Numerical valuation of discrete double barrier options

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\textbf{A R T I C L E I N F O}

\textbf{Article history:}
Received 20 June 2008
Received in revised form 10 February 2009

\textbf{Keywords:}
Discrete barrier options
Black–Scholes model
Quadrature method
Multivariate normal probability evaluation
Exotics

\textbf{A B S T R A C T}

In the present paper we explore the problem for pricing discrete barrier options utilizing the Black–Scholes model for the random movement of the asset price. We postulate the problem as a path integral calculation by choosing approach that is similar to the \textit{quadrature method}. Thus, the problem is reduced to the estimation of a multi-dimensional integral whose dimension corresponds to the number of the monitoring dates.

We propose a fast and accurate numerical algorithm for its valuation. Our results for pricing discretely monitored one and double barrier options are in agreement with those obtained by other numerical and analytical methods in Finance and literature. A desired level of accuracy is very fast achieved for values of the underlying asset close to the strike price or the barriers.

The method has a simple computer implementation and it permits observing the entire life of the option.

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1. Introduction

In the market of financial derivatives the most important problem is the so-called \textit{option valuation problem}, i.e. to compute a fair value for the option. The Black–Scholes analytic model for determining the behavior of the stock price turns out to be fundamental in option pricing. [1]

Peter Carr gives closed form formulas and replication strategies for barrier options, [2]. Analytical formulas using the \textit{method of images} in the case of one barrier applied continuously are presented in [3]. Using reflection principle in Brownian motions, Li expresses the solution in general as summation of an infinite number of normal distribution functions for standard double barrier options, and in many non-trivial cases the solution consists of only \textit{finite terms}, [4]. For more information a detailed comprehensive guide of option pricing formulas is that of Espen Gaarder Haug. [5]. However, unfortunately, in the case of barrier options, most of the frequently presented formulas assumed continuous monitoring of the barrier, i.e., a knock-in or knock-out is presumed to happen if the barrier is touched \textit{at any instant} during the life of the option.

Sometimes, the option price differs substantially between discrete and continuous monitoring, [3]. Broadie found an explicit correction formula for discretely monitored option with one barrier, [6]. However, it has not been still applied in the presence of two barriers, i.e. a discrete double barrier option.

Five different approaches for option pricing have been summarized into a unifying framework in [7]. Often in literature they are listed as \textit{Monte Carlo simulations, binomial and trinomial trees, finite difference schemes, finding an analytical solution,}

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doi:10.1016/j.cam.2009.10.029
and the quadrature method. We will describe shortly their application in the case of discrete barrier options in order to point out the advantages of our algorithm based on the quadrature method.

Path-dependent options could be priced using Monte Carlo simulations but a desired level of accuracy is not achieved quickly. Pricing down-and-in call option with a discretely monitored barrier using an importance sampling technique and a conditional Monte Carlo is presented in [8].

In absence of a valuation formula for non-standard options, binomial and trinomial trees are the simplest means for pricing. Including barrier constraints cause difficulties of adjusting the tree, see Kwok, [3]. The extension of the trinomial scheme to deal with time-dependent barriers is proposed in [9]. Recently, various adaptive mesh mechanisms are used around the payoff region of exercise price at maturity, see Shea in [10].

In the finite difference approach the Black–Scholes equation is resolved backwards or from the initial condition (which is in fact the terminal payoff at expiry), plus extra constraints in case of barriers. The possibility of observing the entire life of the option is advantageous but for numerous grid points this a time-consuming process. In case of discrete barrier options, the numerical solution of some finite difference schemes such as the Crank–Nicolson one suffers from spurious oscillations that derive from an inaccurate approximation of the very sharp gradient produced by the knock-out clause, generating an error that is damp out very slowly, [11]. This drawback could be avoided experimentally if the time-step is prohibitively small, [12], or eliminated theoretically by imposing some extra sufficient conditions, [11].

Tian and Boyle show that discrete monitoring of barrier options could be handled by some tricky modifications of the numerical scheme by arranging the grid so that it is convenient for handling the boundary conditions, [13].

In case of discretely monitored barrier options there are some analytical solutions. For example, Fusai reduces the problem of one barrier to a Wiener–Hopf integral equation and a given z-transform solution of it, [14]. To derive a formula for continuous double barrier knock-out and knock-in options Pelsser inverts analytically the Laplace transform by a contour integration, [15].

Using a probabilistic approach for pricing discretely monitored barrier options seems to be ‘tedious’, because it involves valuation of a multivariate normal distribution functions, [3]. This approach is explored in different forms such as the model of Wai in [16], in [17], or as the quadrature method introduced by Andricopoulos in [18] for single barrier options in 2003 and extended for double barrier options in 2005 in [19].

A more general formulation is given in [7]: ‘The option pricing of path-dependent European options could be reformulated as a path integral that in case of discrete fixing dates reduces to a multi-dimensional integral whose dimension corresponds to the number of observation dates’.

Often, a recursion of one-dimensional integrals is used for valuation of the multiple integral, respectively the option price and Airoldi listed three numerical approaches usually used in literature, [7]. The presented algorithm is similar to that one where the density function is discretized at each time-step and recursively calculated using the values at the previous steps in a forward manner. In contrast, in the quadrature method the option price is found backwards by integrating each node value at each barrier monitoring date starting from the calculated values of the nodes at the maturity date.

The proposed algorithm differs from the recursive numerical integration procedure of AitSahlia, [17] and the tridiagonal probability algorithm of Wai, [16].

In Section 2 it is discussed the model structure for discrete double barrier knock-out options and we expose an analytical formula for the option value.

In Section 3 we expose a numerical algorithm for fast and accurate valuation of the multi-dimensional integral that represents the formula for the option price of Section 2. We present an error estimation of our approximation and derive for discrete barrier options an identity similar to the famous put-call parity.

Section 4 consists of computational results compared with all previously discussed numerical and analytical methods. We explore examples of discrete barrier options frequently used in literature [14,20,10,21,13,22].

We have presented the computational time of our numerical algorithm in order to justify its efficiency. It should be noted that the method has a simple computer implementation and permits also observing the entire life of the option that is a distinctive feature of the finite difference approach.

In the conclusion, we give some final remarks for our method and its possible application to other path-dependent options.

2. Model for discrete double barrier options

In principle, barrier features may be applied in continuous or discrete manner to any option. One example of barrier options with a discrete monitoring clause is the following option:

Definition 2.1. A discrete double barrier knock-out call option is an option with a continuous payoff condition equal to max(S - K, 0) which expires worthless if before the maturity the asset price has fallen outside the barrier corridor [L, U] at the prefixed monitoring dates: at these dates the option becomes zero if the asset falls out of the corridor. If one of the barriers is touched by the asset price at the prefixed dates then the option is canceled, i.e., it becomes zero, but the holder may be compensated by a rebate payment.
We use formula (3). The value of double barrier knock-out call option monitored \( m \)-times is given by the value of the following \( m \)-dimensional integral:

\[
V(S, t) = e^{-rT} \int_0^{\ln\frac{S_0}{U}} \ldots \int_0^{\ln\frac{S_0}{U}} \int_0^{\ln\frac{S_0}{U}} (e^{x_1 + x_2 + \ldots + x_m} - K) f(x_1, x_2, \ldots, x_m) dx_m \ldots dx_2 dx_1
\]

where the density function \( f(x_1, x_2, \ldots, x_m) \) is defined by

\[
\left( \frac{1}{\sigma \sqrt{2\pi \Delta t}} \right)^m e^{-\frac{(x_m - c)^2 + (x_m - c)^2 + \ldots + (x_1 - c)^2}{2\Delta t\sigma^2}}
\]

where \( c = \left( r - \frac{\sigma^2}{2} \right) \sqrt{\Delta t} \) and \( cc = c - \ln\frac{S_0}{U} \).

**Proof.** We use formula (6) in this form in order to develop our numerical algorithm for approximating this multi-dimensional integral.\(^2\)

---

\(^1\) A discrete barrier is one for which the barrier event is considered at discrete times, rather than the normal continuous barrier case. It should be noted that, away from the monitoring dates, the option price can move on the positive real axis interval \([0, +\infty]\).

\(^2\) If \( cc = c \) in (7), \( f(x_1, x_2, \ldots, x_m) \) is a multivariate normal probability density function of the variables \( x_i \), but the integral limits in (6) would be from \( \ln\frac{S_0}{B} \) to \( \ln\frac{S_0}{U} \). In case of a down-and-out call option each integral has an **infinite upper limit**, [16].
Having in mind all the indicators \( I_i \) in formula (4) we have:

\[
A_i = \{ S_i \in (L, U) \} = \{ S_{i,M} \in (L, U) \}
\]

or equivalently, dividing by \( S(0) > 0 \) and then take logarithm:

\[
\left\{ \ln \frac{S_{i,M}}{S_0} \in \left( \ln \frac{L}{S_0}, \ln \frac{U}{S_0} \right) \right\} = \left\{ D_i \in \left( \ln \frac{L}{S_0}, \ln \frac{U}{S_0} \right) \right\}
\]

where \( D_i \) is defined as the sum of random variables \( \xi_i \) in (3). In addition

\[
A_i = \left\{ \left( D_i - \ln \frac{L}{S_0} \right) \in \left( 0, \ln \frac{U}{T} \right) \right\}
\]

(8)

Setting \( \xi = \xi_1 - \ln \frac{L}{S_0} \), \( c = (r - \frac{\sigma^2}{2}) \sqrt{\Delta t} \), \( cc = c - \ln \frac{L}{S_0} \), we have that

\[
S_T = S_{i,M} = S_0 e^{D_m} = L e^{\xi_1 + \xi_2 + \ldots + \xi_m}
\]

and using the conditions in (9), i.e. \( \xi_1 \in (0, \ln \frac{U}{L}) \), \( \xi_2 \in (0, \ln \frac{U}{L}) \) \ldots \( \xi_m \in (0, \ln \frac{U}{L}) \), the problem of estimating the expectation (5) is finally reduced to evaluation of formula (6) in Theorem 2.1.

Analogously, the value of discrete double barrier knock-out put option is:

\[
V_{put}(S, T) = e^{-rT} \int_0^{\ln \frac{U}{L}} \ldots \int_0^{\ln \frac{U}{L}} \int_0^{\ln \frac{U}{L}} (K - L e^{\xi_1 + \xi_2 + \ldots + \xi_m}) f(x_1, x_2, \ldots, x_m) dx_m \ldots dx_2 dx_1
\]

(10)

where the density function \( f(x_1, x_2, \ldots, x_m) \) is defined in (7).

Unfortunately, for large values of \( m \), this \( m \)-dimensional integral could not be quickly estimated on a computer. Experimentally, when \( m = 1, 2, 3 \) the integral could be estimated fast. For \( m = 4 \) the computations take a long period of time (minutes) while for \( m \geq 5 \) hardly a real machine could manage to finish the estimations within a reasonable time.

However, the number \( m \) is the barrier observation frequency and usually it is 25 or 125 in case the option is observed weekly or daily, respectively. □

3 We formally use the term normally distributed for a discrete random variable.

3. Algorithm for numerical valuation

We propose a quick numerical algorithm for pricing formula (6) and thus to overcome the time-obstacle that is frequently met in computations.

The main idea of the numerical algorithm is to substitute the continuous normally distributed random variables \( \xi_i \) in (3) with discrete ones that are ‘normally distributed’ and instead of \( m \)-dimensional integral to valuate \( m \) number of finite sums. Thus, the computations are substantially quicker.

To illustrate our algorithm let us turn to the beginning of our problem (5). If \( \xi_i \) are independent normally distributed random variables with mean 0 and variance 1, \( i = 1, \ldots, m \), we have the system:

\[
L < S_{i,M} = S_0 e^{(r-\frac{\sigma^2}{2})\Delta t + \xi_1} < U
\]

\[
L < S_{2,M} = S_0 e^{(r-\frac{\sigma^2}{2})2\Delta t + (\xi_1 + \xi_2)} < U
\]

\[
\ldots
\]

\[
L < S_{m,M} = S_0 e^{(r-\frac{\sigma^2}{2})m\Delta t + (\xi_1 + \xi_2 + \ldots + \xi_m)} < U.
\]

We divide each row of the system by \( S_0 > 0 \) and then take logarithm. After setting \( c = \left( r - \frac{\sigma^2}{2} \right) \Delta t \), we obtain

\[
\ln \frac{L}{S_0} < c + \xi_1 < \ln \frac{U}{S_0}
\]

\[
\ln \frac{L}{S_0} < 2c + (\xi_1 + \xi_2) < \ln \frac{U}{S_0}
\]

\[
\ldots
\]

\[
\ln \frac{L}{S_0} < mc + (\xi_1 + \xi_2 + \ldots + \xi_m) < \ln \frac{U}{S_0}.
\]
Subtracting \( \ln \frac{1}{n} + \frac{1}{2n} \ln \frac{U}{T} \), where \( n \) is an integer number,\(^4\) we have

\[
\begin{align*}
- \frac{1}{2n} \ln \frac{U}{L} &< c - \left[ \ln \frac{L}{S_0} + \frac{1}{2n} \ln \frac{U}{L} \right] + \epsilon_1 \sigma \sqrt{\Delta t} < \left( \ln \frac{U}{L} \right) \left( 1 - \frac{1}{2n} \right) \\
- \frac{1}{2n} \ln \frac{U}{L} &< 2c - \left[ \ln \frac{L}{S_0} + \frac{1}{2n} \ln \frac{U}{L} \right] + (\epsilon_1 + \epsilon_2) \sigma \sqrt{\Delta t} < \left( \ln \frac{U}{L} \right) \left( 1 - \frac{1}{2n} \right) \\
\ldots \\
- \frac{1}{2n} \ln \frac{U}{L} &< mc - \left[ \ln \frac{L}{S_0} + \frac{1}{2n} \ln \frac{U}{L} \right] + (\epsilon_1 + \ldots + \epsilon_m) \sigma \sqrt{\Delta t} < \left( \ln \frac{U}{L} \right) \left( 1 - \frac{1}{2n} \right).
\end{align*}
\]

Setting \( cc = c - \left[ \ln \frac{L}{S_0} + \frac{1}{2n} \ln \frac{U}{T} \right] \) we have

\[
\begin{align*}
- \frac{1}{2n} \ln \frac{U}{L} &< cc + \epsilon_1 \sigma \sqrt{\Delta t} < \left( \ln \frac{U}{L} \right) \left( 1 - \frac{1}{2n} \right) \\
- \frac{1}{2n} \ln \frac{U}{L} &< cc + c + (\epsilon_1 + \epsilon_2) \sigma \sqrt{\Delta t} < \left( \ln \frac{U}{L} \right) \left( 1 - \frac{1}{2n} \right) \\
\ldots \\
- \frac{1}{2n} \ln \frac{U}{L} &< cc + (m - 1)c + (\epsilon_1 + \ldots + \epsilon_m) \sigma \sqrt{\Delta t} < \left( \ln \frac{U}{L} \right) \left( 1 - \frac{1}{2n} \right).
\end{align*}
\]

Let \( \eta_i = cc + \epsilon_1 \sigma \sqrt{\Delta t}, \) \( \eta_i = c + \epsilon_i \sigma \sqrt{\Delta t}, \) \( i = 2, \ldots, m \) and then \( \epsilon_i = \frac{\eta_i - cc}{\sigma \sqrt{\Delta t}}, \epsilon_i = \frac{\eta_i - c}{\sigma \sqrt{\Delta t}}, i = 2, \ldots, m. \) Setting \( d = \frac{1}{n} \ln \frac{U}{T} > 0 \) we obtain:

\[
\begin{align*}
\frac{d}{2} < \eta_1 < \left( n - \frac{1}{2} \right) d \\
\frac{d}{2} < \eta_1 + \eta_2 < \left( n - \frac{1}{2} \right) d \\
\ldots \\
\frac{d}{2} < \eta_1 + \eta_2 + \ldots + \eta_m < \left( n - \frac{1}{2} \right) d.
\end{align*}
\]

We have \( E(\eta_1) = cc + E(\epsilon_1) \sigma \sqrt{\Delta t} = cc \) and \( E(\eta_i) = c, i = 2, \ldots, m \) because \( \epsilon_i \in N(0, 1), i = 1, \ldots, m. \)\(^5\)

Then for the variances \( D(\eta_1), \eta_2, \ldots, \eta_m, i = 1, \ldots, m, \) is true that \( D(\eta_i) = D(\epsilon)(\sigma \sqrt{\Delta t})^2 = \sigma^2 \Delta t. \) And the density of \( \eta_i \) are:

\[
p_i(x) = \frac{1}{\sqrt{2\pi}D(\eta_i)} e^{-\frac{(x-\eta_i)^2}{2D(\eta_i)^2}} = \begin{cases} 
\frac{1}{\sigma \sqrt{2\pi} \Delta t} e^{-\frac{(x-\eta_i)^2}{2\sigma^2 \Delta t}} & \text{for } i = 2, \ldots, m, \\
\frac{1}{\sigma \sqrt{2\pi} \Delta t} e^{-\frac{(x-\eta_i)^2}{2\sigma^2 \Delta t}} & \text{for } i = 1.
\end{cases}
\]

Then we replace the continuous random variable \( \eta_i \) with a discrete one. Taking in mind the indicator \( 1_{A_i} \) in \((5)\), we are interested only in the following probabilities:

\[
\begin{align*}
\eta_0 &= P \left( \frac{d}{2} < \eta_1 < \frac{d}{2} \right) \\
\eta_1 &= P \left( \left( 1 - \frac{1}{2} \right) d < \eta_1 < \left( 1 + \frac{1}{2} \right) d \right) \\
\ldots \\
\eta_k &= P \left( \left( k - \frac{1}{2} \right) d < \eta_1 < \left( k + \frac{1}{2} \right) d \right) = \frac{1}{\sigma \sqrt{2\pi} t} \int_{(k-\frac{1}{2})d}^{(k+\frac{1}{2})d} e^{-\frac{(x-\eta_i)^2}{2\sigma^2 \Delta t}} dx \\
\ldots \\
\eta_{n-1} &= P \left( \left( n - \frac{1}{2} \right) d < \eta_1 < \left( n - 1 + \frac{1}{2} \right) d \right).
\end{align*}
\]

\(^4\) By adding the term \( \frac{1}{2n} \ln \frac{U}{T} \), we adjust the quadrature solution \((6)\) to be approximated numerically in \( n \) points that could be \( n = 100, n = 200. \)

Respectively, one may think that the error of algorithm depends directly on the value of this term, i.e. the more distant are the two barriers \( L \) and \( U \) the bigger is the error. However, it should be noted that experimentally the dependence of the error and the number of grid points is negligible.

\(^5\) Each of the random variables \( \epsilon_i \) is normally distributed with mean 0 and variance 1.
Thus, we could define the following random variable \( \eta_1 + \eta_2 \):
\[
\eta_1 + \eta_2 = \left( \begin{array}{cccccc}
\ldots & 0 & d & 2d & \ldots & (n-1)d \\
\ldots & b_0 & b_1 & b_2 & \ldots & b_{n-1}
\end{array} \right).
\]

We assume \( b_i := p_i \) for \( i = 0, 1, \ldots, n-1 \) and we have
\[
\eta_1 + \eta_2 := \left( \begin{array}{cccccc}
\ldots & 0 & d & 2d & \ldots & (n-1)d \\
\ldots & p_0 & p_1 & p_2 & \ldots & p_{n-1}
\end{array} \right).
\]

Applying the upper algorithm we find the random variable \( (\eta_1 + \eta_2) + \eta_3 \), and analogously, the random variable \( \eta = \eta_1 + \eta_2 + \cdots + \eta_m \), with
\[
\eta := \left( \begin{array}{cccccc}
\ldots & 0 & d & 2d & \ldots & (n-1)d \\
\ldots & p_0 & p_1 & p_2 & \ldots & p_{n-1}
\end{array} \right).
\]

\footnote{In case of lookback options, similar but more complex idea is used in [17].}
We have that: \( S_T = S_T = S_{m, T} = S_0 e^{mc + (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m) \sigma \sqrt{\Delta t}} \), and 

\[
\eta = \eta_1 + \eta_2 + \cdots + \eta_m = -\left[ \frac{L}{S_0} + \frac{1}{2 n} \ln \frac{U}{L} \right] + mc + (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m) \sigma \sqrt{\Delta t}.
\]

Then 

\[
S_T = S_0 e^{n \left[ \frac{L}{S_0} + \frac{1}{2 n} \ln \frac{U}{T} \right]} = L e^{0 + \frac{1}{2 n} \ln \frac{U}{T}} = e^{\Delta t + \frac{1}{2 n} \ln \frac{U}{T}} = e^{(i + \frac{1}{2}) d}
\]

using that \( d = \frac{1}{n} \ln \frac{U}{T} \). Thus, finally, we obtain the following value for (5):

\[
e^{-r T} \sum_{j=0}^{n-1} p_j \max \left( 0, Le^{(i + \frac{1}{2}) d} - K \right) = e^{-r T} \sum_{j=0}^{n-1} p_j \left( Le^{(i + \frac{1}{2}) d} - K \right),
\]

where \( j_0 = \left\lceil \frac{1}{d} \ln \frac{T}{a} + \frac{1}{2} \right\rceil \) is evaluated by the function \([x]\).\(^7\) Finally, the value of discrete double barrier knock-out call option is:

\[
\hat{V}(S, t) = e^{-r T} \sum_{j=0}^{n-1} p_j \left( Le^{(i + \frac{1}{2}) d} - K \right).
\]

(11)

Analogously, the value of discrete double barrier knock-out put option is:

\[
e^{-r T} \sum_{j=0}^{n-1} p_j \max \left( 0, K - Le^{(i + \frac{1}{2}) d} \right) = e^{-r T} \sum_{j=0}^{n-1} p_j \left( K - Le^{(i + \frac{1}{2}) d} \right).
\]

(12)

**Estimation of the error.** If we denote the price of the discrete double barrier knock-out call option with \( V(S, t) \) given by formula (6) and with \( \hat{V}(S, t) \) the value (11) obtained by using the proposed numerical algorithm applied for \( n \) discrete points the error could be estimated as:

\[
V(S, T) - \hat{V}(S, T) = O \left( \frac{1}{n} \right)
\]

and thus a desired level of accuracy is very fast achieved.\(^8\)

**Proof.** We will prove the relation (13) using mathematical induction for \( m \). First, for discrete double barrier knock-out call option we will make comparison of \( V(S, T) \) and \( \hat{V}(S, t) \) for two observations \( m = 2 \) in the interval \([0; T]\).

We set \( a = \left( r - \frac{\sigma^2}{2} \right) \Delta t, aa = \ln \frac{S_0}{U} + \left( r - \frac{\sigma^2}{2} \right) \Delta t, \eta_1 = aa + \xi_1 \sigma \sqrt{\Delta t}, \eta_2 = a + \xi_2 \sigma \sqrt{\Delta t} \), where \( \xi_{1, 2} \in N(0, 1) \), i.e. \( \xi_{1, 2} \) are normally distributed.

\[
(*) \quad \begin{cases} 0 < \eta_1 < \ln \frac{U}{L}, & E(\eta_1) = aa, E(\eta_2) = a \\ 0 < \eta_1 + \eta_2 < \ln \frac{U}{L}, & D = D(\eta_1) = D(\eta_2) = \sigma^2 \Delta t. \end{cases}
\]

We have \( S_{2, T} = Le^{\eta_1 + \eta_2} = S_0 e^{\left( r - \frac{\sigma^2}{2} \right) 2 \Delta t + (\xi_1 + \xi_2) \sigma \sqrt{\Delta t}} \) and the density \( p(x, y) = \frac{1}{2 \pi \sigma^2} e^{\frac{(x-a)^2 + (y-b)^2}{2 \sigma^2}} \). Then the exact value of the option is:

\[
V(S, T) = \frac{e^{-r T}}{2 \pi D} \int_G \max(Le^{x+y} - K, 0) e^{\frac{(x-a)^2 + (y-b)^2}{2 \sigma^2}} \ dx dy, \quad \text{where} \ G : \begin{cases} 0 < x < \ln \frac{U}{L} \\ 0 < x + y < \ln \frac{U}{L}. \end{cases}
\]

(14)

Following the presented algorithm, in (*) we substitute the continuous random variables \( \eta_1 \) and \( \eta_2 \) respectively with discrete ‘normally distributed’ random variables \( \eta_1 \in (0, kd) \) and \( \eta_2 \in (-kd, kd) \), where \( k = 0, 1, \ldots, n - 1 \) and we should estimate the probabilities \( p_i, q_j \) and \( b_j = \sum_{i=0}^{n-1} p_i q_{j-i} \) for \( i, j = 0, \ldots, n - 1 \).

\(^7\) We mean \([x]\) the biggest integer number smaller or equal to the real number \( x \).

\(^8\) The error of the Monte Carlo simulation is \( O \left( \frac{1}{\sqrt{M}} \right) \), where \( M \) is the number of simulations. Such a low rate of convergence is not quite desirable. [3]
Thus the numerical value of discrete double barrier knock-out call option is:

\[
\hat{V}(S, T) = e^{-rT} \sum_{j=0}^{n-1} \max(Le^{(j+\frac{1}{2})d} - K, 0)b_j
\]

\[
= \frac{e^{-rT}}{2\pi D} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} \max(Le^{(i+\frac{1}{2})d} - K, 0) \int_{id}^{(i+1)d} \left( \int_{ij-(i+\frac{1}{2})d}^{ij+(i+\frac{1}{2})d} e^{-\frac{(x-a)^2}{2\sigma^2}} \, dx \right) \, dy.
\]

(15)

Let denote the following squares with side \(d\) by:

\[G_j = \begin{cases} 
  \text{id} < x < (i+1)d, & i = 0, 1, \ldots, n-1 \\
  \left( j - \frac{1}{2} \right) d < y < \left( j + \frac{1}{2} \right) d, & j = 0, 1, \ldots, n-1 
\end{cases} \]

\[H_i^0 = \begin{cases} 
  \text{id} < x < (i+1)d, & \text{with a diagonal } x + y = 0 \\
  -(i+1)d < y < -id
\end{cases} \]

\[H_i = \begin{cases} 
  \text{id} < x < (i+1)d, & \text{with a diagonal } x + y = nd = \frac{U}{L} \\
  (n-i-1)d < y < (n-i)d
\end{cases} \]

Then for the following region \(G\) is true:

\[
\bigcup_{i=0}^{n-1} \left( \bigcup_{j=0}^{n-1} G_j \cup \left( H_i^0 \cup H_i \right) \right) \subseteq G \subseteq \bigcup_{i=0}^{n-1} \left( \bigcup_{j=0}^{n-1} G_j \cup \left( H_i^0 \cup H_i \right) \right).
\]

(16)

Let denote with \(VV(S, T)\) the following expression

\[
VV(S, T) = \frac{e^{-rT}}{2\pi D} \sum_{j=0}^{n-1} \int_{G_j} \max(Le^{x+y} - K, 0) e^{-\frac{(x-a)^2 + (y-a)^2}{2\sigma^2}} \, dx \, dy.
\]

From (14), having in mind (15) we obtain that:

\[
V(S, T) \leq VV(S, T) + \frac{e^{-rT}}{2\pi D} \sum_{i=0}^{n-1} \int_{H_i^0 \cup H_i} \max(Le^{x+y} - K, 0) e^{-\frac{(x-a)^2 + (y-a)^2}{2\sigma^2}} \, dx \, dy
\]

\[
V(S, T) \geq VV(S, T) - \frac{e^{-rT}}{2\pi D} \sum_{i=0}^{n-1} \int_{H_i^0 \cup H_i} \max(Le^{x+y} - K, 0) e^{-\frac{(x-a)^2 + (y-a)^2}{2\sigma^2}} \, dx \, dy.
\]

Using relation (16) we prove that

\[
|V(S, T) - VV(S, T)| \leq \frac{e^{-rT}}{\sqrt{2\pi D}} \frac{U - K}{n} \ln \frac{U}{L} = O \left( \frac{1}{n} \right).
\]

(17)

Then to prove relation (13), it is sufficient to estimate:

\[
VV(S, T) - \hat{V}(S, T) = \frac{e^{-rT}}{2\pi D} \sum_{j=0}^{n-1} \int_{G_j} \left( e^{-\frac{(x-a)^2}{2\sigma^2}} \right) \left( \max(Le^{x+y} - K, 0) - \max(Le^{(j+\frac{1}{2})d} - K, 0) \right) \, dx \, dy.
\]

Thus using relation (16), it follows that

\[
|VV(S, T) - \hat{V}(S, T)| \leq \frac{C}{n} = O \left( \frac{1}{n} \right)
\]

(18)

where \(C\) is some positive constant.

Finally, from (17) and (18) it follows relation (13) because we have that

\[
|V(S, T) - \hat{V}(S, T)| \leq |V(S, T) - VV(S, T)| + |VV(S, T) - \hat{V}(S, T)| \leq \frac{C}{n}.
\]

Second, we admit that relation (13) is fulfilled for \(m = k\).

Third, the relation (13) for \(m = k + 1\) is proved similarly the case \(m = 2.\)

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The main aim of the paper is to present the algorithm for numerical valuation of pricing formula (6) and for brevity we have only sketched out the proof. Of course, having in mind relation (16), we admit that probably exist alternative ways for proving relation (13).
The low rate of convergence of the Monte Carlo simulations is not quite desirable, [3]. The application of the quadrature method is not always quick from a computational point of view, [7]. For pricing accurately discrete double barrier knock-out options finite difference schemes require a prohibitively small time-step, [12]. The presented algorithm is better than all of the three previously mentioned methods because it turns out to be more efficient in accuracy and speed. One reason for that is that the numerical solution \( V(S, T) \) consists of only a finite sum and a desired accuracy is very quickly achieved. For standard double barrier options monitored continuously, Li expresses the solution in general as summation of an infinite number of normal distribution functions, and in many non-trivial cases the solution consists of only finite terms, [4].

**Put-call parity for discrete double barrier options**

**Definition 3.1.** A discrete double-no-touch option is an option with a payoff condition equal to 1 which expires worthless if before the maturity the asset price has fallen outside the barrier corridor \([L, U]\) at the prefixed monitoring dates: at these dates the option becomes zero if the asset falls out of the corridor. If one of the barriers is touched by the asset price at the prefixed dates then the option is canceled, i.e. it becomes zero.

Thus, using the results (11) and (12), an identity similar to the famous put-call parity could be derived trivially and it illustrates the relationship between different kinds of double barrier options, i.e. the following theorem:

**Theorem 3.1.** Let \( V_{\text{doc}}(S, t, K) \), \( V_{\text{dop}}(S, t, K) \) and \( V_{\text{dnt}}(S, t, K) \) denote the value of discrete double barrier knock-out call, put option and double-no-touch options, respectively with strike price \( K \) and barriers \( L \) and \( U \). Then for all \( t, S \) is true:

\[
V_{\text{doc}}(S, t, K) - V_{\text{dop}}(S, t, K) = V_{\text{doc}}(S, t, L) + (L - K) V_{\text{dnt}}(S, t) \tag{19}
\]

\[
V_{\text{doc}}(S, t, K) - V_{\text{dop}}(S, t, K) = V_{\text{dop}}(S, t, U) + (K - U) V_{\text{dnt}}(S, t) \tag{20}
\]

**Proof.** For the left part of (19) from formulas (11) and (12) we obtain:

\[
V_{\text{doc}}(S, t, K) - V_{\text{dop}}(S, t, K) = e^{-r t} \sum_{j=0}^{n-1} p_j \left( L \left( e^{j+\frac{1}{2}d} - d \right) - K \right). \tag{21}
\]

Presenting in the brackets \(-K = -L + (L - K)\) we re-write (21) as

\[
V_{\text{doc}}(S, t, K) - V_{\text{dop}}(S, t, K) = e^{-r t} \sum_{j=0}^{n-1} p_j \left( L \left( e^{j+\frac{1}{2}d} - d \right) - L \right) + (L - K) V_{\text{dnt}}(S, t).
\]

Using formula (11) in case of discrete double barrier knock-out call option with strike price \( L \), relation (19) is trivial to be concluded. The second relation (20) of Theorem 3.1 is proved in an analogous way using formulas (11) and (12), and presenting \( K = U + (K - U) \) in Eq. (21).

Thus, we demonstrate that our formulas for discrete double barrier call and put options permit obtaining alternatively the relations of Theorem 3.1 that have been proved by using standard arbitrage arguments. \( \square \)

4. Numerical results

In this section we present numerical results. We apply our numerical algorithm to the most explored examples in literature for discrete barrier options that are discretely monitored daily and weekly. We have arranged the results in such order that the distance of the two barriers is increased with each subsequent example. This allows us to observe the numerical error that depends on the number of the discretization points between the barriers.

Formally, there are three examples respectively when the two barriers are \( L = 95 \) and \( U = 110 \), as it is explored in [21], when \( L = 95 \) and \( U = 125 \) used in [20,22], and the third case is when \( L = 95 \) and \( U = 140 \), [13].

We have presented the computational time of our analytical method in order to justify its efficiency. It should be noted that the method has simple computer implementations and permits also observing of the entire life of the option for a reasonable time, i.e we could evaluate the option price simultaneously for different values of the underlying asset price. This property is characterized for the finite difference approach used in [20,11].

We have compared the results with those obtained by other numerical methods in Finance such as the Monte Carlo simulations,\(^{10}\) trinomial trees of Cheuk and Vorst, [9], the Crank–Nicolson method applied in [11,21].

In the second example, i.e. when \( L = 95 \) and \( U = 125 \), we have compared our results with the finite difference scheme of Zvan, [22], and the implicit scheme applied in [20], denoted with HOBIS, where the computational domain is adjusted in advance under some probabilistic hypothesis for the boundaries.

\(^{10}\) We have implemented the Monte Carlo algorithm for pricing two barriers using similar code to that of [23]. In case of one barrier we use the Internal Report done in [24].
In the third example, i.e. when \( L = 95 \) and \( U = 140 \), we compared the result presented in [13], and the Crank–Nicolson method and we should note that in case the barriers are far away from each other the finite difference schemes are able to give satisfactory results (see Table 5) because the error produced by the knock-out clause is damped out quickly, [11].

The accuracy of the quadrature method depends directly from the number of points composing the grid, [7], and often it turns out to be expensive from a computational point of view. However, experimentally, our numerical algorithm does not suffer from this drawback because a small number of grid points (200 or 400) are necessary to achieve highly accurate results.

As we have mentioned in Section 3, we adjust the quadrature solution (6) to be approximated numerically in \( n \) points that could be \( n = 100, n = 200 \). Respectively, one may think that the error of algorithm depends directly on the value of this term, i.e. the more distant are the two barriers \( L \) and \( U \) the bigger is the error. However, it should be noted that experimentally the dependence of the error and the number of grid points is negligible.

For example, in the first example, i.e. when \( L = 95 \) and \( U = 110 \), the numerical results applied with \( N = 200 \) and \( N = 1000 \) grid points differ in the sixth decimal point. The absolute error of our algorithm for \( N = 1000 \) points and the Monte Carlo simulation for \( 10^7 \)-asset paths is the fourth decimal point, see Table 1. Thus, in this case it is enough to be used \( N = 200 \) points in the algorithm.

Increasing the barrier observation frequency from 5 to 25 or 125 times the computational results are not deteriorated as it usually happens with other numerical methods such as finite difference schemes and trinomial trees.

For such a big number of monitoring dates and close barriers obtaining quickly high order accurate results for the option value is not a trivial task for values of the underlying asset close or at the barriers, [21]. The computational time is more than satisfactory, see Table 2. The strength of the present numerical algorithm is particularly demonstrated in valuation of discretely monitored double barrier knock-out options. In the second example, \( N = 400 \) points are enough for obtaining highly quickly accurate results, see Tables 3–4.

In the third example, the distance of the barriers is significant but this is not an obstacle for weekly and daily monitoring, see Table 5. This example leads to the important arguments because it shows that the present numerical algorithm could value also a single barrier down-and-out call option.

In this case the numerical results are compared with the analytical formula of Fusai in [14], i.e. solution of Winer–Hopf integral equation, with the Monte Carlo method with \( 10^3 \) simulations with Mersenne twister pseudorandom generator and antithetic variables in [24].
This phenomenon could be explained by the fact that most of the underlying values that have financial meaning are in the area $[0, 2K]$ where $K$ is the strike price. In fact, in the finite difference approach an usual practice is the computational domain to be truncated at a calculative value sufficiently large such that the computed values are not appreciably affected, [11]. Similarly for our method, if the upper barrier is chosen sufficiently bigger than the strike price it is somehow ‘invisible to the numerical algorithm’. Thus, choosing the upper barrier $U$ such that $U \geq 2K$ is sufficient condition for successful application of the present method for the valuation of a single down-and-out call option, see Tables 6–7. Here we have chosen $U = 2K$.

This fact could be experimentally observed also using the third example when the two barriers are $L = 95$ and $U = 140$. When the upper barrier $U$ takes bigger values, i.e. it is moved higher and higher, at some moment the computational results for the value of the discrete double barrier knock-out options would not change significantly when $U$ is sufficiently large.

In addition, we have compared our results with the these obtained by the tridiagonal probability algorithm proposed in [16], see Table 6. He has also used the quadrature approach but applies different numerical procedure for approximation the value of the multi-dimensional integral that represents the value of option. It is interesting to be noted that his formula differs from our formula (6) because it consists of nested integrals that have infinite upper limits. This fact to some extent confirms the previously discussed phenomena for the invisibility of the upper barrier in case having large values.
The results of the presented algorithm are very close to those of the Monte Carlo report in [24, Tables 6–7]. The accuracy of the results in Table 6 could be compared also by other methods such as the Numerical Recursive Integration of AitSahlia, [25], or the perturbative moment approach of Airoldi, see [7].

A comparison of the presented numerical algorithm with the quadrature method introduced in [10] confirms the high accuracy of our results, see Table 8. ($K_m$ represents the number steps between the barriers).

The obtained values of the proposed numerical algorithm for $N = 200$ and $N = 400$ points are generally the same to the 6th decimal place and in case of high monitoring frequency ($m = 125$) we have accurate results at least to the 4th decimal point for $N = 200$ points, see Table 8.

### 5. Conclusions

The advantage of the presented algorithm is that it has a simple computer implementation and turns out to be very efficient in accuracy and speed for valuation of discrete double barrier knock-out options. This makes it a more competitive

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11 The computations and presented times are done on Pentium IV, Core Duo, 1.8 MHz system, 1 MB RAM, using Maple 6 for implementation of the proposed algorithm.
method than frequently described methods in Finance such as the quadrature method, the Monte Carlo simulations and the Crank–Nicolson scheme. One more advantage of the algorithm is that it permits observing the entire life of the option that is a characteristic feature of the finite difference schemes. The presented algorithm works successfully both for one and double barrier knock-out options that are monitored discretely.

References

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