On the Relationship between Safety and Decision Significance

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Abstract

Risk analysts are often concerned with identifying key safety drivers, i.e., the systems, structures and components (SSCs) that matter the most to safety. SSCs importance is assessed both in the design phase (i.e., before a system is built) and in the implementation phase (i.e., when the system has been built) using the same importance measures. However, in a design phase it would be necessary to appreciate whether the failure/success of a given SSC can cause the overall decision to change from accept to reject (decision significance). This work addresses the search for the conditions under which SSCs which are safety significant are also decision significant. To address this issue, the work proposes the notion of $\theta-$importance measure. We study in detail the relationships among risk importance measures to determine which properties guarantee that the ranking of SSCs does not change before and after the decision is made. An application to a Probabilistic Safety Assessment model developed at NASA illustrates the risk management implications of our work.

Keywords: Risk Analysis; Importance Measures; Probabilistic Risk Assessment.

1 Introduction

The safety categorization of systems, structures and components (SSCs) is a delicate decision making problem involving safety-related, societal and economic considerations. For a risk manager, getting to know, before the plant is actually built, whether an SSC is safety related might mean the ability to streamline the decision process and to achieve a more efficient resource allocation.

In the US nuclear industry, the safety categorization of SSCs follows the procedures of 10 CRF 5059 [1] and is risk-informed, i.e., it makes use of the insights gained from a probabilistic safety assessment (PSA) model[1]. PSA models allow decision makers to estimate the risk metric of interest in a variety of applications [3], from chemical [4] to nuclear [2] to space risk assessment problems [5].

Once a PSA model is developed, analysts gain key insights by using the model to assess the consequences of the failures or success of given SSCs (the failure/success scenarios, henceforth). Several importance measures are built on these scenarios [6].

These success/failure scenarios play a central role in all phases of the risk analysis of a complex system. However, their information contributes differently in different phases of the decision making process. In a post-implementation setting or given an existing plant, the regulator (industry) is interested in the safety significance of an SSC, i.e., in understanding how an SSC contributes to the plant safety. In a pre-implementation setting (when we still have to decide whether to build/licence the plant), the regulator/industry is exposed to a different problem. The decision maker needs to decide whether to accept

[1] In 10 CRF 5059 the Fussell-Vesely and the Risk Achievement Worth are an input to the categorization [17, 6].

(licence) or not a given design. In this phase, an analyst needs indications about whether the success/failure of a given SSC can cause the overall decision to change from accept to reject (or the converse). The decision maker is (or should be) interested also in the decision significance of an SSC.

This then opens the question of whether an SSC can be both decision significant and safety significant. In other words: can an SSC which is important in an accept/reject phase lose importance in favour of other SSCs in a post decision phase? Or can the converse happen? Answering this question has a clear risk-management implication. Quality assurance programs are costly and require a careful consideration by risk managers. A risk manager would have a better way to allocate resources, if she/he could focus on the decision significant SSCs and be reassured that these remain safety significant.

Herein, we investigate under which conditions we are reassured that an SSC which is decision significant (i.e., important in a pre-implementation phase) remains safety significant (i.e., important in the implementation phase[2]). The answer to this question requires a series of non-trivial steps. The first is to establish relationships among traditional risk importance measures and the Decision Worth. In fact, traditional importance measures are used to determine the safety significance of SSCs [19]. The Decision Worth is a value of information-based importance measure introduced in [7] that allows to obtain insights about the decision significance of an SSC. The second is the introduction of the class of θ-importance measures as the class of importance measures whose ranking is invariant in a pre- and post- decision setting. The third step is to address the conditions under which traditional importance measures become θ-importance measures. The existence of these conditions is all but guaranteed. In fact, we would expect that, in general, ranking of SSCs is not preserved if we consider decision or safety significance, given the different probabilistic meaning of the involved risk importance measures.

However, the findings of this part of our investigation reveal the following. If an SSC is structurally more important than another SSC in the sense of [8], then it is both more important in a pre-decision and in a post-decision phase. The caveat is that in a structural analysis all SSCs are set to have equally likely and independent failures, so that only their position in the system determines their importance. If SSCs are not set to have equally likely and independent failures, then the comparison needs to be carried out on a case by case basis. To do so, we introduce a new relationship that binds the Decision Worth, the Risk Achievement Worth and the Fussell-Vesely of an SSC in one equation. This relationship then provides a tool to compare decision significance to safety significance as meant in 10 CFR 5059 [1]. To perform numerical experiments, we utilize the importance measures data coming from the PSA model developed by the Idaho National Laboratories for U.S. National Aeronautics and Space Administration (NASA) in association with the risk assessment of the Constellation project. Implications for the relationship between safety and decision significance will be discussed in detail.

The remainder of the paper is organized as follows. Section 2 provides a brief literature review on importance measures and their relevant role into PSA problems. Section 3 proposes new relationships among the major traditional importance measures and uses such results to develop a one to one correspondence among all the existing importance measures. Section 4 proposes the new notion of θ-importance measures. Section 5 highlights the managerial insights gained by our results through an application to a PSA model developed at NASA. We then conclude with Section 6.

[2]This is relevant in case the preferred alternative is the licence/launch. The question is not posed if the no-licence/no-launch alternative is preferred, of course.

2 Importance Measures and Probabilistic Risk Assessment Modelling: A Synthetic Review

2.1 Risk Importance Measures

PSA models support decision makers in answering the three fundamental questions of Risk Analysis [9]: *1) What can go wrong? 2) How likely is it? and 3) What are the consequences?* The output of a PSA model is the probability of the consequence(s) of interest. For instance, in Level-1 nuclear PSA, the consequences of interest are core damage and large early release of radioactive material. In a space PSA, the consequences of interest are loss of crew or loss of mission [5].

PSA models commonly use a combination of event and fault trees. Their theoretical and mathematical foundations are laid in the works of [10, 11, 12]. Following the classic terminology we refer to the consequence as top event. The top event is the result of the occurrence of a sequence of events characterized by initiating and basic events. Let Ψ be the indicator variable of the *top event*, with $\Psi = 1$ denoting occurrence. The risk metric is then the probability of the top event, $P(\Psi = 1)$. Let φ be a vector of binary variables indicating basic event occurrence (or non occurrence). A basic event represents any event of interest, typically the failure of an SSC. Then, Ψ is a function of the basic events indicator variables, $\Psi = \Psi(\varphi)$. $\Psi(\varphi)$ is a Boolean function called structure function.

Risk importance measures have been developed to help analysts identifying the SSCs that contribute to risk the most. A key question they answer are the change in the overall level of risk that is provoked by the failure or the success of an SSC. That is, we are interested in the conditional risk metrics:

$$P(\Psi_i^+) = P(\Psi = 1|\varphi_i = 1) \text{ and } P(\Psi_i^-) = P(\Psi = 1|\varphi_i = 0), \tag{1}$$

where

- $P(\Psi_i^+)$ is the conditional risk metric given that SSC i has failed;
- $P(\Psi_i^-)$ is the conditional risk metric given that SSC i has not failed.

Intuitively, we expect $P(\Psi_i^+) \geq P(\Psi)$, that is, we expect that the conditional risk metric given the failure of an SSC is higher than the unconditional risk metric.[3] Symmetrically, we expect that the risk decreases when an SSC is set to perfectly working or when we can prevent its failure. Let $P(\Psi = 1) = P(\Psi)$, to simplify the expressions afterwards. In [6], the two quantities:

$$\Delta\Psi_i^+ = P(\Psi_i^+) - P(\Psi) \tag{2}$$

and

$$\Delta\Psi_i^- = P(\Psi) - P(\Psi_i^-) \tag{3}$$

are called risk increase and risk decrease, respectively. Indeed, we observe that risk importance measures capture this intuition since their orginal introduction in [14]. In fact, the Birnbaum importance measure is defined as:

$$B_i = P(\Psi_i^+) - P(\Psi_i^-). \tag{4}$$

[3] A thorough discussion about the calculation of conditional risk metrics is offered by [13].

Note that by adding and subtracting the value of the risk metric, we obtain:

$$B_i = \Delta\Psi_i^+ + \Delta\Psi_i^-. \tag{5}$$

The three most widely used importance measures for regulatory purposes are Risk Achievement Worth, Risk Reduction Worth and Fussell-Vesely [15, 16]. The Risk Achievement Worth (RAW) of basic event i is the ratio between the conditional value of the risk metric given that SSC i has failed, and the base-case value of the risk metric:

$$RAW_i = \frac{P(\Psi_i^+)}{P(\Psi)}. \tag{6}$$

As discussed in [6], RAW_i represents the importance of keeping the current level of reliability for the SSC, since it suggests the potential increase in risk level in case SSC i failed.

Risk Reduction Worth (RRW) is the symmetric concept and is the ratio of the baseline risk metric to the level of the risk metric if the SSC i were perfectly reliable. It represents the decreased risk when SSC i is perfectly reliable or when SSC i does not fail:

$$RRW_i = \frac{P(\Psi)}{P(\Psi_i^-)}. \tag{7}$$

The Fussell-Vesely importance [17] is defined as the conditional probability that a minimal cut set containing the SSC i has failed given that the system has failed. Let $\bigvee\limits_{m:\varphi_i \in C_m} Q_m$ denote the union of the cut sets containing φ_i. For simplicity, let us write $\mathcal{Q}(i) = \bigvee\limits_{m:\varphi_i \in C_m} Q_m$. We observe that $\Psi = \bigvee\limits_{m:\varphi_i \in C_m} Q_m \vee \bigvee\limits_{m:\varphi_i \notin C_m} Q_m$. Then, we have:

$$FV_i = P(\mathcal{Q}(i) = 1 | \Psi = 1).$$

It can be shown (see Proof in Appendix A) that:

$$FV_i \simeq \frac{P(\Psi) - P(\Psi_i^-)}{P(\Psi)} \tag{8}$$

Hence, the Fussell-Vesely importance represents the fractional contribution of the SSC i to the overall model risk when the event is assumed not to fail.

Further importance measures are the differential [18], the Barlow-Proshan [8], the criticality [19, 20]. A complete overview cannot be offered in this work, and we refer to the monograph of [21] for a complete overview.

2.2 The Decision Worth of an SSC

Risk importance measures consider the change in risk metric provoked by the failure (success) of an SSC. However, they do not provide indications about whether, given the change in risk, the overall decision (accept/reject) will change. Thus, in view of the relevant distinction between *decision* and *value sensitivity* [22], risk importance measures are value sensitivity measures. When the probabilistic risk assessment model is used in a design phase, however, it is critical to know whether the failure (success) of an SSC can make the preferred alternative change, for example, from licencing to not licencing. To gather such information one needs to resort to value of information [22].

The notion of value of information appears with the earliest developments of decision analysis and is formalized in [23] and [24]. Hilton [25] provides the relationship between information value and its determinants. Value of information has been studied as a sensitivity measure [22, 26, 27, 28, 29, 30], in the oil and gas industry [31], and in financial contexts [32]. Linkov et al. [33] and Bates et al. [34] apply value of information in nanomaterials and nanotechnology risk assessments — for a detailed review on the use of value of information in several application see [35]. In reliability analysis, value of information has been proposed as an importance measure in [36] and, most recently, in [7].

In several risk assessment problems, the course of action is selected based on *defined thresholds of acceptable risks* [3, p. 221]. Let p_0 denote the acceptable risk (the risk threshold). According to Borgonovo and Cillo [7], the decision problem boils down to a binary choice (license (authorize) vs not license (not authorize)), where the plant is licensed if the risk metric is smaller than the risk threshold, that is, if $P(\Psi) < p_0$. A recent work [7] proposes a value of information based importance measure, called the *Decision Worth* (DW) of an SSC, which can be written as a function of RAW_i and RRW_i:

$$DW_i = p_i \max\{1 - P(\Psi)RAW_i; 1 - p_0\} + (1 - p_i)\max\{1 - \frac{P(\Psi)}{RRW_i}; 1 - p_0\} - \max\{1 - P(\Psi); 1 - p_0\} \quad (9)$$

where p_i is the failure probability of SSC i. This relationship suggests that the Decision Worth of an SSC can be computed without additional runs of the PSA code, because standard PSA software computes the RAW_i and RRW_i of all basic events.

By replacing the definitions of RAW and RRW into Eq. (9), we can write the Decision Worth as:

$$DW_i = p_i \max\{1 - P(\Psi_i^+); 1 - p_0\} + (1 - p_i) \max\{1 - P(\Psi_i^-); 1 - p_0\} - \max\{1 - P(\Psi); 1 - p_0\}. \quad (10)$$

Eq. (10) brings together the conditional risk metric given failure/success of SSC i, the failure probability of SSC i, and the acceptable risk. Eq. (10) evidences that the failure/success scenarios play a central role in the Decision Worth. In this respect, observe that the terms $P(\Psi_i^+)$ and $P(\Psi_i^-)$ appear also in the Birnbaum importance simultaneously (Eq. (4)), while they appear separately in RAW and RRW of SSC i. Also, in risk importance measures the overall level of risk remains somewhat hidden in the background. However, Eq. (10) allows the acceptable risk p_0 to play a direct role in the importance of an SSC.

In Eq. (10) the Decision Worth is null when $p_0 \leq P(\Psi_i^-)$ or $p_0 \geq P(\Psi_i^+)$. In fact, note that $P(\Psi_i^-)$ is smaller than the risk metric $P(\Psi)$. If the acceptable risk is smaller than $P(\Psi_i^-)$, then it is also $p_0 < P(\Psi_i)$. In this case we reject the system and no improvement in the reliability of SSC i can revert the preferred alternative. Similarly, if the acceptable risk is greater than $P(\Psi_i^+)$, then we accept the system in the base case and even when SSC i fails. When positive, the Decision Worth depends on two terms. The first is the difference between the acceptable risk and the probability of system failure given that an SSC has (not) failed, the second is the failure probability of SSC i. The intuition is the following. SSC i has a high Decision Worth if it is such that: a) SSC i has a high failure probability and the conditional system failure probability given that SSC i fails is high compared to the acceptable risk; and b) SSC i has a low failure probability and the conditional probability of system failure given that SSC i is perfectly working is low compared to the acceptable risk.

3 New Relationships for Risk Importance Measures

After the review of importance measures, we carry out our first step towards establishing whether an SSC that is safety significant is also decision significant. This first step consists in obtaining new relationships between risk importance measures.

3.1 Risk Importance Measures: Probabilistic Relationships

In previous works, the links between risk importance measures have been studied mostly from an algebraic viewpoint, that is, exploiting the multilinearity of the risk metric [12, 37]. In this section, we obtain several relationships that rely on the probabilistic meaning of risk importance measures and, thus, are more general in nature. In particular, we show that once for an SSC we have the value of one risk importance measure (say of RAW or Fussell-Vesely or Birnbaum importance) we can obtain the values of all other risk importance measures. Our starting point is to consider the value of the risk metric and the two key scenarios (success/failure of SSC i). By the total probability theorem, we have:

$$P(\Psi) = p_i P(\Psi_i^+) + (1 - p_i) P(\Psi_i^-). \tag{11}$$

Then, dividing both sides by $P(\Psi)$ and normalizing, we get a first relationship (see Appendix A for the proof).

Proposition 1 *For any system, the RAW and RRW of SSC i, are related as:*

$$1 = p_i RAW_i + \frac{1 - p_i}{RRW_i}. \tag{12}$$

The above relationship binds the RAW_i, RRW_i and the failure probability of an SSC. In particular, it suggests that, given p_i and RRW_i, we determine RAW_i or, conversely, given p_i and RAW_i, we determine RRW_i.

Our second relationship binds the Fussell-Vesely and Birnbaum importance measures (see Appendix A for the proof).

Proposition 2 *The Fussell-Vesely and Birnbaum importance measures of SSC i are related as follows:*

$$FV_i = \frac{B_i}{P(\Psi)} p_i. \tag{13}$$

The next proposition shows a general relationship that binds the four most well known risk importance measures (see Appendix A for the proof).

Proposition 3 *Consider SSC i, with failure probability p_i, and let $P(\Psi)$ denote the value of the risk metric. Then, the Birnbaum importance, Fussell-Vesely importance, RAW_i and RRW_i are related by:*

$$p_i RAW_i + \frac{(1 - p_i)}{RRW_i} = \frac{p_i}{P(\Psi)} B_i + (1 - FV_i). \tag{14}$$

Using the above mentioned relationships, we can determine one-to-one relationships between all main risk importance measures, and the Decision Worth as well. Table 1 reports such relationships (see Appendix A for the proofs).

Table 1: Relationships between Risk Importance Measures. Proofs in Appendix A.

	RAW_i	RRW_i	FV_i	B_i
RAW_i	–	$(1-\frac{1-p_i}{RRW_i})\frac{1}{p_i}$	$1+\frac{(1-p_i)FV_i}{p_i}$	$1+\frac{B_i-\Delta\Psi_i^-}{P(\Psi)}$
RRW_i	$\frac{1-p_i}{1-RAW_i p_i}$	–	$\frac{1}{1-FV_i}$	$\frac{P(\Psi)}{P(\Psi_i^+)-B_i}$
FV_i	$\frac{p_i}{1-p_i}(RAW_i-1)$	$1-\frac{1}{RRW_i}$	–	$\frac{B_i}{P(\Psi)}p_i$
B_i	$\frac{P(\Psi)}{1-p_i}(RAW_i-1)$	$\frac{P(\Psi)}{p_i}(1-\frac{1}{RRW_i})$	$\frac{P(\Psi)}{p_i}FV_i$	–

Table 2: Importance measures for the Example 1, with $p_1 = 0.01$, $p_2 = 0.02$, and $p_3 = 0.03$. $P(\Psi) = 0.001088$.

SSCs	**RAW**	**RRW**	**Fussell-Vesely**	**Birnbaum**
1	45.4	1.81	0.45	0.05
2	36.49	3.63	0.73	0.04
3	27.39	5.44	0.82	0.03

In Table 1, some relationships deserve further consideration. For instance, consider the relationship between RAW_i and the Birnbaum importance (second row, fifth column): $RAW_i = 1 + \frac{B_i - \Delta\Psi_i^-}{P(\Psi)}$. We know that, for a coherent system, $RAW_i > 1$. Indeed, this relationship shows that RAW_i is the sum of unity plus an additional term, $\frac{B_i - \Delta\Psi_i^-}{P(\Psi)}$. This term is always positive in a coherent system, and it is what makes RAW_i greater than unity. Another relationship worth noting is FV_i as function of B_i: $FV_i = \frac{B_i}{P(\Psi)}p_i$, namely, the FV_i equals the product between the Birnbaum importance of SSC i and the ratio between its failure probability over the system failure probability. Finally, we note that that the Fussell-Vesely importance of SSC i equals the difference to unity of the inverse of RRW, that is, $1 - \frac{1}{RRW_i}$.

The next Example illustrates the relationships in Table 1 by means of a simple application.

Example 1 *Consider a two out of three system with SSC failure probabilities equal to* $p_1 = 0.01$*,* $p_2 = 0.02$*, and* $p_3 = 0.03$*, respectively. Under independent failures, the probability of system failure is:*

$$P(\Psi) = p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3. \tag{15}$$

Hence, the conditional probabilities of system failure given the failure of each SSC are:

$$P(\Psi_1^+) = p_2 + p_3 - p_2p_3,\ \ P(\Psi_2^+) = p_1 + p_3 - p_1p_3,\ \ P(\Psi_3^+) = p_1 + p_2 - p_1p_2. \tag{16}$$

By Eqs. (15 and 16), we obtain the Birnbaum importance of each SSC, and then, by using the relationships in Table 1, we easily compute all the remaining importance measures. As Table 2 shows, the RAW and the Birnbaum importance measure agree on the ranking of the SSCs: SSC 1 is ranked first, SSC 2 second, and SSC 3 third. Conversely, RRW and Fussell-Vesely importance rank SSC 3 first, SSC 2 second, and SSC 1 third.

The next section provides a relationship between risk importance measures and the Decision Worth.

3.2 Relationships between the Decision Worth and Individual Importance Measures

In this subsection, we move a second step towards answering our research question by establishing relationships between the Decision Worth of an SSC and its risk importance measures.

We start with the relationship between the Decision Worth and the Birnbaum importance. By substituting Eqs. (37 and 40) into the Eq. (9), we obtain:

$$DW_i = p_i \max\{1-P(\Psi)-(1-p_i)B_i; 1-p_0\}+(1-p_i)\max\{1-P(\Psi)+p_iB_i; 1-p_0\}-\max\{1-P(\Psi), 1-p_0\}. \tag{17}$$

The next proposition unfolds such an equation analysing how the Decision Worth depends on the acceptable risk.

Proposition 4 *Consider an* accept-reject *decision problem, with acceptable risk* p_0. *Then, we have:*

$$DW_i(p_0) = \begin{cases} 0 & if \quad p_0 \leq P(\Psi_i^-) \\ (1-p_i)(p_0 + p_iB_i - P(\Psi)) & if \quad P(\Psi_i^-) < p_0 < P(\Psi) \\ p_i(P(\Psi) + B_i(1-p_i) - p_0) & if \quad P(\Psi) \leq p_0 < P(\Psi_i^+) \\ 0 & if \quad p_0 \geq P(\Psi_i^+). \end{cases} \tag{18}$$

Equation (18) suggests that $DW_i(p_0) = 0$ if $p_0 \leq P(\Psi_i^-)$. Then, as the acceptable risk p_0 increases, $DW_i(p_0)$ increases linearly in p_0 and reaches the peak at $p_0 = P(\Psi)$. The Decision Worth then decreases as p_0 increases further, until $DW_i(p_0) = 0$ for all $p_0 \geq P(\Psi_i^+)$. The highest possible value for the Decision Worth of SSC i equals:

$$DW_i^{**} = p_i(1-p_i)B_i.$$

The above discussion shows that if a decision maker has available the Birnbaum importance of an SSC and has specifed the acceptable risk p_0, she can immediately retrieve its Decision Worth.

The same occurs if the decision maker has available the Fussell-Vesely importance of an SSC or its RAW. Precisely, using the relationships as in Table 1 into Eq. (9), we find:

$$DW_i = p_i \max\{1-\frac{P(\Psi)}{p_i}(p_i+FV_i(1-p_i)); 1-p_0\}+(1-p_i)\max\{1-P(\Psi)(1-FV_i); 1-p_0\}-\max\{1-P(\Psi); 1-p_0\}, \tag{19}$$

and

$$DW_i = p_i \max\{1-P(\Psi)RAW_i; 1-p_0\}+(1-p_i)\max\{1-\frac{P(\Psi)(1-RAW_ip_i)}{1-p_i}; 1-p_0\}-\max\{1-P(\Psi); 1-p_0\}. \tag{20}$$

Each of the above mentioned relationships allows us to express the Decision Worth as a function of a single importance measure.

Example 2 *[Example 1 continued] Consider again the two out of three system in Example 1.*

Figure 1 displays the Decision Worth of the three SSCs in Example 1 as a function of the acceptable risk. SSC 3 has the highest Decision Worth for values of $p_0 \leq 0.0099$. *At the peak, for* $p_0 = 0.001088$, *SSC 3 has the highest decision worth,* $DW_3 = 0.00086$. *Then for* $0.01 \leq p_0 < 0.03$ *SSC 2 becomes the most important, while for* $p_0 > 0.0301$ *SSC 1 becomes the most important. If we compare the ranking in Table 2, we note*

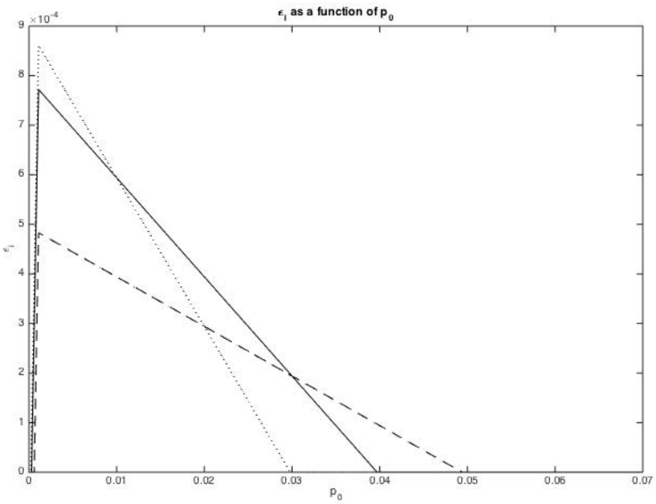

Figure 1: The Decision Worth as a function of the acceptable risk for the SSCs of Example 1 continued. The dashed line (- -) represents $DW_1(p_0)$, the solid line (-) $DW_2(p_0)$ and the dotted line (:) $DW_3(p_0)$.

that the Birnbaum importance ranks SSC 1 first. Thus, the Decision Worth and the Birnbaum importance do not lead to the same SSC ranking, in general.

3.3 Relationships among the Decision Worth, Risk Achievement Worth and Fussell-Vesely

In this section, we address the link between the Decision Worth of an SSC, and, jointly, its RAW and Fussell-Vesely importance. The motivation for emphasizing this relationship originates from the safety categorization process currently in use.

Around twenty years ago [19, 38], as part of the change in US Nuclear power regulation from a deterministic to a risk-informed regulation, a great amount of work has been carried out to come to a categorization of SSCs that would take into account both the deterministic aspects of plant configuration and the insights coming from risk-informed studies. The purpose of such effort has been that of reducing unnecessary burden on the licensees, while maintaining defense in depth principles.

The risk-informed categorization process overlaps to a deterministic classification. In the deterministic classifications, an SSC is deemed either *safety related* or *non-safety related*. Current NRC regulations define the plant equipment necessary to meet the deterministic regulatory basis as safety-related. This equipment is subject to NRC special treatment regulations. [1, p. 3] The SSC categorization in [1] foresees the additional categories of *safety significant* and *low safety significant*. As observed in [1, p. 3], these categories are not a substitute for the deterministic categories. Rather, the safety significant/low safety significant categories divide the safety-related and non-safety-related categories each into two subcategories. Consequently, the risk-informed categorization assigns an SSC into one of the four risk informed safety classes (RISC) in Table 3. The categorization in [1] identifies first the safety related SSCs, and would classify them as either safety significant (RISC-1) or low/no safety significant (RISC-3). The non-safety related SSCs would be categorized as being safety-significant (RISC-2) or low safety significant (RISC-4).

Insights from PSAs may reveal that *certain plant equipment important to the deterministic regulatory*

Table 3: Risk informed safety classification.

	Safety Related	Non-safety Related
Safety Significant	RISC-1	RISC-2
Low Safety Significant	RISC-3	RISC-4

basis is of little significance to safety. Conversely, certain plant equipment is important to safety but is not included in the deterministic regulatory basis [1, p. 2].

For the assignment to the safety significance categories, information coming from risk importance measures is used [1]. These results should form the initial input to the categorization process. For screening, an SSC with a Fussell-Vesely importance strictly smaller than 0.005 and a RAW strictly smaller than 2 is potentially of low safety significance, unless other engineering considerations would recommend otherwise. An SSC with $FV > 0.005$ and $RAW > 2$ is considered of high safety significance while a safety significant SSC would require either $FV > 0.005$ or $RAW > 2$.

Here, we note that by combining Eq. (19) and Eq. (20), we obtain the Decision Worth of SSC i as a function of its RAW and the Fussell-Vesely importance measure:

$$DW_i = p_i \max\{1-P(\Psi)RAW_i; 1-p_0\}+(1-p_i)\max\{1-P(\Psi)(1-FV_i); 1-p_0\}-\max\{1-P(\Psi); 1-p_0\}. \quad (21)$$

Equation (21) links in a single expression the Decision Worth, the Fussell-Vesely importance and the RAW of a given SSC.[4] Then, Equation (21) is an ideal tool to appreciate whether an SSC is safety significant and also decision significant. Moreover, Equation (21) allows us to consider various levels of the acceptable risk p_0, without having to perform additional evaluations of the PSA model.

4 The Family of θ-Importance Measures

As seen before, traditional risk importance measures do not always agree with the Decision Worth. In this section, we investigate whether there exist conditions that allow us to determine in advance whether an SSC which is safety significant is also decision significant.

To investigate this problem, we introduce a family of importance measures, called θ-importance measures. We say that an importance measure is a $\theta-$importance measure if it possesses the following property. Consider two SSCs, say SSC i versus SSC j. SSC i, if failed, increases risk more than when SSC j fails. At the same time, when made perfectly working, SSC i decreases risk more than SSC j.

Then, it would be natural to say that, SSC i is more important than SSC j. If that is the case, we would like the importance measure of interest to rank SSC i as more important than SSC j. We summarize this intuition in the next definition.

Definition 1 *Let* $\Delta^+_{i,j} = P(\Psi^+_i) - P(\Psi^+_j)$ *and* $\Delta^-_{j,i} = P(\Psi^-_j) - P(\Psi^-_i)$. *An importance measure is a* $\theta-$*importance measure if it is such that for any two SSCs (say SSC i and SSC j):*

$$\theta_i \geq \theta_j \Leftrightarrow \Delta^+_{i,j}, \Delta^-_{j,i} \geq 0. \quad (22)$$

[4] [39] propose an index (called SWIM) to combine information from FV_i and RAW_i in order to work with a single list of basic events. The index proposes an empirical strategy to combinate of the values of the two importance measures. Equation (21) can be seen as a generalization of this way of proceeding.

Definition 1 suggests that a $\theta-$importance measure ranks SSC i higher than SSC j if both the risk increase and risk decrease of SSC i are higher than the risk increase and decrease of SSC j.

We now show that these two conditions, in a coherent system, guarantee that SSC i has a higher Decision Worth than SSC j.

Proposition 5 *Consider an* accept-reject *decision problem in a coherent system with n SSCs. Then, for all SSCs i, j and for any value of the acceptable risk p_0:*

$$\theta_i \geq \theta_j \Leftrightarrow DW_i \geq DW_j. \tag{23}$$

The above proposition shows that a $\theta-$importance measure produces a ranking that is equivalent to the one produced by the Decision Worth. In other words, if the conditional probability of system failure given that SSC i has failed is greater (lower) than the same conditional probability given that SSC j has failed (is made perfectly working), then DW_i is greater than DW_j. The opposite holds as well. If the decision worth of SSC i is greater than the Decision Worth of SSC j, for any SSC and acceptable risk, then 1) the failure of SSC i increases the system failure probability more than SSC j, and 2) making SSC i perfectly working reduces risk more than making SSC j perfectly working.

The next question is then which of the traditional risk importance measures is a $\theta-$importance measure. The question is relevant since this would guarantee that such importance measure provides the same ranking in both the pre- and post-implementation phases. Traditional risk importance measures often provide a different ranking from the Decision Worth. For instance, we would not expect RRW_i and RAW_i to be $\theta-$importance measures, because they consider only one of the scenarios (failure or success) at a time.

However, there are some importance measures that appear as natural candidates. One of these is the Birnbaum importance measure which, in its definition, brings together both the SSC success and failure scenarios. In particular, we obtain the following results.

Proposition 6 *Consider a coherent system with n SSCs. Then, for all i, j:*

$$\theta_i \geq \theta_j \Rightarrow B_i \geq B_j. \tag{24}$$

Proposition 6 suggests that if SSC i dominates[5] SSC j, then it has for sure a higher Birnbaum importance.

Proposition 7 *Consider a coherent system with n SSCs. Then, if SSC i and SSC j have the same failure probability:*

$$B_i \geq B_j \Rightarrow \theta_i \geq \theta_j. \tag{25}$$

Proposition 7 shows that if the the Birnbaum importance of SSC i is greater then the Birnbaum importance of SSC j, and the two SSCs have equal failure probabilities, then SSC i dominates SSC j.

We then have immediately the following result.

Corollary 1 *Consider a coherent system with independent and identically distributed (i.i.d.) SSCs. Then, for all i, j:*

$$B_i \geq B_j \Leftrightarrow \theta_i \geq \theta_j. \tag{26}$$

[5] Here, by dominate, we mean that SSC i increases risk more than SSC j when failed and decreases risk more when made perfectly working.

Corollary 1 states that the Birnbaum importance measure is a $\theta-$importance measure for any coherent system with i.i.d. SSCs.

This result finds its natural interpretation in works in reliability analysis dealing with the notion of structural importance. This literature dates back to the seminal works of [14] and [8], and defines the *structural importance* of an SSC as the Birnbaum importance when all SSCs are i.i.d. The rationale is the determination of the importance of an SSC solely based on its position in the system, independently of its failure probability. This assumption is widely used in the literature of stochastic comparisons among systems and is, indeed, the most reasonable assumption to make when one is interested in comparing systems from a structural viewpoint. Then, Corollary 1 suggests that an SSC which is most informative in a design phase (is decision significant) is also the most structurally important SSC in the implementation phase.

Example 3 *Consider a coherent system with three SSCs with failure probabilities* $p_1 = 0.01$*,* $p_2 = 0.02$*,* $p_3 = 0.03$*. Assume the probability of system failure is:*

$$P(\Psi) = p_1 + p_2p_3 - p_1p_2p_3. \tag{27}$$

We compute for both SSCs 1 *and* 2 *the conditional probabilities of system failure given an SSC failing or being perfectly working. We obtain* $\Delta^+_{1,2} = 0.9603$ *and* $\Delta^-_{2,1} = 0.0094$*, that is to say, according to our definition 1,* $\theta_1 > \theta_2$*. Hence, we compute the Birnbaum importance measures, which result in* $B_1 = 0.9994$ *greater than* $B_2 = 0.0297$*. As expected by Proposition 6, if* $\theta_1 > \theta_2$*, then also the Birnbaum measure would rank SSC* 1 *higher than SSC* 2*.*

Example 4 *Consider a (coherent) bridge system with* 5 *SSCs which are i.i.d., with failure probabilities* $p_i = 0.01$ *for* $i = 1, ..5$*. Assume the probability of system failure is:*

$$P(\Psi) = p_1p_2 + p_3p_4 + p_1p_4p_5 + p_2p_3p_5 - p_1p_2p_3p_4 - p_1p_2p_3p_5 - p_1p_2p_4p_5 - p_1p_3p_4p_5 - p_2p_3p_4p_5 + 2p_1p_2p_3p_4p_5. \tag{28}$$

After computing for each SSC the conditional probabilities of system failure given an SSC failing or being perfectly working, we obtain the Birnbaum importance measures $B_i = 0.0101$ *for* $i = 1, 2, 3, 4$ *greater than* $B_5 = 1.9602e - 04$*. Given the same Birnbaum importance values for* $i = 1, 2, 3, 4$*, let us consider SSC* 1 *and SSC* 5*. As expected by Proposition 7, we obtain* $\Delta^+_{1,5} = 0.0098$ *and* $\Delta^-_{5,1} = 9.9000e - 05$*, that is to say, according to our definition 1,* $\theta_1 > \theta_5$*. By Proposition 5,* $\theta_1 > \theta_5$ *guarantees that the Decision Worth ranks SSC* 1 *higher than SSC* 5*:* $DW_1 = 9.9951e - 05$ *is greater than* $DW_5 = 1.9406e - 06$*.*[6]

Let us now come to another importance measure, the Fussell-Vesely importance. The next proposition shows that under the same conditions of the Birnbaum importance, also Fussell-Vesely importance is a θ-importance measure.

Proposition 8 *Consider a coherent system with* n *SSCs, which are i.i.d. Then, for all* i, j*:*

$$\theta_i \geq \theta_j \Leftrightarrow FV_i \geq FV_j. \tag{29}$$

Thus, if SSC i is ranked higher than SSC j by Fussell-Vesely, then the Decision Worth of such SSC will also be higher, for any acceptable risk in an accept-reject decision problem. In other words, in a coherent

[6] By normalizing, we would obtain $DW_1 = 0.4949$ and $DW_5 = 0.0096$.

system with i.i.d. SSCs, the Fussell-Vesely or Birnbaum importance measures preserve the ranking obtained in the pre-implementation phase.

Regarding other importance measures, we easily see that RAW_i and RRW_i cannot be $\theta-$importance measures. The reason is that they consider failure and success separately.

5 Application: A Space PSA

In this section, we make use of a real PSA model to perform numerical experiments. The model is the PSA code developed to support decision making in the design phase of the next generation of lunar space missions as part of the Constellation project [41]. The model relies on a phased based event tree and a fault tree logic structure to model the various states of a lunar mission. The model was developed following the technical procedures in NASA's Probabilistic Risk Assessment Procedures Guide [40]. Two end states are considered: loss of crew (LOC) and loss of mission (LOM). LOC is a safety measure, while LOM is a performance one. LOC and LOM are not mutually exclusive. The mission comprises eight phases, after the launch: Enter Low Earth Orbit, Depart Earth, Lunar Orbit Injection, Vehicle in Lunar Orbit, Lunar Mission, Recrew the Orbiting Vehicle, Return to Earth, and Earth Landing. Each phase can be successful, hence leading to the next phase, or a failure, hence leading to a loss, which can be of crew or mission depending on the scenario considered.

NASA vehicles are composed of seven systems. Each system has to perform some major functions. The systems consist of Propulsion, Avionics, Power Supply, Thermal Control, Environmental Control and Life Support System, the Launch Abort System, and Pyrotechnic Devices. Each system in each phase is modelled via fault trees. The PSA model results in 150 fault trees and 872 basic events. After a truncation of 10^{-15}, 393 basic events survive. The fault trees suggest what has to fail in order to cause either LOC or LOM at each phase. For further details, we refer to [41, 42]. Here, we are interested in understanding whether a basic event which is significant in a pre-decision phase remains safety significant, i.e., remains relevant once the alternative "launch"is implemented. Indeed, a very powerful application of probabilistic risk assessment (PRA) consists of supporting the mission design, development, and planning. Even though some of this information is based on pre-design data and assumptions, it is also true that during the development phases some other design solutions and requirements can be identified to manage risk.

Let us start with a structural analysis. Here, we would consider that all basic events are equally likely. As mentioned, this analysis would suggest us the basic events that are most important due to their position in the PSA model, disregarding the likelihood of their failure. With these distributional assumptions, we know that if the θ-conditions are satisfied, namely, $\Delta^{+}_{i,j} = \Delta^{-}_{j,i} \geq 0$, then we expect that a higher Birnbaum importance implies a higher Decision Worth. The results of the calculations show that this is indeed the case. To illustrate, Table 4 shows the ten basic events ranked top by the Birnbaum importance. These are the top ranked by the Decision Worth as well, confirming Proposition 6 and Corollary 7. For example, consider basic events 67 and 392. The θ-conditions for these basic events are satisfied. Indeed, $\Delta^{+}_{67,392} = 3.3e - 06$ and $\Delta^{-}_{392,67} = 3.3e - 13$, which are both greater than 0. Hence, we expect that the Birnbaum importance and the Decision Worth rank basic event 67 higher than basic event 392. Indeed, we register that basic event 67 ranks first and basic event 392 ranks second.

Despite the fact that these results hold based on our theoretical findings, the managerial implications of these findings are important. For example, our results suggest that basic event 67 [7] is not only relevant in

[7] The explicit function of the basic events is not displayed for privacy reasons.

Table 4: The top ten ranked basic events by the Decision Worth and the Birnbaum importance, when the basic events are equally likely.

Rank	Basic Event
1	67
2	392
3	393
4	91
5	83
6	86
7	10
8	14
9	15
10	17

Table 5: The top ten ranked basic events according to the Decision Worth (with $p_0 = P(\Psi)$) and Birnbaum importance when the true failure probabilities are assigned to the basic events.

Rank	Birnbaum Importance	Decision Worth
1	371	371
2	49	48
3	67	52
4	68	49
5	126	67
6	376	68
7	377	126
8	69	376
9	91	377
10	83	69

a post-implementation phase, as suggested by the Birnbaum importance, but also in a pre-decision phase. Thus, this is an important safety driver before and after launch. Hence, resources should be spent to make the element corresponding to this basic event as reliable as possible.

These considerations, however, are valid within a structural framework only, that is, in the hypothetical case in which all basic events are equally likely. When basic events are assigned different failure probabilities, we are not reassured that the basic events with a high Birnbaum importance are also decision significant. Indeed, if we perform a similar analysis with the true failure probabilities assigned for this space PSA, the basic event ranking induced by the Birnbaum importance and the Decision Worth differ. Table 5 shows the results.

Except for basic event 371, which is ranked first both by the Birnbaum importance and the Decision Worth, the other basic events are ranked differently. For example, while the Birnbaum importance ranks basic event 67 as third, the Decision Worth ranks it fifth. Moreover, two of the top ten ranked basic events under the Decision Worth are ranked outside the top ten Birnbaum most important basic events. These are basic event 48 (which ranks 2nd according to the Decision Worth and 53rd by the Birnbaum importance) and basic event 52 (which ranks 3rd according to the Decision Worth and 77th by the Birnbaum importance). Similarly, two of the top ten ranked basic events by the Birnbaum importance fall outside the top ten ranked by the Decision Worth: these are basic event 91 (9th with Birnbaum importance), which ranks 11th and basic event 83 (10th with Birnbaum importance), which ranks 16th. Additionally, if we have a closer look at the top ten events ranked by the Birnbaum importance in Table 4 and 5, we note that only three out of

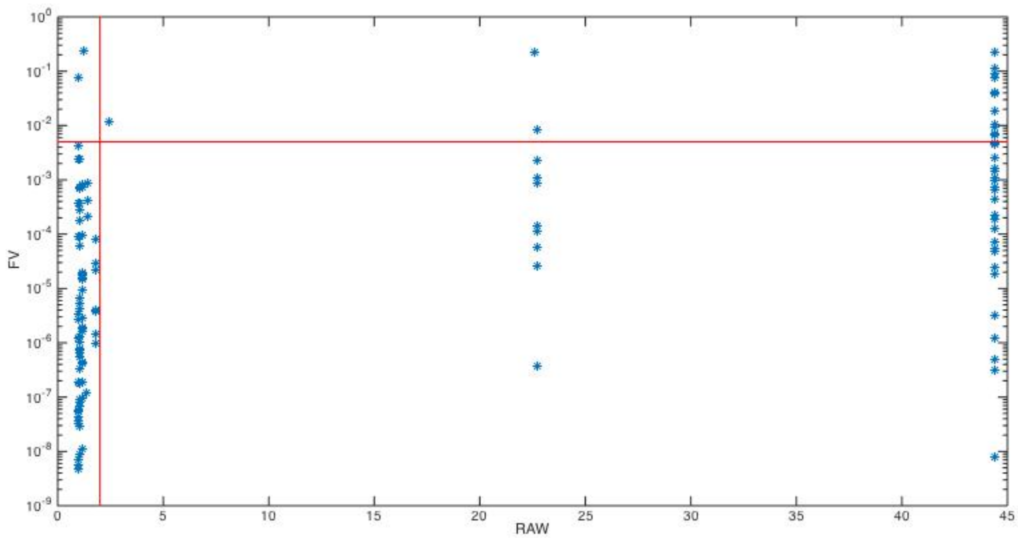

Figure 2: Safety significance categorization plane for the basic events of our case study. Each $*$ represent one of the 393 events. The horizontal axis reports the basic event RAWs, while the vertical axis reports the Fussell-Vesely values. The RAW-FV plane is divided into four quadrants by the two thresholds on RAW and Fussell-Vesely — $RAW = 2$ (horizontal axis) and $FV = 0.005$ (vertical axis), respectively.

ten events appear in both top ten lists. These are basic events 67, 91 and 83. Note that basic event 67 is the only event in common between the top ten events ranked by the Decision Worth in Tables 4 and 5. These results have a twofold interpretation. On the one hand, the basic events that are important in a structural analysis and remain such when probabilities are assigned should deserve managerial attention. On the other hand, these are only a few, confirming that results of a structural analysis have a preliminary validity, but indications might drastically change once non-identical failure distributions are assigned.

To understand which basic events are safety significant, we form the safety significance categorization plane as foreseen by 10 CFR 5069 [1].

Figure 2 reports the safety categorization plane resulting from the values of the Fussell-Vesely and RAW importance measures of the 393 basic events of the PSA model under concern. The red lines indicate the threshold values $RAW = 2$ (horizontal axis) and $FV = 0.005$ (vertical axis), dividing the plane in four quadrants. Quadrant I includes the eighteen basic events that have both $RAW > 2$ and $FV > 0.005$. These basic events are highly safety significant. Quadrant II includes basic events that have $RAW > 2$ and $FV < 0.005$: there are fourty-nine events with these features. Quadrant III includes two basic events that have $FV > 0.005$ and $RAW < 2$. The set of safety significant basic events comprises all the events in quadrants I, II, and III, counting for sixty-nine events out of a total of 393 basic events. The remaining 324 basic events are non-safety significant, as they fall in in quadrant IV.

To answer the question of which basic events are decision significant, we use the relationship between the Decision Worth, Fussell-Vesely and RAW given in Eq. (21). In this numerical experiment, we set $p_0 = P(\Psi)$. Here, out of the 393 basic events, fifty basic events are decision significant, i.e., have a non-null Decision Worth. Then, we have sixty-nine safety significant basic events and fifty decision significant ones. Among these fifty decision significant events, we find all the highly safety significant events, i.e., all the eighteen events in the quadrant I in Figure 2. We also find eighteen of the events in the quadrants II and III. However, fourteen events have a strictly positive Decision Worth (that is, are decision significant), but lie in quadrant IV. This finding has an interesting implication: there are non-safety significant basic events that are decision significant. Thus, a basic event (SSC) important in a pre-decision phase might become non-important in the implementation phase. Conversely, among the sixty-nine safety-significant basic events, thirty-three basic events have a null Decision Worth. Thus, a basic event (an SSC) that is not important in a pre-decision phase can become important in a post-decision phase.

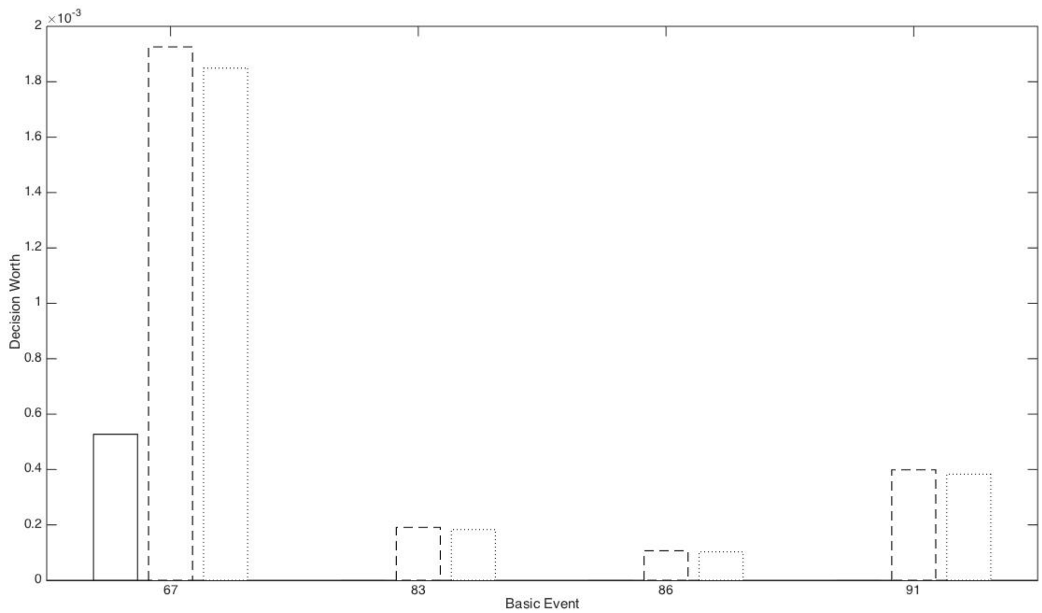

Figure 3: Decision Worth for basic events 67, 83, 86, and 91, for different levels of the threshold probability. The solid bar (-) for $p_0 = 0.02 < P(\Psi)$, the dashed bar (- -) for $p_0 = P(\Psi)$, and the dotted bar (:) for $p_0 = 0.1 > P(\Psi)$.

Now, let us consider the case in which the threshold probability is greater than the system failure probability: For example, let us set $p_0 = 0.1 > P(\Psi)$. In this case, the preferred alternative is to authorize the mission. With this threshold, the number of decision-significant events drops to fourty-four. We still register sixty-nine safety-significant basic events — in fact the number of safety-significant basic events is independent of the threshold. Among the fourty-four decision significant events, fifteen are highly safety significant, and seventeen have either RAW>2 or Fussell-Vesely >0.005. The remaining twelve basic events, despite having a strictly positive Decision Worth, are not safety significant. Conversely, among the sixty-nine safety-significant basic events, thirty-seven basic events have a null Decision Worth and are, therefore, not decision-significant. If we consider the case in which the threshold probability is lower than the system failure probability, the results differ again. For example, let us consider the case $p_0 = 0.02 < P(\Psi)$. In this case, the preferred alternative is to not authorize the mission. The number of decision significant basic events drops dramatically to six. Of these events, all of them are highly safety significant, with the exception of basic event 52, which only has $FV > 0.005$ (2.36e-01).

These results show that decision significance does not necessarily coincide with safety significance. Moreover, the bigger the difference between the safety threshold and the system failure probability, the more we expect the ranking induced by the Decision Worth and by the combination of RAW and Fussell-Vesely importance to diverge. The explanation is that the Decision Worth of any basic event (SSC) has a peak at $p_0 = P(\Psi)$. Thus, as we move away from the peak value, the decision worth of all basic events decreases. For example, Figure 3 shows the Decision Worth of basic events 67, 83, 86 and 91 for $p_0 = 0.02 < P(\Psi)$ (solid bar), $p_0 = P(\Psi)$ (dashed bar), and $p_0 = 0.1 > P(\Psi)$ (dotted bar). Consider basic event 67. The three bars denote that the Decision Worth of basic event 67 is strictly greater than zero for all three values of the threshold. For basic event 83 the Decision Worth is null when $p_0 = 0.02 < P(\Psi)$, then it reaches its peak at $p_0 = P(\Psi)$, and decreases at $p_0 = 0.1 > P(\Psi)$. The same occurs for basic events 86 and 91. Then, at the peak all these basic events are decision significant, while far from the peak only basic event 67 remains decision significant. The reason is that if $P(\Psi)$ differs significantly from the threshold, then we have a stable decision (either positive or negative) and it becomes unlikely that the occurrence of a single basic event can cause the preferred alternative to change. In an extreme case, we may have only safety-significant basic events, with no decision-significant ones.

6 Conclusions

What are the conditions such that an SSC which is important in a post-decision setting keeps the relevance in a pre-decision setting? And viceversa? We note that, intuitively, given the intrinsic differences between the pre- and post-decision phases and the difference between the definitions of importance measures, the existence of such conditions is not necessarily to be expected. In a sense, it would be more intuitive that no such conditions exist.

Herein, we have proposed the notion of θ-importance as the notion of a class of importance measures that guarantee an equivalent ranking of an SSC in a pre- and post-decision setting. These $\theta-$conditions require that SSC i has a greater effect on the system success and failure than SSC j. We have proven that if the $\theta-$conditions are satisfied, then if an SSC i is decision significant it is also the most important one according to the Birnbaum importance. Moreover, if the system has i.i.d. SSCs, we are also reassured that $B_i \geq B_j$ implies that the $\theta-$condition is verified. This finding signals that in a structural importance analysis, an SSC which is structurally important, remains such both in a pre- and post-decision phase.

However, a structural analysis considers solely the position of an SSC in the system and not in the associated failure probability. We have then examined how to answer the question of whether an SSC which is decision significant is also safety significant in a realistic application. To do so, we have considered the safety-categorization procedure of SSCs as foreseen in 10 CFR 5069 [1]. In particular, the analysis can be streamlined by a new relationship between the Decision Worth, the Fussell-Vesely and the RAW of an SSC. This relationship brings together the two importance measures used to create the safety-significance plane and the importance measure to be used in a pre-decision setting. As a test, we have applied the relationship to data coming from the PSA code developed for the assessment of the Constellation space missions. It has emerged that the highly-safety significant basic events are also decision significant. However, not all the basic events of intermediate safety-significance are also decision significant and, surprisingly, some non-safety significant basic events are decision significant. Thus, in realistic applications, the investigation of whether an SSC can be both safety and decision significant needs to be carried out on a case by case basis. For that, however, one can profit of the relationships established here which allow one to obtain the decision worth of basic events from knowledge of the other importance measures.

7 Appendix A: Proofs

Proof. Equation 8. Using Bayes rule, and since $P(\Psi = 1|\mathcal{Q}(i) = 1) = 1$, we can rewrite it as:

$$FV_i = \frac{P(\mathcal{Q}(i))}{P(\Psi)} = \frac{P(\bigvee_{m:\varphi_i \in C_m} Q_m)}{P(\bigvee_{m=1}^{M} Q_m)}. \tag{30}$$

Now, if $\varphi_i = 0$, then we have system failure only if a minimal cut sets (MCS) not contaning φ_i is realized. The collection of all MCS is partitioned into two subsets, the subset containing φ_i and the subset not containing φ_i. Hence, we have:

$$P(\Psi) = P(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m) + P(\bigvee_{m:\varphi_i \in C_m}^{M} Q_m) - 2P(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m \cap \bigvee_{m:\varphi_i \in C_m}^{M} Q_m)$$

from which follows:

$$P(\bigvee_{m:\varphi_i \in C_m}^{M} Q_m) = P(\Psi) - P(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m) + 2P(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m \cap \bigvee_{m:\varphi_i \in C_m}^{M} Q_m).$$

Substituting into Eq. (30), we can write:

$$FV_i = \frac{P(\Psi) - P(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m) + 2P(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m \cap \bigvee_{m:\varphi_i \in C_m}^{M} Q_m)}{P(\Psi)}. \tag{31}$$

Now, one notes that, especially for reliable systems, $2\Pr(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m \cap \bigvee_{m:\varphi_i \in C_m}^{M} Q_m) << \Pr(\Psi)$ (rare event approximation) so that we obtain:

$$FV_i \simeq \frac{P(\Psi) - P(\bigvee_{m:\varphi_i \notin C_m}^{M} Q_m)}{P(\Psi)},$$

which is equivalent to Eq. (8). ■

Proof. Proposition 1. By applying the definition of RRW and RAW, and dividing Eq. (11) by $P(\Psi)$, we get:

$$1 = p_i RAW_i + \frac{1 - p_i}{RRW_i},$$

which is what we wanted to prove. ■

Proof. Proposition 2. We can rewrite Eq. (8) as:

$$FV_i = \frac{P(\Psi) - P(\Psi_i^-)}{P(\Psi)} = \frac{P(\Psi_i^+)p_i - P(\Psi_i^-)p_i}{P(\Psi)}, \tag{32}$$

which is Eq. (13). ■

Proof. Proposition 3. Dividing Eq. (11) by $P(\Psi)$, we get $1 = \frac{p_i}{P(\Psi)} B_i + (1 - FV_i)$. Because we have that both $p_i RAW_i + \dfrac{(1-p_i)}{RRW_i}$ and $\frac{p_i}{P(\Psi)} B_i + (1 - FV_i)$ are equal to unity, we can set them equal. This ends the proof. ■

Proofs of relationships in Table 1 *RAW as a function of other importance measures. Consider SSC i, with failure probability p_i, and given values of $P(\Psi)$, RRW_i, FV_i, and BI_i. Then:*
1) RAW as a function of RRW. Rewriting Eq. (12), we have:

$$RAW_i = \frac{1}{p_i}\left(1 - \frac{1-p_i}{RRW_i}\right). \tag{33}$$

2) RAW as a function of the Fussell-Vesely importance measure. By rearranging Eq. (41) we obtain:

$$RAW_i = 1 + \frac{FV_i(1-p_i)}{p_i}. \tag{34}$$

3) RAW as a function of the Birnbaum importance. By using Eq. (4) we can rewrite Eq. (6) as:

$$RAW_i = \frac{B_i + P(\Psi_i^-)}{P(\Psi)}. \tag{35}$$

From Eq. (3) we get:

$$P(\Psi_i^-) = P(\Psi) - \Delta\Psi_i^-. \tag{36}$$

Replacing Eq. (36) into Eq. (35), we then obtain:

$$RAW_i = 1 + \frac{B_i - \Delta\Psi_i^-}{P(\Psi)}. \tag{37}$$

RRW as a function of other importance measures. Consider SSC i, with failure probability p_i, and given values of $P(\Psi)$, RAW_i, FV_i, and B_i. Then, the following relationships must hold:
1) RRW as a function of RAW. By using Eq. (12), we obtain:

$$RRW_i = \frac{1-p_i}{1 - RAW_i p_i}. \tag{38}$$

2) RRW as a function of the Fussell-Vesely importance. By rearranging Eq. (42) we obtain:

$$RRW_i = \frac{1}{1 - FV_i}. \tag{39}$$

3) RRW as a function of the Birnbaum importance. By using Eq. (4) we can rewrite Eq. (7) as:

$$RRW_i = \frac{P(\Psi)}{P(\Psi_i^+) - B_i}. \tag{40}$$

The Fussell-Vesely importance measure as a function of other importance measures. Consider SSC i, with failure probability p_i, and given values of $P(\Psi)$, RAW_i, RRW_i, BI_i and CR_i. Then, the following relationships must hold:
1) The Fussell-Vesely importance measure as a function of RAW. By applying the Bayes rule, we can rewrite Eq. (8) as:

$$FV_i = 1 - \frac{P(\Psi = 1|\varphi_i = 0)}{P(\Psi)} = 1 - \frac{P(\varphi_i = 0|\Psi = 1)}{1-p_i} = 1 - \frac{1 - P(\varphi_i = 1|\Psi = 1)}{1-p_i}$$

which can be rewritten as:

$$FV_i = \frac{-p_i + P(\varphi_i = 1|\Psi = 1)}{1 - p_i} = \frac{-p_i}{1 - p_i} + \frac{P(\Psi = 1|\varphi_i = 1)p_i}{(1 - p_i)P(\Psi)}$$

which, by using Eq. (6), is equal to:

$$FV_i = \frac{p_i}{1 - p_i}(RAW_i - 1). \tag{41}$$

2) The Fussell-Vesely importance measure as a function of RRW. By replacing Eq. (7) into Eq. (8), we obtain:

$$FV_i = 1 - \frac{P(\Psi_i^-)}{P(\Psi)} = 1 - \frac{1}{RRW_i}. \tag{42}$$

3) The Fussell-Vesely importance measure as a function of Birnbaum importance, see Proposition (2).

The Birnbaum importance measure as a function of other importance measures. Consider SSC i, with failure probability p_i, and given values of $P(\Psi)$, RAW_i, RRW_i, FV_i and CR_i. Then, the following relationships must hold:

1) The Birnbaum importance measure as a function of RAW. Eq. (4) can be rewritten as:

$$B_i = P(\Psi)(RAW_i - \frac{1}{RRW_i}). \tag{43}$$

Substituting Eq. (38) into Eq. (43), we obtain:

$$B_i = P(\Psi)(RAW_i - \frac{1 - RAW_i p_i}{1 - p_i}), \tag{44}$$

which gives:

$$B_i = \frac{P(\Psi)}{1 - p_i}(RAW_i - 1). \tag{45}$$

2) The Birnbaum importance measure as a function of RRW. By substituting Eq. (33) in Eq. (43), we obtain:

$$B_i = P(\Psi)[(1 - \frac{1 - p_i}{RRW_i})\frac{1}{p_i} - \frac{1}{RRW_i}], \tag{46}$$

which can be rearranged as:

$$B_i = \frac{P(\Psi)}{p_i}(1 - \frac{1}{RRW_i}). \tag{47}$$

3) The Birnbaum importance measure as a function of Fussell-Vesely importance. By rearranging Eq. (13), we obtain:

$$B_i = \frac{P(\Psi)}{p_i}FV_i. \tag{48}$$

The Decision Worth as a function of other importance measures. Consider SSC i, with failure probability p_i, and given values of $P(\Psi)$, RAW_i, RRW_i, FV_i , and B_i. Then, the following relationships must hold:

1) Decision Worth as a function of RAW. See Eq. (20).

2) Decision Worth as a function of RRW. By substituting Eq. (33) into Eq. (9), we obtain:

$$DW_i = p_i \max\{1 - \frac{P(\Psi)}{p_i}(1 - \frac{1 - p_i}{RRW_i}); 1 - p_0\} + (1 - p_i)\max\{1 - \frac{P(\Psi)}{RRW_i}; 1 - p_0\} - \max\{1 - P(\Psi); 1 - p_0\}. \tag{49}$$

3) Decision Worth as a function of Fussell-Vesely importance. See Eq. (19).

4) Decision Worth as a function of Birnbaum importance. See Eq. (17).

Proof. Proposition 4. If $P(\Psi^-) < P(\Psi) < P(\Psi^+)$, we have four possibles cases:

1) If $p_0 \leq P(\Psi_i^-)$ then Eq. (17) can be rewritten as:

$$DW_i = p_i(1 - p_0) + (1 - p_i)(1 - p_0) - 1 + p_0 \tag{50}$$

from which the first line of Eq. (18) follows.

2) If $P(\Psi_i^-) < p_0 < P(\Psi)$ then Eq. (17) can be rewritten as:

$$DW_i = p_i(1 - p_0) + (1 - p_i)(1 - P(\Psi) + p_i B_i) - 1 + p_0 \tag{51}$$

from which the second line of Eq. (18) follows.

3) If $P(\Psi) \leq p_0 < P(\Psi_i^+)$ then Eq. (17) can be rewritten as:

$$DW_i = p_i(1 - p_0) + (1 - p_i)(1 - P(\Psi) + p_i B_i) - 1 + P(\Psi) \tag{52}$$

from which the third line of Eq. (18) follows.

4) If $p_0 \geq P(\Psi_i^+)$ then Eq. (17) can be rewritten as:

$$DW_i = p_i(1 - P(\Psi) - (1 - p_i)B_i) + (1 - p_i)(1 - P(\Psi) + p_i B_i) - 1 + P(\Psi) \tag{53}$$

from which the fourth line of Eq. (18) follows. This ends the proof. ■

Proof. Proposition 5. Sufficiency. Let us obtain first a useful relationship. By Definition 1, $\theta_i \geq \theta_j$ implies $\Delta_{i,j}^+, \Delta_{j,i}^- \geq 0$. Because the system is coherent we have the following ordering of the conditional probabilities:

$$P(\Psi_i^-) \leq P(\Psi_j^-) < P(\Psi) < P(\Psi_j^+) \leq P(\Psi_i^+). \tag{54}$$

By the total probability theorem we obtain:

$$p_i P(\Psi_i^+) + (1 - p_i) P(\Psi_i^-) = p_j P(\Psi_j^+) + (1 - p_j) P(\Psi_j^-),$$

whence:

$$p_i(P(\Psi_i^+) - P(\Psi_i^-)) + p_j(P(\Psi_j^-) - P(\Psi_j^+)) = P(\Psi_j^-) - P(\Psi_i^-). \tag{55}$$

Let now $\Delta DW_{i,j} = DW_i - DW_j$ denote the difference between DW_i and DW_j. From Eq. (10) we obtain the following six possible cases:

1) If $p_0 < P(\Psi_i^-)$ then $DW_i = 0$ and $DW_j = 0$, so that $\Delta DW_{i,j} = 0$.

2) If $P(\Psi_i^-) \leq p_0 < P(\Psi_j^-)$ then $\Delta DW_{i,j} = (1 - p_i)(p_0 - P(\Psi_i^-)) \geq 0$ because $1 - p_i \geq 0$ and $p_0 > P(\Psi_i^-)$ by the ordering in Eq. (54).

3) If $P(\Psi_j^-) \leq p_0 < P(\Psi)$ then $\Delta DW_{i,j} = (P(\Psi_j^-) - P(\Psi_i^-)) + p_j(p_0 - P(\Psi_j^-)) - p_i(p_0 - P(\Psi_i^-))$, which can be rewritten as

$$\Delta DW_{i,j} = (P(\Psi_j^-) - P(\Psi_i^-)) + p_j(p_0 - P(\Psi_j^-)) - p_i(P(\Psi_j^-) - P(\Psi_i^-)) - p_i(p_0 - P(\Psi_j^-)). \tag{56}$$

If $p_j \geq p_i$, $P(\Psi_j^-) - P(\Psi_i^-) + p_j(p_0 - P(\Psi_j^-)) \geq p_i(P(\Psi_j^-) - P(\Psi_i^-)) + p_i(p_0 - P(\Psi_j^-))$ because $0 \leq p_i \leq 1$, hence we obtain $\Delta DW_{i,j} \geq 0$. If $p_j \leq p_i$, by using Eq. (55), we can rewrite $\Delta DW_{i,j} = p_i(P(\Psi_i^+) - P(\Psi_i^-)) + p_j(P(\Psi_j^-) - P(\Psi_j^+)) + p_j(p_0 - P(\Psi_j^-)) - p_i(p_0 - P(\Psi_i^-))$, which can be further arranged as $p_i(P(\Psi_i^+) - p_0) - p_j(P(\Psi_j^+) - p_0) \geq 0$ since $P(\Psi_i^+) > P(\Psi_j^+)$ by assumption and $p_j \leq p_i$ by construction.

4) If $P(\Psi) \leq p_0 < P(\Psi_j^+)$ we proceed in the same way as in 3.

5) If $P(\Psi_j^+) \leq p_0 < P(\Psi_i^+)$ then $\Delta DW_{i,j} = (P(\Psi_j^-) - P(\Psi_i^-)) + p_j(P(\Psi_j^+) - P(\Psi_j^-)) - p_i(p_0 - P(\Psi_i^-))$. By Eq. (55), we obtain $\Delta DW_{i,j} = p_i(P(\Psi_i^+) - P(\Psi_i^-)) - p_j(P(\Psi_j^+) - P(\Psi_j^-)) + p_j(P(\Psi_j^+) - P(\Psi_j^-)) - p_i(p_0 - P(\Psi_i^-))$, which can be simplified as $\Delta DW_{i,j} = p_i(P(\Psi_i^+) - p_0) > 0$ because probabilities are non-negative and $p_0 < P(\Psi_i^+)$ by construction.

6) If $P(\Psi_i^+) \leq p_0$ then $\Delta DW_{i,j} = (P(\Psi_j^-) - P(\Psi_i^-)) + p_j(P(\Psi_j^+) - P(\Psi_j^-)) - p_i(P(\Psi_i^+) - P(\Psi_i^-))$, which by Eq. (55) can be rewritten as $\Delta DW_{i,j} = p_i(P(\Psi_i^+) - P(\Psi_i^-)) - p_j(P(\Psi_j^+) - P(\Psi_j^-)) + p_j(P(\Psi_j^+) - P(\Psi_j^-)) - p_i(P(\Psi_i^+) - P(\Psi_i^-)) = 0$.

Necessity. We now show that if $\Delta DW_{i,j} \geq 0$ then $\theta_i \geq \theta_j$. By Eq. (10) we know that $DW_i = 0$ for $p_0 \leq P(\Psi_i^-)$, then $DW_i = 0$ increases linearly as a function of p_0 with slope $(1 - p_i)$, reaches a peak at $p_0 = P(\Psi)$, then decreases with slope $-p_i$, and then it reaches the value of 0 at $p_0 = P(\Psi_i^+)$ and remains null for $p_0 > P(\Psi_i^+)$. DW_j has the same pattern as DW_i as a function of p_0. Because $\Delta DW_{i,j} \geq 0$, and by the ordering in Eq. (54), we have that $P(\Psi_j^-) \geq P(\Psi_i^-)$ and $P(\Psi_i^+) \geq P(\Psi_j^+)$. Thus, $\theta_i \geq \theta_j$ for all values of p_0. ■

Proof. Proposition 6. If $\theta_i \geq \theta_j$ then $\Delta_{i,j}^+, \Delta_{j,i}^- \geq 0$. This together with the coherence of the system imposes the following ranking $P(\Psi_i^-) \leq P(\Psi_j^-) \leq P(\Psi) \leq P(\Psi_j^+) \leq P(\Psi_i^+)$, which implies $P(\Psi_i^+) - P(\Psi_i^-) \geq P(\Psi_j^+) - P(\Psi_j^-)$, namely, $B_i \geq B_j$. ■

Proof. Proposition 7. We now show that if $B_i \geq B_j$ then $\theta_i \geq \theta_j$. Because the system is coherent, $P(\Psi_j^+) \geq P(\Psi_j^-)$ and $P(\Psi_i^+) \geq P(\Psi_i^-)$. By rearranging Eq. (55), we obtain $p_i[P(\Psi_i^+) - P(\Psi_i^-)] + P(\Psi_i^-) = p_j[P(\Psi_j^+) - P(\Psi_j^-)] + P(\Psi_j^-)$. Since $p_i = p_j$, we obtain $p_i(B_i - B_j) = P(\Psi_j^-) - P(\Psi_i^-)$. Because $B_i \geq B_j$, we get $P(\Psi_j^-) - P(\Psi_i^-) \geq 0$, namely, $\Delta_{j,i}^- \geq 0$. Since $p_i = p_j$, we can also rearrange Eq. (55) as $p_i[P(\Psi_i^+) - P(\Psi_j^+)] = (1 - p_i)[P(\Psi_j^-) - P(\Psi_i^-)]$. Because $\Delta_{j,i}^- \geq 0$ as we just proved, we have $p_i[P(\Psi_i^+) - P(\Psi_j^+)] \geq 0$, namely, $\Delta_{i,j}^+ \geq 0$. This ends the proof. ■

Proof. Proposition 8. Sufficiency. By Proposition (6) if a system is coherent and $\theta_i \geq \theta_j$ then $B_i \geq B_j$. By substituting this result and the fact that the SSCS are i.i.d. into Eq. (13) we get $FV_i = \dfrac{B_i}{P(\Psi)} p_i \geq FV_j = \dfrac{B_j}{P(\Psi)} p_i$.

Necessity. Because of the proportionality of the FV importance and the Birnbaum importance and since SSCs are i.i.d., the proof is the same of the one for Proposition (7). ■

References

[1] NEI. 10 CFR 50.69 SSC Categorization Guideline. *00-04 (Rev 0)*, pages 1–90, 2005.

[2] G.E. Apostolakis. How useful is quantitative risk assessment? *Risk Analysis*, 24(3):515–520, 2004.

[3] M.E. Paté-Cornell and R.L. Dillon. The respective roles of risk and decision analyses in decision support. *Decision Analysis*, 3(4):220–232, 2006.

[4] R.F. Boykin, R.A. Freeman, and R.R. Levary. Risk assessment in a chemical storage facility. *Management Science*, 30(4):512–517, 1984.

[5] H. Dezfuli, A. Benjamin, C. Everett, G. Maggio, M.G. Stamatelatos, and R. Youngblood. NASA Risk Management Handbook. *NASA SP 2011 3422*, 2011.

[6] W.E. Vesely, T.C. Davis, R.S. Denning, and N. Saltos. Measures of Risk Importance and Their Applications. Technical report, US Nuclear Regulatory Commission, 1983.

[7] E. Borgonovo and A. Cillo. Deciding with thresholds: Importance measures and value of information. *Risk Analysis*, forthcoming, 2017, 10.1111/risa.12732.

[8] R.E. Barlow and F. Proschan. Importance of system components and fault tree events. *Stochastic Processes with their Applications*, 3:153–173, 1975.

[9] S. Kaplan and B.J. Garrick. On the quantitative definition of risk. *Risk Analysis*, I(I):11–27, 1981.

[10] N.D. Singpurwalla. Foundational issues in reliability and risk analysis. *SIAM Review*, 30(2):264–282, 1988.

[11] G.E. Apostolakis. The concept of probability in safety assessments of technological systems. *Science*, 250(4986):1359–1364, 1990.

[12] I. Papazoglou. Mathematical foundations of event trees. *Reliability Engineering & System Safety*, 61:169–183, 1998.

[13] C.L. Smith. Calculating conditional core damage probabilities for nuclear power plant operations. *Reliability Engineering & System Safety*, 59(3):299–307, 1998.

[14] L.W. Birnbaum. On the importance of different elements in a multielement system. *Multivariate analysis, New York Academic Press*, 2:1–15, 1969.

[15] W.E. Vesely, R.E. Kurth, and S.M. Scalzo. Evaluations of core melt frequency effects due to component aging and maintenance. *NUREG/CR-5510, US Nuclear Regulatory Commission, Washington, DC*, 1990.

[16] NRC. An Approach for Using Probabilistic Risk Assessment in Risk-Informed Decisions on Plant Specific Changes to the Licensing Basis. *Regulatory Guide DG-1226, Proposed Revision of Regulatory Guide 1174*, pages 1–34, 2009.

[17] J. Fussell. How to calculate system reliability and safety characteristics. *IEEE Transactions on Reliability*, 24 (3):169–174, 1975.

[18] E Borgonovo and G E Apostolakis. A new importance measure for risk-informed decision making. *Reliability Engineering & System Safety*, 72(2):193–212, 2001.

[19] M.C. Cheok, G.W. Parry, and R.R. Sherry. Use of importance measures in risk-informed regulatory applications. *Reliability Engineering & System Safety*, 60(3):213–226, 1998.

[20] E. Borgonovo. Differential, criticality and Birnbaum importance measures: An application to basic event, groups and SSCs in event trees and binary decision diagrams. *Reliability Engineering and System Safety*, 92(10):1458–1467, 2007.

[21] W. Kuo and X. Zhu. *Importance Measures in Reliability, Risk and Optimization*. Wiley and Sons, UK, 2012.

[22] J.C. Felli and G. Hazen. Sensitivity analysis and the expected value of perfect information. *Medical Decision Making*, 18:95–109, 1998.

[23] R.A. Howard. Information value theory. *IEEE Transactions on Systems Science and Cybernetics*, 2(1):22–26, 1966.

[24] H. Raiffa. Decision Analysis: Introductory Lectures on Choices Under Uncertainty. *Random House, New York*, 1968.

[25] R.W. Hilton. The determinants of information value: synthesizing some general results. *Management Science*, 27(1):57–64, 1981.

[26] J.C. Felli and G.B. Hazen. A Bayesian approach to sensitivity analysis. *Health Economics*, 8:263–268, 1999.

[27] J.E. Oakley. Decision-theoretic sensitivity analysis for complex computer models. *Technometrics*, 51(2):121–129, 2009.

[28] J.E. Oakley, A. Brennan, P. Tappenden, and J.B. Chilcott. Sample sizes for Monte Carlo partial EVPI calculations. *Journal of Health Economics*, 29(3):468–77, 2010.

[29] M. Strong and J.E. Oakley. An efficient method for computing partial expected value of perfect information for correlated inputs. *Medical Decision-Making*, 33:755–766, 2013.

[30] M. Strong, J.E. Oakley, and A. Brennan. Estimating multiparameter partial expected value of perfect Information from a Probabilistic Sensitivity Analysis Sample: A nonparametric regression approach. *Medical Decision-Making*, 34:311–326, 2014.

[31] R.B. Bratvold, J.E. Bickel, and H.P. Lohne. Value of information in the oil and gas industry: Past, present and future. *SPE Reservoir Evaluation and Engineering*, 12(4):630–638, 2009.

[32] G.C. Pflug. A value-of-information approach to measuring risk in multi-period economic activity. *Journal of Banking & Finance*, 30(2):695–715, 2006.

[33] I. Linkov, M.E. Bates, L.J. Canis, T.P. Seager, and J.M. Keisler. A decision-directed approach for prioritizing research into the impact of nanomaterials on the environment and human health. *Nature Nanotechnology*, 6(12):784–787, 2011.

[34] M.E. Bates, J.M. Keisler, N.P. Zussblatt, K.J. Plourde, B.A. Wender, and I. Linkov. Balancing research and funding using value of information and portfolio tools for nanomaterial risk classification. *Nature Nanotechnology*, 11(2):198–203, 2016.

[35] J.M. Keisler, Z.A. Collier, E. Chu, N. Sinatra, and I. Linkov. Value of information analysis: The state of application. *Environment Systems and Decisions*, 34(1):3–23, 2014.

[36] K. Poern. A decision-oriented measure of uncertainty importance for use in PSA. *Reliability Engineering & System Safety*, 56:17–27, 1997.

[37] D. Vasseur and M. Llory. International survey on PSA figures of merit. *Reliability Engineering and System Safety*, 66:261–274, 1999.

[38] H.D. Brewer and K.S. Canady. Probabilistic safety assessment support for the maintenance rule at Duke Power Company. *Reliability Engineering & System Safety*, 63:243–249, 1999.

[39] C.L. Smith, V. N. Shah, T. Kao, and G.E. Apostolakis. Incorporating Aging Rffects into Probabilistic Risk Assessment: A Feasibility Study Utilizing Reliability Physics Models. *NUREG/CR-5632*, 2000.

[40] M. Stamatelatos and H. Dezfuli. NASA Probabilistic Risk Assessment Procedures Guide for NASA Managers and Practitioners. *NASA/SP-2011-3421*, Second Edi, 2011.

[41] E. Borgonovo and C. L. Smith. Composite multilinearity, epistemic uncertainty and risk achievement worth. *European Journal of Operational Research*, 222:301–311, 2012.

[42] E. Borgonovo and C.L. Smith. A study of interactions in the risk assessment of complex engineering systems: An application to space PSA. *Operations Research*, 59(6):1461–1476, 2011.