Expectations, Disappointment, and Rank-Dependent Probability Weighting

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June 2005

Abstract

We develop a model of Disappointment in which disappointment and elation arise from comparing the outcome received, not with an expected value as in previous models, but rather with the other individual outcomes of the lottery. This approach may better reflect the way individuals are liable to experience disappointment. The model obtained accounts for classic behavioral deviations from the normative theory, offers a richer structure than previous disappointment models, and leads to a Rank-Dependent Utility formulation in a transparent way. Thus, our disappointment model may provide a clear psychological rationale for the subjective transformation of probabilities.

Key words: Disappointment theory; Expected Utility violations; Probability weighting; Rank-dependent utility

This paper was presented at the 11th conference on the Foundations and Applications of Utility, Risk, and Decision Theory (FUR XI), Cachan, July 2004.

Several descriptive theories of choice under risk have been proposed to explain behavioral departures from the Expected Utility (EU) model. Among them, Regret theory, proposed by Bell (1982) and Loomes and Sugden (1982), and Disappointment theory subsequently developed by these same authors (Bell, 1985; Loomes & Sugden, 1986) provide attractive alternatives because they are based on simple, intuitively compelling psychological postulates.

The basic proposition of Regret theory is that, in choosing between two options, individuals will evaluate the outcome of one option by comparison with the outcome that the other option would have yielded. If the outcome of the forgone alternative is superior, the individual will experience regret, if it is inferior, he/she will experience rejoice. A key assumption of Regret theory is that individuals will anticipate the possible feelings of regret *ex ante* and factor them into their decision making. Disappointment theory is based on a similar idea: instead of comparing the outcome received with what would have been obtained had another alternative been chosen, the hypothesis here is that individuals compare the outcome they receive to a prior expectation. If the outcome is inferior to their expectation, individuals will be disappointed; if it is superior, they will be relieved. Again, the proposition is that individuals take these feelings into account while evaluating choices. Both regret and disappointment may influence individuals' preferences concurrently. For example, Inman, Dyer and Jia (1997) propose a model of consumer satisfaction incorporating both regret and disappointment.

Previous Disappointment Models

Previous approaches to Disappointment have used the expected (subjective) value of the lottery as a benchmark for categorizing the outcomes as "disappointing" or "elating." For instance, Loomes and Sugden (1986) propose the following. Denote by X the lottery that delivers payoff x_i with probability p_i, $i = 1,..., n$. The individual's subjective valuation of payoffs is captured by a function $v(.)$, and his/her prior expectation about X is taken to be the expected subjective value of the lottery, $\mathrm{E}[v(X)] = \sum_{i=1}^{n} p_i v(x_i)$. Disappointment and elation arise from comparing the subjective value of the outcome obtained, $v(x_i)$, with $\mathrm{E}[v(X)]$. Therefore, the utility experienced upon receiving outcome x_i is given by:

$$u(x_i) = v(x_i) + D(v(x_i) - \mathrm{E}[v(X)]), \qquad (1)$$

where $D(.)$ is an increasing function capturing the effects of disappointment or elation, depending on whether $v(x_i) - \mathrm{E}[v(X)]$ is less than or greater than 0, respectively. The lottery's overall utility is just the mathematical expectation of the modified utility expressed in (1):

$$U(X) = \mathrm{E}[u(X)] = \sum_{i=1}^{n} p_i (v(x_i) + D(v(x_i) - \mathrm{E}[v(X)])). \qquad (2)$$

All previous models of Disappointment that we are aware of use a summary measure of the distribution (e.g., the expected value, or certainty equivalent) as a basis for measuring disappointment (Bell 1985; Loomes & Sugden 1986; Gul 1991; Jia, Dyer & Butler 2001). Here we propose another approach to capturing disappointment and elation, as explained in the next section.

An Alternative Model of Disappointment

The expected utility of a lottery is akin to an overall measure of worth of the lottery, but why should it qualify as a "prior" expectation about the lottery's outcome? We feel that it could be rather artificial for an individual to compare his/her payoff with an expected value and more natural to compare it to actual, tangible outcomes that the lottery could have produced. For example, even if the outcome received is above the expected value, an individual may still feel some disappointment simply because he/she had a decent probability of getting something even better out of the lottery. Rather than focusing on a particular reference point, e.g. $\mathrm{E}[v(X)]$, to measure disappointment or elation, we propose that individuals may compare outcomes to one another in evaluating a lottery.

Our hypothesis for modeling disappointment is that individuals are liable to compare the outcomes they obtain, not to a prior expectation, but to the other outcomes that they did not get. Specifically, upon obtaining a particular outcome, x_i, an individual may be disappointed with regard to the outcomes better than x_i, but comforted with regard to the outcomes worse than x_i.

Model structure

Without loss of generality, let us assume that outcomes are rank ordered by decreasing preference, that is, $x_1 \geq x_2 \geq ... \geq x_n$, and that the valuation of outcomes is linear, that is, $v(x_i) = x_i$. This assumption is also made by Bell (1985), and adopted by Loomes and Sugden (1986) as well in deriving results for their model. Now, the intensity of disappointment experienced from receiving a given x_i as compared to a better x_k, $k < i$, should increase with both the difference between these two outcomes ($x_k - x_i$) and the probability p_k of the missed outcome. Besides, if either of these terms is 0, no disappointment should be felt. Therefore, the disappointment of receiving x_i as opposed to x_k is driven by $p_k(x_k - x_i)$, which is non-negative and increasing in both p_k and x_k. Taking all the outcomes better than x_i, the total potential for disappointment in x_i can be measured as: $\sum_{k=1}^{i-1} p_k(x_k - x_i)$. In a similar fashion, the total elation potential resulting from getting x_i will be measured by: $\sum_{k=i+1}^{n} p_k(x_i - x_k)$. Upon obtaining an outcome x_i in lottery X, the individual is liable to experience the following utility: the value associated with having x_i itself; the disappointment arising from having x_i instead of better outcomes, x_k, $k < i$; and the elation from having x_i rather than worse outcomes, i.e., x_k, $k > i$. Combining these three elements, the utility of an outcome incorporating disappointment and elation is defined as follows:

$$u(x_i) = x_i - D(\sum_{k=1}^{i-1} p_k(x_k - x_i)) + E(\sum_{k=i+1}^{n} p_k(x_i - x_k)), \quad (3)$$

where $D(.)$ and $E(.)$ are positively valued, non decreasing functions capturing the effects of disappointment and elation, respectively. Note from (3) that any outcome can give rise to a rich mixture of anticipated disappointment and elation feelings, unlike previous models. Thus, the overall utility of the gamble:

$$U(X) = \sum_{i=1}^{n} p_i\left(x_i - D(\sum_{k<i} p_k(x_k - x_i)) + E(\sum_{k>i} p_k(x_i - x_k))\right). \quad (4)$$

Let us now review some features of the disappointment model in (4).

Comparison to previous models

First, note that our model will be equivalent to Loomes and Sugden's (1986) if and only if *D*(.) in (1) is linear, and our functions *D*(.) and *E*(.) in (3) are linear and equal. In that case, both our and Loomes and Sugden's models reduce to the Expected Utility model. Outside of that particular case, our model in (4) has a distinct structure from that of Loomes and Sugden (1986). Next, our model will coincide with Bell's (1985) model for two-outcome lotteries if we take linear functions *D*(.) and *E*(.) in (3), even though our model is not built on the assumption of single prior expectation, as Bell's model is. The immediate implication is that our model explains all of the results derived by Bell (1985) for two-outcome lotteries. Moreover, unlike Bell's (1985, p. 8), our model applies to any number of outcomes.

Qualitatively speaking, our modeling of disappointment is more parallel to the approach followed in modeling regret (Bell, 1982; Loomes & Sugden, 1982) than previous disappointment models. Indeed, regret is based on comparing the outcome obtained under a particular act with the outcome that would be obtained under another act, for a given state of nature. In a similar fashion, disappointment in our model is based on comparing the outcome obtained under a particular state of nature with the outcome that would be obtained under another state of nature (as opposed to an average outcome), for a given act.

Explaining certainty effects

The model developed in (4) can explain Allais Paradox (Allais, 1953) and related certainty effects. As an example, let us consider the following choice problems proposed by Kahneman and Tversky (1979, p. 265), which is a variant of Allais' (1953) example:

Problem 1: Choose between *A* and *B*

A:	\$2,400 for sure	*B*:	0.33 chance of \$2,500 0.66 chance of \$2,400 0.01 chance of \$0

Problem 2: Choose between *C* and *D*

C:	0.34 chance of \$2,400 0.66 chance of \$0	*D*:	0.33 chance of \$2,500 0.67 chance of \$0

Kahneman and Tversky find that 82% of their respondents prefer *A* over *B*, while 83% prefer *D* to *C*. This pattern of choices is unallowable under Expected Utility theory. Assuming *D*(*x*) = 0.4·*x*, *E*(*x*) = 0 in (4), we obtain: *U*(*A*) = 2400, *U*(*B*) = 2391, thus $A \succ B$; and *U*(*C*) = 601, *U*(*D*) = 604, thus $D \succ C$, as observed experimentally.

Explaining common ratio effects

Consider the following pair of problems proposed by Kahneman and Tversky (1979, p. 266):

Problem 3: Choose between *A* and *B*

A: $3,000 for sure	*B*: 0.8 chance of $4,000 0.2 chance of $0

Problem 4: Choose between *C* and *D*

C: 0.25 chance of $3,000 0.75 chance of $0	*D*: 0.2 chance of $4,000 0.8 chance of $0

Here, Kahneman and Tversky find that a majority of subjects exhibit the following pattern of preferences: $A \succ B$ and $D \succ C$. This pattern is, again, incompatible with the EU model. Our disappointment model, however, can account for this so-called common ratio effect. Indeed, assuming *D*(*x*) = *d*·*x* and *E*(*x*) = *e*·*x* in (4), we get *U*(*A*) > *U*(*B*) and *U*(*D*) > *U*(*C*) whenever 0.3125 < *d* – *e* < 0.6451.

Further power in accounting for choice behavior may be obtained if we bring in non linear valuation of outcomes and non linear *D* and *E* functions, so as to exploit the full flexibility of the model in (4).

Stochastic dominance

Compliance with the Stochastic Dominance (SD) principle is a property that theorists have always been reluctant to sacrifice in building descriptive models of choice under risk. Indeed, the very development of Rank-Dependent Utility (RDU) theory (Quiggin, 1982) was motivated by a desire to avoid violations of stochastic dominance due to subjective probability weighting, as they are liable to occur in, e.g., Prospect theory (Kahneman & Tversky, 1979). The issue here is that if disappointment (or elation) were allowed to have too much leverage on utility in (4), this could lead to violations of SD. Indeed, if the influence of disappointment was too strong in the evaluation of a gamble, it may become possible to construct a pair of gambles such that one dominates the other, but also carries sufficiently more potential for disappointment, so that the dominating gamble ends up having a lower utility.

Putting proper bounds on the variations of the functions D and E may ensure that SD will always hold. In Appendix 1 we show that a sufficient condition for this is: $0 \le D'(x), E'(x) \le 1$ everywhere, and D, E concave. This closely resembles the condition that Loomes and Sugden (1986) obtained for their disappointment/elation function.

In the next section, we show that the special case of linear disappointment and elation in (4) leads to a probability weighting representation of our disappointment model.

From Disappointment to Probability Weighting

In this section, we assume linear functions for disappointment and elation, that is, $D(x) = d \cdot x$, $E(x) = e \cdot x$, with $d, e \ge 0$. The same assumption is made by Bell (1985) and Jia, Dyer and Butler (2001) throughout their analysis of disappointment. With linear functions, Expression (4) becomes:

$$U(X) = \sum_{i=1}^{n} p_i \left(x_i - d \cdot \sum_{k=1}^{i-1} p_k (x_k - x_i) + e \cdot \sum_{k=i+1}^{n} p_k (x_i - x_k) \right). \qquad (5)$$

This model form may seem daunting, but in Appendix 2 we show that it can be rearranged as:

$$U(X) = \sum_{i=1}^{n} p_i \left(1 + (d - e) \left(\sum_{k=1}^{i-1} p_k - \sum_{k=i+1}^{n} p_k \right) \right) x_i , \qquad (6)$$

which nicely separates the probabilities and outcomes in a multiplicative form.

Define the following transformation:

$$p_i \mapsto \pi_i = p_i \left(1 + (d - e) \cdot \left(\sum_{k=1}^{i-1} p_k - \sum_{k=i+1}^{n} p_k \right) \right). \qquad (7)$$

The mapping in (7) will constitute a valid weighting of probabilities if and only if $\pi_i \geq 0$ for i = 1, …, n, and $\sum_{i=1}^{n} \pi_i = 1$. One can verify that the π_is defined in (7) will always add up to 1, for any values of d and e. Furthermore, it is easy to show that $\pi_i \geq 0$ if and only if $-1 \leq d - e \leq 1$. Now, remember that d and e are ≥ 0 since the disappointment and elation functions are defined to be positively valued; additionally, to satisfy stochastic dominance we require $d, e \leq 1$. These together imply precisely $-1 \leq d - e \leq 1$. Therefore, if our linear disappointment model in (5) should satisfy SD, it can be rewritten as $U(X) = \sum_{i=1}^{n} \pi_i \cdot x_i$ where π_i, i = 1, …, n, are subjectively weighted probabilities as defined in (7).

We can see from (7) that the probability weight π_i of a given outcome depends on the *whole* set of probabilities and on the *rank* of the outcome in the distribution, very much like in RDU. It would be desirable to identify an underlying cumulative probability weighting function that yields the probability transformation of (7), that is, a function $w(.)$ such that: $p_i = w(\sum_{k=1}^{i} p_k) - w(\sum_{k=1}^{i-1} p_k)$ (Quiggin, 1982). Recall that this function operates on cumulative probabilities, that is, the argument p of $w(p)$ is not the probability of an outcome, but rather the probability of obtaining a given outcome or better. The extent of the transformation in (7) is directly controlled by the coefficient (d – e). For some cases, we are able to exhibit the relevant $w(.)$. When d – e = 1, that is, disappointment weighs more than elation, we find that the associated weighting function is: $w(p) = p^2$. This function is convex, therefore, as would be expected, it reflects pessimism or risk aversion in RDU. For $(d - e) = -1$, elation is stronger than disappointment, we arrive at: $w(p) = 2p - p^2$, which is concave, reflecting optimism or risk seeking in RDU. Of course, when $d - e = 0$, our model reduces to EU, in which case there is no probability distortion. These cases are summarized in Table 1. For other values of $(d - e)$ between –1 and +1, the implicit $w(.)$ function should lie somewhere in the envelope defined by $w(p) = 2p - p^2$ and $w(p) = p^2$. The form of $w(.)$ for the general case will be investigated in future work.

Table 1. Correspondence between linear disappointment and the RDU weighting function

Linear Coefficient of Disappointment/Elation		*Corresponding RDU Weighting Function*	*Implied Risk Attitude*
▪ $d - e = 0$	→ No disappointment / no elation, or disappointment and elation cancel out	$w(p) = p$	Risk-Neutral
▪ $d - e = 1$	→ All disappointment / no elation	$w(p) = p^2$	Risk-Averse
▪ $d - e = -1$	→ No disappointment / all elation	$w(p) = 2p - p^2$	Risk-Seeking

It is a quite remarkable result that a rank-dependent representation of preferences naturally emerges from a purely psychological hypothesis about disappointment.

Conclusion

Our approach to modeling disappointment avoids the delicate issue of choosing an adequate prior expectation altogether. We assume that each and every outcome may, in a sense, act as an expectation. This behavioral hypothesis seems quite defensible, as we argued. The resulting model shares the appealing descriptive features of previous models, offers more flexibility, and links naturally to RDU in some special cases. The cumulative probability transformation of RDU is often interpreted as reflecting pessimism/optimism, but it has also been criticized for lacking clear psychological origins. Our model suggests an underlying psychological factor for the rank-dependent transformation of probabilities: a desire to protect oneself against disappointment.

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Appendix 1

Theorem 1 *Let us assume that $D(\cdot)$ and $E(\cdot)$ are non-decreasing, concave functions. If $0 \leq D'(\cdot), E'(\cdot) \leq 1$ then first order stochastic dominance holds.*

Proof. Based on Machina (1982), the local utility function $V(x_i; X) = \frac{\partial U(X)}{\partial p_i}$ can be interpreted as the rate at which valuation changes as the probability associated with x_i changes. Machina (1982) shows that the preference functional $U(X)$ satisfies first order stochastic dominance if $V(x_i; X)$ is non-decreasing in x_i.

The local utility function of the Expression (4) is:

$$\begin{aligned}\frac{\partial U(X)}{\partial p_i} &= x_i - D\left(\sum_{k=1}^{i-1} p_k(x_k - x_i)\right) + E\left(\sum_{k=i+1}^{n} p_k(x_i - x_k)\right) \\ &\quad + \sum_{k=1}^{i-1} p_k(x_k - x_i)E'\left(\sum_{j=k+1}^{n} p_j(x_k - x_j)\right) - \sum_{k=i+1}^{n} p_k(x_i - x_k)D'\left(\sum_{j=1}^{k-1} p_j(x_j - x_k)\right).\end{aligned} \tag{8}$$

The derivative of (8) with respect to x_i is given by:

$$\begin{aligned}&1 + \sum_{k=1}^{i-1} p_k D'\left(\sum_{k=1}^{i-1} p_k(x_k - x_i)\right) + \sum_{k=i+1}^{n} p_k E'\left(\sum_{k=i+1}^{n} p_k(x_i - x_k)\right) \\ &- \sum_{k=1}^{i-1} p_k E'\left(\sum_{j=k+1}^{n} p_j(x_k - x_j)\right) - \sum_{k=1}^{i-1} p_i p_k(x_k - x_i)E''\left(\sum_{j=k+1}^{n} p_j(x_k - x_j)\right) \\ &- \sum_{k=i+1}^{n} p_k D'\left(\sum_{j=1}^{k-1} p_j(x_j - x_k)\right) - \sum_{k=i+1}^{n} p_i p_k(x_i - x_k)D''\left(\sum_{j=1}^{k-1} p_j(x_j - x_k)\right).\end{aligned} \tag{9}$$

For the Expression (9) to be ≥ 0, it must be that:

$$\begin{aligned}1 \geq &- \sum_{k=1}^{i-1} p_k D'\left(\sum_{k=1}^{i-1} p_k(x_k - x_i)\right) - \sum_{k=i+1}^{n} p_k E'\left(\sum_{k=i+1}^{n} p_k(x_i - x_k)\right) \\ &+ \sum_{k=1}^{i-1} p_k E'\left(\sum_{j=k+1}^{n} p_j(x_k - x_j)\right) + \sum_{k=1}^{i-1} p_i p_k(x_k - x_i)E''\left(\sum_{j=k+1}^{n} p_j(x_k - x_j)\right) \\ &+ \sum_{k=i+1}^{n} p_k D'\left(\sum_{j=1}^{k-1} p_j(x_j - x_k)\right) + \sum_{k=i+1}^{n} p_i p_k(x_i - x_k)D''\left(\sum_{j=1}^{k-1} p_j(x_j - x_k)\right).\end{aligned}$$

Since $D(\cdot)$ and $E(\cdot)$ are non-decreasing and concave functions, $\sum_{k=1}^{i-1} p_k E'\left(\sum_{j=k+1}^{n} p_j(x_k - x_j)\right)$ and $\sum_{k=i+1}^{n} p_k D'\left(\sum_{j=1}^{k-1} p_j(x_j - x_k)\right)$ are the only two positive terms in the right side of the inequality. Therefore, the assumption $0 \leq D'(\cdot), E'(\cdot) \leq 1$ is sufficient to make the above inequality hold.

Appendix 2

Assume that the disappointment and elation functions are linear, that is, $D(x) = dx$ and $E(x) = ex$, with $d, e \geq 0$. We show that the Expression (5), that is:

$$U(X) = \sum_{i=1}^{n} p_i \left(x_i - d \sum_{k=1}^{i-1} p_k (x_k - x_i) + e \sum_{k=i+1}^{n} p_k (x_i - x_k) \right),$$

can be equivalently written as the Expression (6).

Let us develop the summation in (5):

$$\begin{aligned}
&p_1 (x_1 + ep_2(x_1 - x_2) + ep_3(x_1 - x_3) + ... + ep_n (x_1 - x_n)) + \\
&p_2 (x_2 - dp_1(x_1 - x_2) + ep_3(x_2 - x_3) + ... + ep_n(x_2 - x_n)) + \\
&p_3 (x_3 - dp_1(x_1 - x_3) - dp_2(x_2 - x_3) + ep_4(x_3 - x_4) + ... + ep_n(x_3 - x_n)) + \\
&....... + \\
&p_n (x_n - dp_1(x_1 - x_n) - dp_2(x_2 - x_n) - dp_3(x_3 - x_n) - ... - dp_{n-1}(x_{n-1} - x_n)).
\end{aligned} \tag{10}$$

Let us rearrange all elements in (10) by gathering – on the same line – all terms containing $p_i x_i$ for $i = 1, ..., n$:

$$\begin{aligned}
&p_1x_1 - dp_1p_2x_1 - dp_1p_3x_1 - .... - dp_1p_nx_1 + ep_1p_2x_1 + ... + ep_1p_nx_1 + \\
&p_2x_2 + dp_1p_2x_2 - dp_2p_3x_2 - .... - dp_2p_nx_2 + ep_2p_3x_2 + ... + ep_2p_nx_2 - ep_1p_2x_2 + \\
&p_3x_3 + dp_1p_3x_3 + dp_2p_3x_3 - dp_3p_4x_3 - .. - dp_3p_nx_3 + ep_3p_4x_3 + .. + ep_3p_nx_3 - ep_1p_3x_3 - ep_2p_3x_3 + \\
&..... + \\
&p_nx_n + dp_1p_nx_n + dp_2p_nx_n + ... + dp_{n-1}p_nx_n - ep_1p_nx_n - .... - ep_{n-1}p_nx_n.
\end{aligned}$$

We can factorize the $p_i x_i$ for $i = 1, ..., n$ to obtain:

$$\begin{aligned}
&p_1x_1 (1 + d(-p_2 - .... - p_n) - e(-p_2 - .... - p_n)) + \\
&p_2x_2 (1 + d(p_1 - p_3 - ...... - p_n) - e(p_1 - p_3 - .... - p_n)) + \\
&p_3x_3 (1 + d(p_1 + p_2 - p_4 - .. - p_n) - e(p_1 + p_2 - p_4 - .. - p_n)) + \\
&..... + \\
&p_nx_n (1 + d(p_1 + p_2 + ... + p_{n-1}) - e(p_1 + p_2 + ... + p_{n-1})).
\end{aligned}$$

Rearranging further, we get $U(X)$ as displayed in (6).