An inverse coefficient problem related to elastic–plastic torsion of a circular cross-section bar

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A B S T R A C T

An inverse coefficient problem related to identification of the plasticity function \(g(\eta)\) from a given torque \(\tau\) is studied for a circular section bar. Within the deformation theory of plasticity the mathematical model of torsion leads to the nonlinear Dirichlet problem

\[
-\nabla \cdot (g(\|\nabla u\|^2)\nabla u) = 2\varphi, \quad x \in \Omega \subset \mathbb{R}^2; \quad u(s) = 0, \quad s \in \partial \Omega.
\]

(1)

related to the elastoplastic torsion of a circular cross-section bar. Here \(u(x)\) is the Prandtl stress function, \(g = g(T^2)\) is the plasticity function, \(T^2 := \|\nabla u\|^2\) is the stress intensity and \(\varphi > 0\) is the angle of twist per unit length.

The boundary value problem (1) represents an elastoplastic torsion of a strain hardening bar, represented by the cylinder with base \(\bar{\Omega} = \Omega \cup \partial \Omega\) and with generators parallel to the axis \(Ox_3\). The base of a bar is assumed to be fixed, i.e. rigid clamped.

The inverse coefficient problem here consists of the determination of the unknown coefficient \(g(T^2)\), from the experimentally given value \(\tau = \tau(\varphi)\) of torque (or torsional stiffness), under the action of a moment \(M\), during the quasi-static process of torsion, given by the angle of twist per unit length \(\varphi \in [\varphi_*, \varphi^*], \varphi_* > 0\). Accordingly, for the given coefficient \(g = g(T^2)\) and the angle of twist \(\varphi \in [\varphi_*, \varphi^*]\), the boundary value problem (1) will be referred as the direct problem. The solution of this problem will be defined as \(u(x; \varphi)\), when the parameter \(\varphi > 0\) varies in \([\varphi_*, \varphi^*]\). Then, by definition, the

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1. Introduction

In this paper, we study the inverse coefficient problem associated with the nonlinear boundary value problem

\[
\begin{align*}
-\nabla \cdot (g(\|\nabla u\|^2)\nabla u) &= 2\varphi, \quad x \in \Omega \subset \mathbb{R}^2; \\
u(s) &= 0, \quad s \in \partial \Omega,
\end{align*}
\]

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torque is defined to be as the integral of the Prandtl stress function \( u(x) \) over the domain \( \Omega \subset \mathbb{R}^2 \):

\[
\tau(\varphi) = 2 \int_\Omega u(x; \varphi) \, dx, \quad \varphi \in [\varphi_*, \varphi^*].
\]  

(2)

Hence the considered inverse coefficient problem can be formulated as the problem of finding the pair \((u, g)\) from the following equations:

\[
\begin{cases}
-\nabla \cdot (g(|\nabla u|^2) \nabla u) = 2\varphi, & x \in \Omega \subset \mathbb{R}^2; \\
u(s) = 0, & s \in \partial \Omega; \\
2 \int_\Omega u(x; \varphi) \, dx = \tau(\varphi), & \varphi \in [\varphi_*, \varphi^*].
\end{cases}
\]  

(3)

Elastic–plastic torsion of hardening bars has been investigated in numerous engineering [1–5] and mathematical [6–9] studies. Thus, mathematical models and engineering analysis of torsional problems for circular and non-circular cross-section bars have been proposed in [1]. The torsion problem of a circular bar with fixed ends is solved in [5] using a finite deformation constitutive model. Large elastic–plastic torsion of uniform circular bars has numerically been investigated in [4], using special finite elements. In the analysis here either linear hardening or power law hardening uniaxial stress–strain curves are assumed in the physical model. An existence of the weak solution in the Sobolev spaces \( W^1_p(\Omega) \), \( p \geq 2 \), of the nonlinear boundary value problem for the torsion of a hardening bar has been given in [6]. This approach has then been extended in [10] as a variational method. Mathematical models of elastic–plastic torsion of hardening bars have been proposed in [11–13]. A maximum principle for the nonlinear boundary value problem (1) is derived in [9].

The first attempt to identify the unknown coefficient \( g = g(T^2) \) in (1) from the measured torque, has been given in [8]. In this study the numerical inversion algorithm, based on finite element approximation, is proposed for a prismatic bar of rectangular cross-section. Mathematical analysis of the inverse problem (3) has then been given in [2]. Numerical inversion algorithm for reconstruction of the unknown coefficient \( g = g(T^2) \) from noisy data \( \tau(\varphi) \), with various noise levels \( \gamma > 0 \), has been proposed in [14]. However, to the best of our knowledge, for a circular cross-section bar the inverse problem (3) has not been studied so far.

The paper is organized as follows. Solvability of the direct and inverse problems is discussed in Section 2. An analysis of the inverse coefficient problem for a circular cross-section bar is given in Section 3. It is shown here that the coefficient-to-torque (i.e., input–output) map is explicitly invertible. Then an explicit formula relating the plasticity function \( g(\eta) \) and the torque \( \tau \) is derived. In Section 4, the well-known formula between the elastic shear modulus \( G > 0 \) and the torque for pure elastic torsion of circular cross-section bar is obtained from this explicit formula.

2. The main assumptions and solvability of the direct and inverse problems

The torsion of elastic–plastic shafts with work-hardening has generally been studied by means of \( J_2 \)-deformation theory [11–13], it being argued that under monotonic (simple) loading such a treatment would provide a good approximation to the solutions of the incremental theory. It is well known that these two theories coincide exactly only when the bar (shaft) is circular [13]. In the deformation theory, the relationship \( T = f(G) \Gamma \), between the intensity of the shear strain tensor \( \Gamma \) and the intensity of the tangential stress \( T = (u_1 + u_2)/2 \), is a continuous, piecewise differentiable monotone and concave curve [12,7,13]:

\[
\begin{cases}
T \in C^1[\Gamma_*, \Gamma^*] \cap C^1_p(\Gamma_*, \Gamma^*), & \Gamma_* \geq 0; \\
\frac{dT}{d\Gamma} > 0, & \Gamma \in (\Gamma_*, \Gamma^*); \\
T(t\Gamma_1 + (1-t)\Gamma_2) \geq tT(\Gamma_1) + (1-t)T(\Gamma_2), & \forall \Gamma_1, \Gamma_2 \in (\Gamma_*, \Gamma^*), \forall t \in [0, 1].
\end{cases}
\]  

(4)

Here \( C^1_p(\Gamma_*, \Gamma^*) \) is the class of piecewise differentiable functions. The function \( f(\Gamma) \) is called the modulus of plasticity. Experiments show that the concave curve \( T = f(\Gamma) \Gamma \) consists of two phases: initially the material follows the Hooke’s law \( T = G\Gamma, \quad \Gamma \in [\Gamma_0, \Gamma^*] \), where \( G > 0 \) is the elastic shear modulus, and \( \Gamma_0 > 0 \) is the elasticity limit. This phase is called the linear elastic phase. The part of the diagram \( T = f(\Gamma) \Gamma \), corresponding to \( \Gamma \in [\Gamma_0, \Gamma^*] \) is defined to be as the strain–hardening phase. Due to assumptions (4), in both cases the \( T = f(\Gamma) \Gamma \) function is invertible, and these inverses are defined as follows [7,8]:

\[
\Gamma = \begin{cases}
\frac{1}{G} T, & T \in [T_0, T_*] ;
\frac{G}{\tau} T, & T \in [T_0, T^*].
\end{cases}
\]  

(5)

where \( g = g(T^2) \) is the above defined plasticity function. Further, the plasticity function \( g(T^2) \) satisfies the following conditions of deformation theory [11,12]:

\[
\begin{cases}
c_1 \geq g(T^2) \geq c_0 > 0; \\
g(T^2) + 2T^2 g'(T^2) \geq c_2 > 0, & T \in [T_0, T^*].
\end{cases}
\]  

(6)
The case \( T \in [T_0, T_*] \), corresponding to the linear elastic torsion case, will be considered later, separately. Hence we will assume that \( T \in [T_0, T^*] \). The set of continuous and piecewise differentiable functions \( g(T^2) \) satisfying the conditions (6) will be defined as the set of admissible coefficients \( \mathcal{G} \subset H^1[T_0, T^*] \) for the inverse problem (3).

Note that the equation \(- \nabla \cdot (g(|\nabla u|^2) \nabla u) = F(x)\), with the diffusion coefficient \( g(|\nabla u|^2) \) satisfying the same conditions (6), is a steady state analogue of the nonlinear diffusion equation \( u_t = \nabla \cdot (g(|\nabla u|^2) \nabla u) \), arising in nonlinear isotropic diffusion filtering [15, 14].

Let \( g(T^2) \in \mathcal{G} \) be a given function from a set of admissible coefficients and \( \Omega \subset \mathbb{R}^2 \) is a domain with a piecewise smooth boundary. Then the weak solution \( u(x; \psi; g) \in \mathcal{H}^1(\Omega) \) of the direct problem (1) satisfies the following integral identity:

\[
\int_{\Omega} g(|\nabla u|^2) \nabla u \nabla v dx = 2\omega \int_{\Omega} v dx, \quad \forall v \in \mathcal{H}^1(\Omega),
\]

where \( \mathcal{H}^1(\Omega) := \{ v \in \mathcal{H}^1(\Omega) : \| v \| = 0, \quad s \in \partial \Omega \} \) and \( \mathcal{H}^1(\Omega) \) is the Sobolev space of functions [16]. Under conditions (6), this solution exists and is unique (see [2, 6]).

We define the coefficient-to-torque map \( \mathcal{T}[] : \mathcal{G} \mapsto \mathbb{T} \) as follows:

\[
\mathcal{T}[g](\psi) := 2 \int_{\Omega} u(x; \psi; g) dx, \quad \psi \in [\varphi_*, \varphi^*].
\]

(7)

where \( u(x; \psi; g) \in \mathcal{H}^1(\Omega) \) is the solution of the direct problem (1) for a given coefficient \( g \in \mathcal{G} \). Then the inverse problem (3) with the given measured output data \( \tau = \tau(\varphi) \) can be reduced to the solution of the nonlinear operator equation

\[
\mathcal{T}[g] = \tau, \quad g \in \mathcal{G},
\]

(8)

or to inverting the coefficient-to-torque map \( \mathcal{T}[] : \mathcal{G} \mapsto \mathbb{T} \).

Evidently, in practice the measured output data \( \tau = \tau(\varphi) \) can only be given with some measurement error (noise), which means that exact fulfillment of the equality in (8) is not possible. For this reason one needs to introduce the auxiliary (cost) functional

\[
I(g) = \max_{\varphi \in [\varphi_*, \varphi^*]} \left\| 2 \int_{\Omega} u(x; \psi; g) dx - \tau(\varphi) \right\|^2, \quad g \in \mathcal{G},
\]

and consider the following minimum problem:

\[
I(g_*) = \inf_{g \in \mathcal{G}} I(g).
\]

(9)

A solution of this minimum problem is defined to be a quasi-solution of the inverse problem (3). An existence of a quasi-solution in the set of admissible coefficients \( g \) has been proved in [2].

3. Invertibility of the coefficient-to-torque map for a circular section bar

Let us consider the elastic–plastic torsion of a circular cross-section bar: \( \Omega_{\vartheta} := \{ (r, \theta) \in \mathbb{R}^2 \} : r := \sqrt{x_1^2 + x_2^2} < r_0, \quad \theta \in [0, 2\pi] \}, r_0 > 0 \) (Fig. 1). Then in \( \Omega_{\vartheta} \), the Prandtl stress function \( u(r, \theta; \varphi) \) does not depend on \( \theta \), for each \( \varphi \in [\varphi_*, \varphi^*] \). In this polar coordinates we have: \( \nabla u = u_r r_1 + u_\theta r_2 \), \( x = x_1 x_1 + x_2 x_2 \), and \( |\nabla u|^2 := u_1^2 + u_2^2 = u_r^2 \). Hence the differential operator on the left hand side of Eq. (1) has the form:

\[
\nabla \cdot (g(|\nabla u|^2) \nabla u) = \nabla \cdot (g(u_\theta^2) \nabla u),
\]

Since \( \nabla (x/r) = 1/r \) we have: \( \nabla \cdot (g(|\nabla u|^2) \nabla u) = (g(u_\theta^2) \nabla u)_r + g(u_\theta^2) u_r = 2\varphi \). For this second order differential equation one needs to impose two boundary conditions. The first one is a result of the Dirichlet condition in (3): \( u(r; \varphi)|_{r=r_0} = 0 \). Since the bar is of circular cross-section with no holes, it is natural to assume that \( u_r(r; \varphi) = 0, \quad \text{when} \quad r = 0 \).

Thus, for a circular cross-section bar the inverse coefficient problem can be formulated as the problem of finding the pair \( (u, g) \), \( u \in \mathcal{H}^1[0, r_0] := \{ v \in \mathcal{H}^1(\Omega) : v(r_0; \varphi) = 0, g \in \mathcal{G} \}, \) from the following equations:

\[
\begin{cases}
-g(u^2_\theta) u_r - \frac{1}{r} g(u^2_\theta) u_r = 2\varphi, & r \in (0, r_0); \\
u(r, \varphi)|_{r=r_0} = 0, & u_r(r, \varphi)|_{r=0} = 0; \\
4\pi \int_0^{r_0} u(r, \varphi) rdr = \tau(\varphi), & \varphi \in [\varphi_*, \varphi^*].
\end{cases}
\]

(10)

**Theorem 1.** Let the function \( T = f(\Gamma) \Gamma \) satisfy conditions (4) and \( \tau \in C^1(\varphi_*, \varphi^*) \). Then the coefficient-to-torque map \( \mathcal{T}[] : \mathcal{G} \mapsto \mathbb{T} \), defined for a circular cross-section bar as

\[
\mathcal{T}[g](\psi) := 4\pi \int_0^{r_0} u(r; \psi; g) rdr, \quad g \in \mathcal{G},
\]

(11)
is uniquely invertible, i.e. \( \exists T^{-1}[\tau] : \mathbb{T} \mapsto \mathcal{G} \). Moreover, the unknown coefficient \( g \in \mathcal{G} \) can be defined via the torque \( \tau = \tau(\varphi) \) as follows:

\[
g(T^2) = 2\pi \frac{\Gamma^3}{\hat{\tau}(\Gamma)}, \quad \Gamma \in [\Gamma^*, \Gamma^*], \quad \hat{\tau}(\Gamma) := \left( \varphi^3 \tau(\varphi) \right)_{\varphi = \Gamma/\rho_0}.
\] (12)

**Proof.** Let \( g \in \mathcal{G} \) be a given coefficient. Then for a circular section bar, the direct problem can be formulated as the following two-point boundary value problem

\[
\begin{cases}
-(\varphi \varphi) \psi = \varphi, & r \in (0, r_0); \\
\frac{\psi}{\varphi} = 0, & u(r; \varphi) |_{r=r_0} = 0,
\end{cases}
\] (13)

for each \( \varphi \in [\varphi_*, \varphi^*] \). To solve problem (13) we introduce the function \( u[g](r; \varphi) := g(\varphi^2)u_0(r, \varphi) \). This function solves the problem

\[
\begin{cases}
\frac{du}{dr} - \frac{1}{r} u = 2\varphi, & r \in (0, r_0); \\
u(r; \varphi) |_{r=r_0} = 0,
\end{cases}
\] (14)

whose solution is \( u(r; \varphi) = -\varphi r, r \in [0, r_0] \). Since \( T \ := |u_0| \) and \( u_0(r; \varphi) < 0, r \in [0, r_0] \), we have \( T(r; \varphi) = -u_0(r; \varphi) \). On the other hand \( u(r; \varphi) := g(\varphi^2)u_0(r, \varphi) \), and due to the conditions of the theorem, the function \( \Gamma = g(T^2)T \) is invertible, with the inverse \( T = f(\Gamma) \). Hence \( u(r, \varphi) = (f(\Gamma)T)|_{\Gamma=\varphi} \). Using the boundary condition \( u(r; \varphi) |_{r=r_0} = 0 \) we conclude that

\[
u(r; \varphi) = \int_0^{r_0} f(\varphi) \varphi r^2 d\rho, \quad r \in (0, r_0), \quad \varphi \in [\varphi_*, \varphi^*].
\]

Using this in definition (11) we may rewrite the operator equation (8) as follows:

\[
4\pi \int_0^{r_0} \left( \int_0^{\rho} f(\varphi) \varphi r^2 d\rho \right) \varphi r^2 d\rho = \tau(\varphi), \quad \varphi \in [\varphi_*, \varphi^*],
\]

where \( \tau = \tau(\varphi) \) is the given (noise free) torque, defined by (2). To solve this equation with respect to the function \( f(\Gamma) \) we use integration by parts to compute the left integral. Then we have:

\[
2\pi \varphi \int_0^{r_0} f(\varphi) r^2 dr = \tau(\varphi), \quad \varphi \in [\varphi_*, \varphi^*].
\]

We use now the change of variables \( r \varphi = \gamma \) in the left integral and denote \( r_0 \varphi = \Gamma \). As a result, we obtain the formula

\[
\int_0^\Gamma f(\gamma) \gamma^3 d\gamma = \frac{1}{2\pi} \left( \varphi^3 \tau(\varphi) \right)_{\varphi = \Gamma/\rho_0}, \quad \Gamma \in [\Gamma_*, \Gamma^*] \equiv [\varphi, \rho_0 \varphi_0].
\] (15)

Now we differentiate both sides of the above equality with respect to the new variable (shear strain) \( \Gamma \in [\Gamma_*, \Gamma^*] \), to derive the explicit formula between modulus of plasticity \( f(\Gamma) \) and the torque \( \tau(\varphi) \):

\[
f(\Gamma) = \frac{1}{2\pi \Gamma^3} \hat{\tau}(\Gamma), \quad \hat{\tau}(\Gamma) := \left( \varphi^3 \tau(\varphi) \right)_{\varphi = \Gamma/\rho_0}.
\] (16)

Taking into account the relationship \( f(\Gamma) = 1/g(T^2) \) in (16) we obtain the required formula (12). This completes the proof. □
Corollary 1. For a circular cross-section bar the shear strain tensor \( \Gamma \) \( \Gamma := \Gamma(r; \varphi) \) satisfies the following nonlinear boundary value problem

\[
\begin{aligned}
\frac{d\Gamma}{dr} + \frac{1}{G}\Gamma &= 2\varphi, \quad r \in (0, r_0);
\left[\frac{d\Gamma}{dr}\right]_{r=0} &= 0,
\end{aligned}
\]

as a function of \( r \in [0, r_0] \).

This result follows from the above found solution \( w = -\varphi r \) of Eq. (14) and from the definitions \( w := g(u_0^2)u_r, T = -u_r \) and \( \Gamma = g(T^2)T \).

Corollary 2. Let conditions of Theorem 1 hold. Then the main relationship \( T(\Gamma) := f(\Gamma)\Gamma \) of the deformation theory of plasticity for a circular cross section bar can be determined via the torque by the following formula:

\[
T(\Gamma) = \frac{1}{2\pi r_0^2} \left[ 3\tau(\varphi) + \varphi \tau'(\varphi) \right]_{\varphi = \Gamma / r_0}.
\]

4. Relationship between the elastic shear modulus \( G > 0 \) and the torque: an analytical solution of an inverse coefficient problem in the linear elastic torsion case

Consider the linear elastic torsion of a circular cross section bar. In this case \( \varphi \in [\varphi_*, \varphi_0] \), and \( g(T^2) = 1/G \), by (5). The inverse problem can be formulated as the problem of finding the unknown elastic shear modulus \( G > 0 \) in

\[
\begin{aligned}
\frac{1}{G} \left[ u_{x_1 x_1} + u_{x_2 x_2} \right] &= 2\varphi, \quad (x_1, x_2) \in \Omega_{\varphi};
\end{aligned}
\]

from the measured torque, given by (11). Evidently, an analytical solution of the above problem is the function \( u(r; \varphi) = G\varphi(r_0^2 - r^2)/2 \). Calculating the torque we have:

\[
\tau(\varphi) = 4\pi \int_0^{r_0} u(r, \varphi) r dr = \frac{1}{2} \pi Gr_0^4 \varphi, \quad \varphi \in [\varphi_*, \varphi_0].
\]

Hence for a given angle of twist \( \varphi \in [\varphi_*, \varphi_0] \), the elastic shear modulus \( G > 0 \) can be found from a given torque \( \tau = \tau(\varphi) \) by the following formula:

\[
G = \frac{2\tau}{\pi r_0^4 \varphi}.
\]

As an application of Theorem 1, we derive this well-known formula [1] from the inversion formulas (12) and (15). In the linear elastic torsion of a circular cross section bar \( \varphi \in [\varphi_*, \varphi_0] \), and \( f(\Gamma) = G \), according to (4) and (5). Then formula (15) yields:

\[
G \int_0^r y^3 dy = \frac{1}{2\pi} \varphi^3 \tau(\varphi), \quad \Gamma = r_0 \varphi, \quad \Gamma \in [\Gamma_*, \Gamma_0] \equiv [\varphi_* r_0, \varphi_0 r_0].
\]

Calculating the left integral and substituting here \( \Gamma = r_0 \varphi \) we obtain:

\[
\frac{1}{4} Gr_0^4 \varphi^4 = \frac{1}{2\pi} \varphi^3 \tau(\varphi), \quad \varphi \in [\varphi_*, \varphi_0],
\]

which implies formula (18).

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