial^3 \delta_{ij}}{\partial \lambda \partial \lambda} \right], \quad i, j, k, q = 1, 2, 3
\]

(18)

The second set of fundamental solutions, the tractions Green’s functions \( \tilde{T}^D_{ij}(r, \kappa_z, \omega) \), are obtained as

\[
\tilde{T}^D_{ij}(r, \kappa_z, \omega) = \tilde{\sigma}^D_{ijk}(r, \kappa_z, \omega) n_k(x), \quad i, j, k = 1, 2, 3
\]

(19)

being \( n_k(x) \) the \( k \)th component of the outward normal at \( x \in \partial \Omega_b \) and

\[
\tilde{\sigma}^D_{ijk}(r, \kappa_z, \omega) = \lambda \tilde{\sigma}^D_{ijk}(r, \kappa_z, \omega) \delta_{jk} + 2\mu \tilde{\sigma}^D_{ijk}(r, \kappa_z, \omega), \quad i, j, k, = 1, 2, 3
\]

(20)

the \( j \)th component of the Cauchy stress tensor at \( x \) when the line load of projection \( x' \) is acting in direction \( i \), while

\[
\tilde{\sigma}^D_{ijk}(r, \kappa_z, \omega) = \frac{1}{\mu} \left[ \frac{\partial \tilde{U}^D_{ij}(r, \kappa_z, \omega)}{\partial \lambda} + \frac{\partial \tilde{U}^D_{ik}(r, \kappa_z, \omega)}{\partial \lambda} \right], \quad i, j, k = 1, 2, 3
\]

(21)
is the associated Green tensor of linear strains. Detailed expressions of Eqs. (17) and (19) can be found in the work of Gunawan and Hirose [40].

The static displacements and tractions Green tensors in Eq. (16), $U^S_{ij}(r)$ and $T^S_{ij}(r)$, respectively, correspond to the fundamental solutions for the in-plane line load problem in plane strain [18, 60]

$$U^S_{ij}(r) = \frac{1}{8\pi \text{Re}(\bar{\mu})(1 - \text{Re}(\bar{v}))} \left\{ 3 - 4\text{Re}(\bar{\mu}) \right\} \ln \left( \frac{1}{r} \right) \delta_{ij} + \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j},$$  \tag{22}

$$T^S_{ij}(r) = -\frac{1}{4\pi (1 - \text{Re}(\bar{\mu}))} \frac{\partial r}{\partial x_k} \left\{ 2\text{Re}(\bar{\mu}) \delta_{ij} + 2 \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j} \right\} - (1 - 2\text{Re}(\bar{\mu})) \left\{ \frac{\partial r}{\partial x_i} n_j - \frac{\partial r}{\partial x_j} n_i \right\}, \quad i, j, k = 1, 2 \tag{23}$$

and those for the anti-plane line load problem in plane strain [18, 60]

$$U^S_{33}(r) = \frac{1}{2\pi \text{Re}(\bar{\mu})} \ln \left( \frac{1}{r} \right),$$  \tag{24}

$$T^S_{33}(r) = -\frac{1}{2\pi r} \frac{\partial r}{\partial x_k} n_k, \quad k = 1, 2 \tag{25}$$

Since the asymptotic behaviour of the dynamic and static fundamental solutions correspond when $r \to 0$ [60, 35], the dominant singularities of the kernel functions in the first and last integral of Eq. (16) cancel each other out when the source point $x'$ approaches the receiver point $x$. Consequently, these integrals can be evaluated numerically using the standard Gauss-Legendre quadrature formula [83]. The second integral in Eq. (16) behaves asymptotically as $\ln(1/r)$ for $r \to 0$ and can therefore be evaluated using the Gauss-Laguerre quadrature formula [83, 36].

The boundary $\partial \Omega$ is discretized by subdividing it in $N_b$ quadratic semi-isoparametric monodimensional elements. In order to satisfy the compatibility conditions Eq. (1), the nodes of the generic boundary element $\partial \Omega^q_b$ are chosen to coincide with those belonging to one edge of an adjacent semi-analytical finite element $\Omega^q_s$. Displacements and tractions are interpolated at the generic boundary point $x \in \partial \Omega^q_b$ as follows

$$\bar{u}^q(\eta, \kappa_z, \omega) = \mathbf{N}(\eta) \bar{q}^q(\kappa_z, \omega),$$  \tag{26}

$$\bar{t}^q(\eta, \kappa_z, \omega) = \mathbf{N}(\eta) \bar{h}^q(\kappa_z, \omega),$$  \tag{27}

where $\mathbf{N}(\eta)$ is the matrix containing the quadratic shape functions in the natural coordinate $\eta \in \partial \Omega^q_b$, while $\bar{q}^q(\kappa_z, \omega)$ and $\bar{h}^q(\kappa_z, \omega)$ are the vectors of nodal displacements and tractions, respectively.

The regularized boundary integral formulation Eq. (16) is rewritten in discretized form by applying a point collocation scheme [18], where collocation points $x'$ are assumed to be coincident with the nodes of the boundary element mesh. Denoting by $x_c$ the $c$th collocation node and introducing Eqs. (26) and (27) into Eq. (16), the recursive collocation procedure over the total number of nodes $N_n = N_b \times 2$ of the boundary element mesh allows to rewrite Eq. (16) in the following discrete form
\[
\begin{aligned}
&\bigcup_{c=1}^{N_c} \left\{ \bigcup_{q=1}^{N_q} \left[ \bar{U}^q_c (r_c (\eta), \kappa_c, \omega) + U^q_c (r_c (\eta)) \right] \tilde{h}^q_r (\kappa_c, \omega) \right. \\
&\bigcup_{c=1}^{N_c} \left[ \bar{T}^q_0 (r_c (\eta), \kappa_c, \omega) \mathbf{q}^q (\kappa_c, \omega) \right] \\
&\bigcup_{c=1}^{N_c} \left[ \bar{T}^q_2 (r_c (\eta), \kappa_c, \omega) \mathbf{q}^q (\kappa_c, \omega) \right] + \bigcup_{c=1}^{N_c} \left[ T^q_1 (r_c (\eta)) \bar{U}_c (\kappa_c, \omega) \right] \biggr) \\
= \bigcup_{c=1}^{N_c} \bar{U}_c (\kappa_c, \omega), \quad (\mathbf{x}, \mathbf{x}_c) \in \partial \Omega_b
\end{aligned}
\]

where \( \mathbf{u}_c (\kappa_c, \omega) \) is the displacement vector at \( \mathbf{x}_c \) and

\[
\begin{aligned}
\bar{U}^q_c (r_c (\eta), \kappa_c, \omega) &= \int_{\partial \Omega_b^c} \left[ \bar{U}^D (r_c (\eta), \kappa_c, \omega) - U^D (r_c (\eta)) \right] \mathbf{N} (\eta) J^q_b (\eta) \, d\eta, \\
U^q_c (r_c (\eta)) &= \int_{\partial \Omega_b^c} U^S (r_c (\eta)) \mathbf{N} (\eta) J^q_b (\eta) \, d\eta, \\
\bar{T}^q_0 (r_c (\eta), \kappa_c, \omega) &= \int_{\partial \Omega_b^c} \bar{T}^D (r_c (\eta), \kappa_c, \omega) \mathbf{N} (\eta) J^q_b (\eta) \, d\eta, \\
T^q_2 (r_c (\eta), \kappa_c, \omega) &= \int_{\partial \Omega_b^c} \left[ \bar{T}^D (r_c (\eta), \kappa_c, \omega) \mathbf{N} (\eta) - T^S (r_c (\eta)) \mathbf{N} (\eta) \right] J^q_b (\eta) \, d\eta, \\
T^q_1 (r_c (\eta)) &= \int_{\partial \Omega_b^c} T^S (r_c (\eta)) J^q_b (\eta) \, d\eta.
\end{aligned}
\]

are influence operators, in which \( r_c (\eta) = |\mathbf{x} (\eta) - \mathbf{x}_c| \) denotes the in-plane distance between the integration point \( \mathbf{x} (\eta) \) and the collocation point \( \mathbf{x}_c, \eta_c \) is the adimensional coordinate evaluated at the element’s node coincident with \( \mathbf{x}_c \) and \( J^q_b (\eta) = |\partial \mathbf{x} (\eta) / \partial \eta| \) is the Jacobian of the semi-isoparametric transformation.

From Eq. (28), by grouping the displacements and tractions operators into the global influence operators \( \bar{U}_b (\kappa_c, \omega) = \bigcup_{c=1}^{N_c} \sum_{j=1}^{N_q} \bar{U}^q_c (r_c, \kappa_c, \omega) \) and \( T_b (\kappa_c, \omega) = \bigcup_{c=1}^{N_c} \sum_{j=1}^{N_q} T^q_1 (r_c, \kappa_c, \omega) \), and by assembling the displacements and tractions vectors into the global vectors \( \bar{Q}_b (\kappa_c, \omega) = \bigcup_{c=1}^{N_c} \sum_{j=1}^{N_q} \mathbf{q}^q (\kappa_c, \omega) \) and \( \bar{H}_b (\kappa_c, \omega) = \bigcup_{c=1}^{N_c} \sum_{j=1}^{N_q} \tilde{h}^j (\kappa_c, \omega) \) according to the mesh topology, the following set of linear algebraic equations is obtained

\[
\begin{aligned}
&\bigl[ \bar{T}_b (\kappa_c, \omega) + I \bigr] \bar{Q}_b (\kappa_c, \omega) = \bar{U}_b (\kappa_c, \omega) \bar{H}_b (\kappa_c, \omega), \quad (\mathbf{x}, \mathbf{x}_c) \in \partial \Omega_b
\end{aligned}
\]

which is defined only for source points \( \mathbf{x}_c \) belonging to the boundary. Once the vectors of boundary displacements \( \bar{Q}_b (\kappa_c, \omega) \) and tractions \( \bar{H}_b (\kappa_c, \omega) \) have been determined from Eq. (30), the radiated wavefield \( \bar{u}_d (\mathbf{x'}, \kappa_c, \omega) \) at any \( \mathbf{x'} \in \Omega_b \) can be computed using the relation [60, 35]

\[
\begin{aligned}
\bar{u}_d (\mathbf{x'}, \kappa_c, \omega) &= \bar{U}_d (\kappa_c, \omega) \bar{H}_b (\kappa_c, \omega) - \bar{T}_d (\kappa_c, \omega) \bar{Q}_b (\kappa_c, \omega), \quad \mathbf{x'} \in \Omega_b
\end{aligned}
\]
where the influence operators \( \mathbf{U}_d (\kappa, \omega) \) and \( \mathbf{T}_d (\kappa, \omega) \) result from the following element assembling procedure

\[
\mathbf{U}_d (\kappa, \omega) = \sum_{q=1}^{N_q} \int_{\partial \Omega_q} \mathbf{D} (r' (\eta), \kappa, \omega) \mathbf{N} (\eta) J_b^q (\eta) \, d\eta, \tag{32}
\]

\[
\mathbf{T}_d (\kappa, \omega) = \sum_{q=1}^{N_q} \int_{\partial \Omega_q} \mathbf{D} (r' (\eta), \kappa, \omega) \mathbf{N} (\eta) J_b^q (\eta) \, d\eta, \tag{33}
\]

in which \( r' (\eta) = x (\eta) - x' \). Since the dynamic Green functions are nonsingular for \( x' \in \Omega_b \), the integrals in Eqs. (32) and (33) can be evaluated numerically using the standard Gauss-Legendre quadrature formula.

2.3. SAFE-BE coupling

The coupling between the SAFE and the 2.5D BEM formulations is established via the compatibility conditions Eq. (1) and is carried out in a finite element sense [3]. On these bases, the infinite boundary element domain is converted into a single, wavenumber and frequency dependent, finite element-like domain with \( N_n \) nodes. The dynamic stiffness matrix of this pseudo finite element, relating nodal tractions to nodal displacements, is obtained by recasting Eq. (30) in the following form

\[
\mathbf{\tilde{K}}_b (\kappa, \omega) = \mathbf{\bar{K}}_b (\kappa, \omega) \mathbf{\bar{Q}}_b (\kappa, \omega), \tag{34}
\]

where \( \mathbf{\bar{K}}_b (\kappa, \omega) = \mathbf{U}_b^{-1} (\kappa, \omega) [\mathbf{T}_b (\kappa, \omega) + \mathbf{I}] \) is complex and non-symmetric. The nodal tractions are then converted into nodal forces

\[
\mathbf{\tilde{F}}_b (\kappa, \omega) = \mathbf{T}_b \mathbf{\tilde{K}}_b (\kappa, \omega) = \mathbf{T}_b \mathbf{\bar{K}}_b (\kappa, \omega) \mathbf{\bar{Q}}_b (\kappa, \omega), \tag{35}
\]

where \( \mathbf{T}_b = \sum_{q=1}^{N_q} \int_{\partial \Omega_q} \mathbf{N}^T (\eta) \mathbf{N} (\eta) J_b^q (\eta) \, d\eta \) is a distribution matrix obtained by substituting Eq. (27) into Eq. (15) and making use of the correspondence \( \mathbf{N} (\xi (\eta)) = \mathbf{N} (\eta) \). Eq. (35) is finally introduced in Eq. (11), leading to the following complex and non-symmetric \( N \)-dimensional linear system

\[
\begin{bmatrix} \kappa^2 \mathbf{K}_3 + i \kappa \mathbf{K}_2 + \mathbf{K}_1 + \mathbf{L}_b^T \mathbf{T}_b \mathbf{\tilde{K}}_b (\kappa, \omega) \mathbf{L}_b - \omega^2 \mathbf{M} \end{bmatrix} \mathbf{\bar{Q}} (\kappa, \omega) = -\mathbf{\tilde{F}}_b (\kappa, \omega), \tag{36}
\]

where \( \mathbf{L}_b \) is a collocation matrix so that \( \mathbf{\bar{Q}}_b (\kappa, \omega) = \mathbf{L}_b \mathbf{\bar{Q}} (\kappa, \omega) \). The displacement field at any \( x \in (\Omega_b \cup \partial \Omega_b) \) can be obtained by solving the \( N \times N \) linear system Eq. (36) in the unknown nodal displacements \( \mathbf{\bar{Q}} (\kappa, \omega) \) and using the interpolation in Eq. (5). In addition, substituting Eq. (34) into Eq. (31) leads to the following \( 3 \times N \) linear system of equations

\[
\mathbf{\tilde{U}}_b (x', \kappa, \omega) = \left[ \mathbf{\tilde{U}}_d (\kappa, \omega) \mathbf{\bar{K}}_b (\kappa, \omega) - \mathbf{\tilde{T}}_d (\kappa, \omega) \right] \mathbf{L}_b \mathbf{\bar{Q}} (\kappa, \omega), \quad x' \in \Omega_b \tag{37}
\]

which allows to compute the radiated displacement wavefield at any source point belonging to the surrounding domain.
3. Dispersion analysis

The dispersion properties of guided modes are determined in terms of complex wavenumbers \( \kappa_z(\omega) \) for any fixed \( \omega > 0 \) in absence of external forces applied at the interface \( \partial \Omega_s \). Substituting \( \bar{\mathbf{F}}_s(\kappa_z, \omega) = \mathbf{0} \) in Eq. (36), the dispersive equation for the open waveguide of domain \( \Omega_s \cup \Omega_b \) results in the following nonlinear eigenvalue problem in \( \kappa_z(\omega) \)

\[
\mathbf{Z}(\kappa_z, \omega) \mathbf{\dot{Q}}(\kappa_z, \omega) = \mathbf{0},
\]

where \( \mathbf{Z}(\kappa_z, \omega) = \{\kappa_z^2 \mathbf{K}_1 + i \kappa_z \mathbf{K}_2 + \mathbf{K}_1 + \mathbf{Z}_b^T \mathbf{T}_b \mathbf{K}_b(\kappa_z, \omega) \mathbf{L}_b - \omega^2 \mathbf{M} \} \in \mathbb{C}^{N \times N} \) is the dynamic stiffness matrix of the coupled SAFE-2.5D BEM model.

In this work, a contour integral method \([2, 5, 15]\) is applied to transform the nonlinear eigenvalue problem Eq. (38) into a linear one inside a simple closed curve \( \Gamma(\kappa_z) \in \mathbb{C} \) where poles of the normal modes must be sought. Compared to classical algorithms such as, for example, the Newton-Raphson method \([40]\) or the bisection method \([56]\), the adopted strategy presents the advantage that complex eigenvalues and eigenvectors are found without the need for initial estimation or derivatives computation, while their algebraic multiplicities are completely retained.

In order to separate physical from unphysical solutions and to fulfill the holomorphicity requirement for \( \mathbf{Z}(\kappa_z, \omega) \) inside \( \Gamma(\kappa_z) \), the multi valued feature of the operator \( \mathbf{Z}(\kappa_z, \omega) \) is investigated. Finally, eigensolutions are post-processed to extract dispersive characteristics of guided waves at the frequency of interest.

3.1. Contour integral method

The nonlinear eigenvalue problem Eq. (38) is solved using the contour integral method proposed by Beyn \([15]\). The algorithm is initialized by evaluating the two moment matrices

\[
\mathbf{A}_0 = \frac{1}{2\pi i} \oint_{\Gamma(\kappa_z)} \mathbf{Z}^{-1}(\kappa_z, \omega) \mathbf{V} \mathbf{d}\kappa_z \in \mathbb{C}^{N \times L},
\]

\[
\mathbf{A}_1 = \frac{1}{2\pi i} \oint_{\Gamma(\kappa_z)} \mathbf{k} \mathbf{Z}^{-1}(\kappa_z, \omega) \mathbf{V} \mathbf{d}\kappa_z \in \mathbb{C}^{N \times L},
\]

where \( \mathbf{V} \in \mathbb{C}^{N \times L} \) is chosen randomly and the positive integer \( L \) is assumed to satisfy the requirement \( K \leq L \leq N \), where \( K \) denotes the supposed number of eigenvalues inside \( \Gamma(\kappa_z) \).

A singular value decomposition (SVD) \( \mathbf{A}_0 = \mathbf{V} \Sigma \mathbf{W}^H \) \([74]\) is then performed, followed by a rank test in which singular values of \( \Sigma \) lower than a fixed tolerance \( \text{tol}_{\text{rank}} \) are eliminated along with their corresponding columns in \( \mathbf{V} \) and \( \mathbf{W} \), being responsible of a bad conditioning of the eigenvalue computation. After the rank test has been performed, the number of singular values inside \( \Gamma(\kappa_z) \) is reduced from \( L \) to \( M \) and the eigenvalues \( \kappa_z(\omega) \) and corresponding eigenvectors \( \mathbf{y}_m(\omega) \) \( (m = 1, 2, ..., M) \) are computed from the operator

\[
\mathbf{B} = \mathbf{V}_0^H \mathbf{A}_1 \mathbf{W}_0 \Sigma_0^{-1} \in \mathbb{C}^{M \times M},
\]

where \( \Sigma_0 = \text{diag}(\Sigma_1, ..., \Sigma_M) \), \( \mathbf{V}_0 = \mathbf{V}(1 : N, 1 : M) \) and \( \mathbf{W}_0 = \mathbf{W}(1 : L, 1 : M) \). Although it is expected \( M = K \) after the rank test, generally \( K \leq M \) due to the choice of \( \text{tol}_{\text{rank}} \), leading to \( M - K \) spurious eigensolutions. Some of these may lie outside the contour and can therefore be directly discarded. The remaining spurious roots can be detected by performing the following residual test on the full set of eigensolutions.
\( \left\| \mathbf{Z}_{\kappa}^{m}(\omega) \mathbf{Q}_{\omega}^{m} \right\|_{\infty} \leq \text{tol}_{res}, \) (42)

where \( \mathbf{Q}_{\omega}^{m}(\omega) = \mathbf{V}_{0} y^{m}(\omega) \) is the true eigenvector associated to \( \kappa^{m}_{\omega}(\omega) \), \( \| \cdot \|_{\infty} \) denotes the infinity norm and \( \text{tol}_{res} \) is an arbitrary threshold value used to discriminate well-conditioned from ill-conditioned eigensolutions. If solutions with algebraic multiplicity higher than one are expected, ill-conditioning may also occur. In this case, a Schur decomposition \( \mathbf{B} \mathbf{Q} = \mathbf{Q} \mathbf{R} [39] \) can be performed, where \( \mathbf{R} \) is a block-diagonal matrix such that diagonal blocks belong to different eigenvalues. The eigenvectors \( y^{m}(\omega) \) are then selected from the first column of each \( m \)th diagonal-block in \( \mathbf{R} \) to compute the associated true eigenvector \( \mathbf{Q}_{\omega}^{m}(\omega) \). The eigenpairs \( [\kappa^{m}_{\omega}(\omega), \mathbf{Q}_{\omega}^{m}(\omega)] \) that satisfy the inequality Eq. (42) are finally accepted as final solution.

3.2. Single-valued definition of the dynamic stiffness matrix

The procedure reported in Sec. 3.1 allows to extract all the eigenvalues for a holomorphic problem \( \mathbf{Z}(\kappa, \omega) \in \mathcal{H}(\Omega^{*}, \mathbb{C}^{K \times K}) \), where \( \Omega^{*} \) denotes the region of the complex \( \kappa \)-plane enclosed by \( \Gamma(\kappa) \). However, this condition is not generally satisfied as \( \mathbf{Z}(\kappa, \omega) \) is singular and multi-valued due to the properties of the Hankel functions \( H_{n}^{(1)}(\cdot) \) as well as the two wavenumbers \( \kappa_{a} = \pm(\kappa_{0}^{2} - \kappa_{1}^{2})^{1/2} \) and \( \kappa_{b} = \pm(\kappa_{0}^{2} - \kappa_{2}^{2})^{1/2} \). Therefore, before performing the contour integration in Eqs. (39) and (40), the operator \( \mathbf{Z}(\kappa, \omega) \) must be made single valued and analytic everywhere inside \( \Omega^{*} \). This task is accomplished by choosing the phase of \( \kappa_{a} \) and \( \kappa_{b} \) consistently with the nature of the existing partial bulk waves in the surrounding medium, and by removing points of singularity and discontinuity in the \( \kappa \)-plane.

The signs of the wavenumbers normal to the interface, with reference to the more general viscoelastic case of Fig. 2(a), are established as in the following:

- for \( \text{Re}(\kappa_{a}) > \text{Re}(\kappa_{b}) \), the Snell-Descartes law [76, 17] enforces a total reflection, with possible mode conversion at the interface, of the longitudinal and shear bulk waves traveling inside the waveguide (non-leaky region). In this case, the particles motion in the surrounding medium remains confined in proximity of the interface, with amplitude decaying exponentially in the direction normal to the interface [6, 71]. Since the propagation process is represented by the Hankel functions \( H_{n}^{(1)}(\kappa_{a} \rho) \) and \( H_{n}^{(1)}(\kappa_{b} \rho) \) and assumes a dependence \( \exp[i(\kappa_{a} \rho - \omega \tau)] \), in order to have outgoing waves satisfying the radiation condition at infinity, the signs of \( \kappa_{a} \) and \( \kappa_{b} \) must be chosen so that \( \text{Im}(\kappa_{a}) > 0 \) and \( \text{Im}(\kappa_{b}) > 0 \).

- In the range \( \text{Re}(\kappa_{a}) < \text{Re}(\kappa_{b}) < \text{Re}(\kappa_{s}) \), the longitudinal bulk waves are still totally reflected at the interface, and the sign of \( \kappa_{a} \) is then selected in order to preserve the positivity of its imaginary component, which satisfies the radiation condition at infinity. On the other hand, shear bulk waves are also refracted at some leakage angle \( \theta_{\text{Leak}} \approx \sin^{-1} \left( \frac{\text{Re}(\kappa_{s})}{|\text{Re}(\kappa_{b})|} \right) \) with respect to the normal at the interface [6, 20] and therefore, for the properties of the \( n \)th order Hankel function of the first kind, \( \text{sgn}(\kappa_{b}) \) must be chosen in order to satisfy the condition \( \text{Re}(\kappa_{b}) > 0 \).

Regarding the imaginary component of \( \kappa_{b} \), it must be observed that for any fixed positive \( \text{Re}(\kappa_{a}) \in [0, \text{Re}(\kappa_{s})] \), \( \text{Im}(\kappa_{b}) \) changes monotonically as a function of \( \text{Im}(\kappa_{s}) \) and vanishes for values of \( \text{Im}(\kappa_{s}) = \text{Re}(\kappa_{s}) \text{Im}(\kappa_{s}) / \text{Re}(\kappa_{s}) \), which define a branch of hyperbola passing through the point \( \kappa_{s} \). This branch of hyperbola, indicated with \( q_{b} \) in Fig. 2(a), determines the transition between an outgoing growing (\( \text{Im}(\kappa_{b}) < 0 \)) and an outgoing decaying
(Im(κβ) > 0) shear waves wavefield along the orthogonal direction to the interface. These physical states are represented by points P1 and P3 in Fig. 2(a), respectively, while the transition state (Im(κβ) = 0) is represented by point P2 on qβ.

The wavevector configurations for points P1, P2 and P3 for a planar interface, are shown in Fig. 3 in terms of propagation and attenuation vectors, \( \mathbf{k}_{\text{Re}}^\text{S} \) and \( \mathbf{k}_{\text{Im}}^\text{S} \), respectively, with \( \mathbf{k}_{\text{S}}^\text{Re} = \mathbf{k}_{\text{S}}^\text{Re} \cdot \mathbf{\cos}(\gamma_S) = \mathbf{k}_{\text{Re}}^\text{S} \cdot \mathbf{k}_{\text{Im}}^\text{S} \) and \( 0 < \gamma_S < \pi/2 \) [19, 21]. The attenuation vector is given by \( \mathbf{k}_{\text{Im}}^\text{S} = \mathbf{k}_{\text{S}}^\text{Re} + \mathbf{k}_{\text{D}}^\text{Re} \), where \( \mathbf{k}_{\text{S}}^\text{Im} \) is the component due to material damping (homogeneous component), parallel to \( \mathbf{k}_{\text{Re}}^\text{S} \) according to the material damping model of Eq. (10), and \( \mathbf{k}_{\text{D}}^\text{Re} \) is the component associated to energy radiation (inhomogeneous component), which is normal to \( \mathbf{k}_{\text{Re}}^\text{S} \) [22]. All the wavenumber vectors lie on the plane containing the \( z \)-axis and the outward normal \( \mathbf{n} \) at the interface.

For any \( \kappa_S \) above \( q_{\beta} \) (point P1), imposition of \( \text{Re}(\kappa_\beta) > 0 \) implies that \( \text{Im}(\kappa_\beta) < 0 \) and the propagation and attenuation vectors normal to the interface, \( \mathbf{k}_{\text{Re}}^\text{S} \) and \( \mathbf{k}_{\text{Im}}^\text{S} \), respectively, result in opposite directions (Fig. 3(a)). Since \( \mathbf{k}_{\text{Im}}^\text{S} \) is perpendicular to the lines of constant amplitudes in the \( z - \mathbf{n} \) plane and is oriented in the direction of the maximum decay of amplitude [22], a well known characteristic of leaky waves can be observed: while material damping (homogeneous component) causes the amplitude of the partial shear wave to decrease along the radiation direction (dashed lines), due to the inhomogeneous component the wave amplitude increases in direction \( \mathbf{n} \). This behaviour can be observed from the intersections of the equi-amplitude lines (solid lines) with the normal to the interface and has been already discussed by different authors [81, 88, 89] in the special case of isotropic elastic open waveguides, for which \( \mathbf{k}_{\text{Im}}^\text{S} = 0 \) and \( \gamma_S = \pi/2 \) [19, 21].

If \( \kappa_S \) lies below \( q_{\beta} \) (point P3), then \( \text{Re}(\kappa_\beta) > 0 \) implies that \( \text{Im}(\kappa_\beta) > 0 \), and \( H_\alpha^{(3)}(\kappa_\beta r) \) is therefore convergent. In this case, the shear wave amplitude decreases along both the radiation direction and the direction normal to the interface (Fig. 3(c)).

- Analogous considerations apply in the range \( \text{Re}(\kappa_\beta) < \text{Re}(\kappa_\alpha) \), where both longitudinal and shear bulk waves are radiated in the surrounding medium.

Once the signs for \( \kappa_\alpha \) and \( \kappa_\beta \) have been determined, the operator \( \mathbf{\tilde{Z}}(\kappa_\gamma, \omega) \) results single-valued everywhere on \( \Omega^+ \). To make it also analytical, it is necessary to remove points corresponding to singularities and discontinuities. In this work, an approach similar to the Vertical Branch Cut Integration (VBCI) method [48, 52, 91, 43] has been adopted.

For the given choices of \( \text{sgn}(\kappa_\alpha) \) and \( \text{sgn}(\kappa_\beta) \), points of discontinuities are represented by the two vertical lines departing from \( \kappa_L = \omega/\bar{c}_L \) and \( \kappa_S = \omega/\bar{c}_S \) and extending along the positive direction of the imaginary axis (see Fig. 2(a)). These lines delimit zones of \( \Omega^+ \) where \( \kappa_\alpha \) and \( \kappa_\beta \) change sign in both their real and imaginary components. The two vertical cuts are therefore introduced to remove these discontinuities. These cuts also include the bulk wavenumbers, since \( H_\alpha^{(3)}(\kappa_\alpha, \mu r) \) are not defined for \( \kappa_z \rightarrow \kappa_\alpha, \beta \). The last branch cut is represented by the whole negative real axis, which is a branch cut of the Hankel function, and is easily avoided by restricting the contour \( \Gamma(\kappa_\gamma) \) only to positive real values of the axial wavenumbers (right propagating waves).

The integral path in Fig. 2(b) represents a special case of that in Fig. 2(a) when an isotropic elastic surrounding medium is considered. In this case, \( \kappa_L \) and \( \kappa_S \) are purely real and the hyperbolic lines \( q_{\alpha} \) and \( q_{\beta} \) collapse on the positive imaginary axis and part of the real axis. Since in this case \( \mathbf{k}_{\text{Im}}^\text{S} = 0 \), lines of constant amplitude become parallel to the radiation direction (\( \gamma_L = \pi/2 \)), causing the displacement field to grow with distance in the direction normal to the interface [89].
3.3. Dispersion characteristics extraction

Once the complete set of eigensolutions $[\kappa^m_z(\omega), \bar{Q}^m(\omega)]$ has been determined from Eq. (38) for the frequency of interest, the dispersion characteristics for the $m$th guided mode are computed as [71, 14, 7]

**phase velocity:**
$$c^m_p(\omega) = \frac{\omega}{\text{Re} \left \{ \kappa^m_z(\omega) \right \}} \quad (43)$$

**attenuation:**
$$\alpha^m(\omega) = \text{Im} \left \{ \kappa^m_z(\omega) \right \} \quad (44)$$

**energy velocity:**

$$c^m_e(\omega) = \frac{\frac{1}{4} \omega \text{Im} \left \{ \int_{\Omega} \bar{\sigma}^m_{ij}(\omega) \text{conj} \left \{ \bar{u}^m_i(\omega) \right \} dxdy \right \}}{\frac{1}{4} \text{Re} \left \{ \int_{\Omega} \omega^2 \bar{\bar{u}}^m_i(\omega) \text{conj} \left \{ \bar{u}^m_i(\omega) \right \} + \bar{\sigma}^m_{ij}(\omega) \text{conj} \left \{ \bar{\epsilon}^m_{ij}(\omega) \right \} dxdy \right \}} \quad (45)$$

where the displacements $\bar{u}^m_i(\omega)$, strains $\bar{\epsilon}^m_{ij}(\omega)$ and stresses $\bar{\sigma}^m_{ij}(\omega)$ on $\Omega_c$ are recovered from $\bar{Q}^m(\omega)$ using the interpolations Eq. (5), the strain-displacement relations Eq. (6) and the constitutive relations Eq. (9) [7, 61]. It should be noted that Eq. (45) does not represent the exact expression of the energy velocity for leaky guided modes. In this case, energy flow curves bend away from the waveguide into the surrounding medium, determining an axial component of the energy flow on $\Omega_b$ [6, 81, 62, 20]. In such circumstances, the domain integrals in Eq. (45) should be rigorously evaluated on $\Omega_c \cup \Omega_b$, which has infinite extension. However, Eq. (45) is commonly accepted as sufficiently accurate in GUW applications [71, 72] and becomes exact for non-leaky modes ($\text{Re}(\kappa_z) > \text{Re}(\kappa_S)$), being the wavefield on $\Omega_b$ constituted by evanescent waves [6]. In this case there is no energy flux through $\Omega_b$, with the total energy remaining confined within $\Omega_c$ and flowing parallel to the interface.

4. Numerical applications

In this section, four numerical applications are presented. The first two, which have been studied in literature using the Global Matrix Method and the SAFE method with absorbing regions, are used as validation cases, while the remaining two applications are proposed to show the unique capabilities of the proposed tool to compute dispersive properties of leaky waves in embedded waveguides of arbitrary cross-section. The material properties used in the analyses are listed in Tab. 1. Since only the Maxwell rheological model is considered in this work, the material constants are independent from frequency [7]. The settings of the contour algorithm have been defined on the basis of single analysis performed at few frequencies, by changing the parameters ($N_p$, tol_rank and tol_res) until a stable trend was observable in the separation of the singular values as well as the relative residuals of eigensolutions.

4.1. Elastic steel bar of circular cross section embedded in elastic concrete

In the first example, the coupled SAFE-BEM formulation is validated with respect to the FEM solution proposed by Castaings and Lowe [20] for a 20 mm diameter elastic steel (st) bar embedded in elastic concrete (co). The SAFE mesh used in the analysis is composed of 48 six-node triangular elements and 32 nine-node quadrilateral elements, as shown in Fig. 4(a). The BEM mesh matches the SAFE mesh at the interface and is composed of 32 three-node monodimensional elements. The steel longitudinal and shear bulk wave attenuations listed in
Tab. 1, $\beta_{L}^{\mu}$ and $\beta_{S}^{\mu}$, respectively, are neglected. The dispersion curves, represented in Fig. 4 in terms of phase velocity, attenuation and energy velocity, have been obtained by considering the upper limit of the integration path in Fig. 2(b) equal to 200 Np/m (1737.18 dB/m), while the horizontal extension has been limited to $\text{Re}(\kappa_{S}^{\mu})$ at each frequency step. The attenuation value has been added to the phase and energy velocity curves filling the circular markers with different blue levels (color online). Light and dark levels denote higher and lower values of the attenuations, respectively.

The results for the $L(0, 1)$, $F(1, 1)$ and $F(1, 2)$ modes are in very good agreement with those in Ref. [20]. Of the remaining modes, it is interesting to observe the global behaviour of the $F(2, 1)$, which experiences three discontinuities in the range $0-200$ kHz. The first discontinuity is located at about 40 kHz, where the mode becomes leaky ($c_{p}^{F(2, 1)} > c_{S}^{\mu}$). Moreover, the energy velocity in the frequency range corresponding to $c_{S}^{\mu} < c_{p}^{F(2, 1)} < c_{L}^{\mu}$ is negative. The second discontinuity occurs when the mode crosses the longitudinal bulk velocity of the concrete. In the frequency range $82-130$ kHz, where the mode is indicated as $F(2, 1)'$, both longitudinal and shear bulk waves are leaked in the concrete. The third discontinuity occurs in the frequency range $130-136$ kHz, where the phase velocity becomes lower than the longitudinal bulk velocity of the concrete, thus corresponding to radiation of shear bulk waves only.

4.2. Viscoelastic steel bar of circular cross section embedded in viscoelastic grout

In this second example, a 20 mm diameter viscoelastic steel bar ($st$) embedded in viscoelastic grout ($gr$) is considered. This example is used to validate the proposed SAFE-BEM formulation for a case in which all the materials are viscoelastic. The cross section is discretized with the same type and number of elements used in Sec. 4.1. The obtained dispersion curves, shown in Fig. 5, are very similar to those in Ref. [73], in which the same problem has been solved by using the software DISPERSE [70]. In the analysis, the imaginary part of the integral path in Fig. 2(a) has been limited to 200 Np/m (1737.18 dB/m). The dispersion curves for the fundamental longitudinal mode, $L(0, 1)$, and fundamental flexural mode, $F(1, 1)$, are in very good agreement with those in the Ref. [73]. It is also worth noting that the contour integral method is able to detect the portion of the $F(1, 1)$ mode in the frequency range $0-15$ kHz, although the non-leaky poles lie in this case very close to $\kappa_{S}^{gr}$. As indicated by Beyn [15], the contour integral method is indeed able to detect the roots if they lie outside but close to the contour, although the accuracy becomes strongly dependent on the number of integration points used in proximity of the same roots. Since $Z(\kappa, \omega)$ is not defined for $\kappa_{S} \to \kappa_{S}^{gr}$, the solutions provided by the contour integral method can be inaccurate. In fact, some of these solutions have been found to lie in the non-leaky region, while Pavlakovic et al. [73] have excluded the existence of the $F(1, 1)$ mode in this region. Therefore, to get precise and reliable solutions for $\kappa_{S} \to \kappa_{S}^{gr}$, the roots obtained by the contour integral method were improved by using them as initial guesses in the Muller’s root finding algorithm [74].

As in the elastic case of Sec. 4.1, discontinuities occur when the modes cross the bulk velocities of the external medium. The discontinuities for the $F(1, 1)$ mode in the phase velocity spectra are mild compared with those of the $F(2, 1)$ mode. The corresponding jumps in attenuation are clearly observable. As for the $F(2, 1)$ mode in Sec. 4.1, the branch of the mode that satisfies the condition $c_{S}^{gr} < c_{p}^{F(2, 1)} < c_{L}^{gr}$ shows a negative energy velocity.

4.3. Viscoelastic steel bar of square cross section embedded in viscoelastic grout

The third example considers a square viscoelastic steel bar ($st$), of 20 mm side length, embedded in viscoelastic grout ($gr$) (see Fig. 6(a)). Actually, in the considered frequency range...
the steel does not generally exhibit material damping, thus, to test the presented method a small
damping was artificially added by considering the material as viscoelastic. The bar is discretized
with 100 eight-node quadrilateral elements, while a boundary mesh of 40 three-node monodi-
dimensional elements is adopted to model the surrounding space. A maximum attenuation of 200
Np/m (1737.18 dB/m) has been considered in the analysis. The modes in the dispersion spectra
of Fig. 6 have been labeled as in Ref. [40], where a square waveguide in vacuum was considered.
It can be observed that, in the frequency range 0 − 13 kHz, the first flexural mode, \( F_1 \),
behaves similarly to the \( F(1,1) \) mode for the circular bar in Sec. 4.2. Also in this case the Muller’s
method has been applied to improve the accuracy of these solutions. It is also interesting noting
the existence of a non-leaky section for the two skew modes \( S^1_p \) and \( S^2_p \). Similarly to the \( F(2,1) \)
mode in both the examples of Secs. 4.1 and 4.2, the energy velocity of these modes becomes
positive for values of \( c_L' > c_S' \) and \( c_L'' > c_S'' \).

4.4. Viscoelastic HP200 steel beam embedded in viscoelastic soil

Despite the fact that non-destructive evaluation of pile-integrity is an important topic in
technical engineering [66, 31, 53], dispersion analyses of guided waves propagating in founda-
tion piles seem to be limited in literature to simple geometries [34, 33]. In this example, the
proposed formulation is exploited to predict the dispersion curves for an HP200 steel (st) pile
embedded in soil (so). Both steel and surrounding soil are treated as linear viscoelastic materi-
als. The attenuations of L and S waves in soils have been investigated by Ketcham et al. [45]
and are reported in Tab. 1 for a surface soil layer. The pile is discretized with 52 eight-node
quadrilateral elements and 2 six-node triangular elements, as shown in Fig. 7(a). The BEM
mesh is composed of 108 three-node monodimensional elements. Since only the first low order
modes are of interest in practical applications, the analysis has been carried out by considering
a maximum attenuation of 9.2 Np/m (80.86 dB/m), where these modes have been found to exist
in the frequency range 0 − 1000 Hz. The low order modes are indicated with \( m1, m2, m3 \) and
\( m4 \) in Fig. 7. It can be noted that all these modes are discontinuous in correspondence of the
soil bulk velocities. The flexural-like mode \( m4 \), which is indicated with \( m4' \) for \( c_L'^{m4'} > c_S'^{m4'} \)
and with \( m4'' \) for \( c_L''^{m4''} > c_S''^{m4''} \), exists in both the two leaky zones of the spectra in the frequency range
600 − 870 Hz. The longitudinal-like mode \( m1 \) becomes almost non-dispersive in the frequency range
700 − 1000 Hz, while its attenuation remains almost constant in the frequency range cor-
responding to \( c_L^{m1} > c_S^{m1} \). Since this mode also shows the highest energy velocity combined with the
minimum attenuation if compared to the remaining low order modes, it can be particularly
suitable in practical inspection applications.

The behaviour of the radiated wavefield for the flexo-torsional \( m2 \) mode is examined with
reference to the analysis of Sec. 3.2. From Fig. 7(c), it can be noted that the attenuation of this
mode is always greater than the attenuations of both longitudinal and shear bulk waves,
\( \alpha_L^{m2}(\omega) = \text{Im}[\kappa_L^{m2}(\omega)] \) and \( \alpha_S^{m2}(\omega) = \text{Im}[\kappa_S^{m2}(\omega)] \), respectively. It is therefore expected that the
amplitudes of both longitudinal and shear waves in their corresponding leaky zones must increase
with distance along the direction normal to the boundary. On the other hand, for \( c_L^{m2} < c_S^{m2} \), the
radiated wavefield must decay with distance from the pile-soil interface. These behaviours can
be observed in Fig. 8, where the wavestructures \( Q^{m2}(\omega) \) of the \( m2 \) mode at various frequencies
have been substituted in Eq. (37) to compute the radiated wavefield at the nodes \( x' \) of an external
mesh. As can be noted, the wavefield in soil for the \( m2 \) mode at 88.38 Hz decays rapidly away
from the interface, while the wavefield amplitudes for the \( m2' \) mode at 616.16 Hz (radiated S
waves) and the $m 2''$ mode at 952.02 Hz (radiated S and L waves) increase with distance from the interface.

It should be kept in mind that the modes attenuations resulting from the present analysis are probably overestimated due to the assumptions in Eq. (1). Lower and more realistic attenuations could be predicted by inserting an appropriate thin layer between the two media as previously done by Nayfeh and Nagy [65] and by Pavlakovic [71].

5. Conclusions

A Semi-Analytical Finite Element (SAFE) method coupled with a regularized 2.5D Boundary Element Method (BEM) has been applied to derive the dispersive equation for a viscoelastic waveguide of arbitrary cross-section embedded in a viscoelastic isotropic unbounded medium. The coupling between the SAFE and BEM domains has been established in a finite element sense, by converting the infinite BEM domain into a single, wavenumber and frequency dependent, SAFE-like element. The discretized wave equation, which is configured as a nonlinear eigenvalue problem in the complex axial wavenumber, has been solved using a Contour Integral Method. In order to fulfill the requirement of holomorphicity of the dynamic stiffness matrix $\bar{Z}(\kappa_z, \omega)$ inside the complex contour, the phase of the wavenumbers normal to the interface have been chosen consistently with the nature of the waves existing in the surrounding medium.

The method has been first validated against literature results for an elastic circular steel bar embedded in elastic concrete and a viscoelastic circular steel bar embedded in viscoelastic grout, for which results relative to some modes were available. In both cases, a very good agreement between the solutions has been observed.

Next, two new cases have been investigated. The dispersion curves obtained for a viscoelastic steel bar of square cross section embedded in viscoelastic grout show some analogies with those of the viscoelastic steel bar of circular cross section embedded in viscoelastic grout, especially for the longitudinal, torsional and flexural modes. The dispersion analysis performed for a HP200 viscoelastic steel beam embedded in viscoelastic soil show that the first longitudinal mode is the most suitable in practical guided waves-based inspections. Due to the structure of the 2.5D BEM, the proposed approach requires higher computational times in comparison with SAFE formulations that use infinite elements or PMLs. The coupled SAFE-2.5D BEM formulation can be further extended to problems with embedded thin walled sections [77], immersed waveguides [38], poroelastic surrounding media [59, 60] and waveguides embedded in both isotropic and layered half spaces [35, 75].

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New York, NY, USA.


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* Pavlakovic et al. [73]
* Castaings and Lowe [20]
† Ketcham et al. [45]

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