LINS–MANDEL CRYSTALIZATIONS

Alberto CAVICCHIOLI*

Istituto Matematico "G. Vitali" della Università di Modena, 41100 Modena, Italy

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We prove that each closed connected 3-manifold admits particular crystallizations with nice properties. As a consequence all 3-manifolds can be represented by a positive integer $b$ and a fixed-point-free involutory permutation on $\mathbb{Z}_b \times \mathbb{Z}_c$. Other 3-manifold representations are also obtained. Finally, a family of closed connected 3-manifolds is studied.

1. Preliminaries

By $\Delta_n$ (resp. $N_n$) we will denote the set $\{0, 1, \ldots, n\}$ (resp. $\Delta_n - \{0\}$). If $X$ is an arbitrary set, $\#X$ means the cardinality of $X$. All spaces and maps will be in the PL (piecewise-linear) category in the sense of [9] or [15]. The prefix PL will always be omitted. All the spaces that we consider will be compact. For basic graph theory see [8], [6]. The term graph will always be used instead of multigraph (without loops). Given a graph $\Gamma$, $V(\Gamma)$ (resp. $E(\Gamma)$) is the set of vertices (resp. edges) of $\Gamma$. An edge-coloration on $\Gamma$ is defined to be a map $\gamma: E(\Gamma) \rightarrow \mathcal{C}$ ($\mathcal{C}$ being an arbitrary set, called the colour-set) such that $\gamma(e_1) \neq \gamma(e_2)$ for any two adjacent edges $e_1, e_2$. An $(n + 1)$-coloured graph $(\Gamma, \gamma)$ is a connected graph $\Gamma = (V(\Gamma), E(\Gamma))$ regular of degree $n + 1$, endowed with an edge-coloration $\gamma: E(\Gamma) \rightarrow \Delta_n$. A colour-isomorphism between two $(n + 1)$-coloured graphs $(\Gamma, \gamma)$ and $(\Gamma^*, \gamma^*)$ is a pair $(\psi, \phi)$, where $\psi$ is a permutation on $\Delta_n$ and $\phi: \Gamma \rightarrow \Gamma^*$ is a graph-isomorphism such that $\psi \circ \gamma = \gamma^* \circ \phi$. For every subset $\mathcal{B}$ of $\Delta_n$, we set $\Gamma_{\mathcal{B}} = (V(\Gamma), \gamma^{-1}(\mathcal{B}))$. Each connected component of $\Gamma_{\mathcal{B}}$ is said to be $\mathcal{B}$-coloured. The cardinality of $\Gamma_{\mathcal{B}}$ will be denoted by $g_{\mathcal{B}}$. If $\mathcal{B} = \{c\}$, $g_{\mathcal{B}}$ will also be written $g_c$. $C_{ij}$ will denote a $\{i, j\}$-coloured connected component of $\Gamma_{\{i,j\}}$, for $i, j \in \Delta_n$ and $i \neq j$. We shall call the connected components of $\Gamma_{\{i\}}$ also $i$-coloured edges, for $i \in \Delta_n$. For every $i \in \Delta_n$, we set $\hat{i} = \Delta_n - \{i\}$. $(\Gamma, \gamma)$ is said to be contracted if $\Gamma_{\hat{i}}$ is connected for each $i \in \Delta_n$.

A pseudocomplex $K$ is a principal ball complex in which each $h$-ball is abstractly isomorphic with a $h$-simplex (see [11]). We shall call the balls of a pseudocomplex also simplexes. An $n$-dimensional pseudocomplex $K$ is said to be contracted if the

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set of its 0-simplexes (vertices) has cardinality $n + 1$. A \textit{contracted triangulation} of a polyhedron $P$ is a pair $(K, f)$, where $K$ is a contracted pseudocomplex and $f: |K| \rightarrow P$ is a homeomorphism.

Given an $(n + 1)$-coloured graph $(\Gamma, \gamma)$, the associated pseudocomplex $K(\Gamma)$ is defined by the following rules:

1. Take an $n$-simplex $\sigma^n(v)$, for each $v \in V(\Gamma)$ and label its vertices by $\Delta_n$;
2. If $v, w \in V(\Gamma)$ are joined by an $i$-coloured edge, then identify the $(n-1)$-faces of $\sigma^n(v)$ and $\sigma^n(w)$ opposite to the vertices labelled by $i$, so that equally labelled vertices are identified.

The graph $(\Gamma, \gamma)$ will be said to represent $|K(\Gamma)|$ and every homeomorphic space. A contracted graph representing a closed connected $n$-manifold $M$ is said to be a \textit{crystallization} of $M$. Every closed connected $n$-manifold $M$ admits a crystallization [13]. For a general survey on crystallizations see [5].

Recall now some definitions listed in [2]. Let $(\Gamma, \gamma)$ be a 4-coloured graph with colour set $\Delta_3$ and let $E_k(\Gamma)$ be the set of all $k$-coloured edges of $\Gamma$ ($k \in \Delta_3$). For each triple $(i, j, k)$ of distinct colours in $\Delta_3$ and each component $C_{ij}$ of $\Gamma_{(i,j)}$, we define:

- $\mathcal{C}_k(C_{ij}) = \{\alpha \in E_k(\Gamma): \text{the end-points of } \alpha \text{ do not lie in } C_{ij}\};$
- $\mathcal{R}_k(C_{ij}) = \{\alpha \in E_k(\Gamma): \text{the end-points of } \alpha \text{ lie in } C_{ij}\};$
- $\mathcal{A}_k(C_{ij}) = E_k(\Gamma) - (\mathcal{C}_k(C_{ij}) \cup \mathcal{R}_k(C_{ij})).$

$(\Gamma, \gamma)$ is said to be $(0, 1, 2)$ $p$-normal (partially normal) if $\Gamma_3$ represents the 2-sphere and there exists a connected component $C_{01}$ such that $\mathcal{A}_2(C_{01}) = E_2(\Gamma)$. $C_{01}$ will be called a \textit{base component}. Fix a connected component $C_{01}$ and consider the regular imbedding of $\Gamma_3$ in the standard 2-sphere $S^2$ (unique, up to isotopy) such that $C_{01}$ is mapped on the equatorial line and $\Gamma_3$ is imbedded in the upper hemisphere $\Sigma$ of $S^2$. By orthogonally projecting $\Sigma$ on the equatorial plane, we get a planar representation $\omega$ of $\Gamma_3$ such that all the other components of $\Gamma_{(0, 1)}$ are contained in the 2-cell $D$ bordered by $C_{01}$. $C_{01}$ will be called the \textit{external component} of $\Gamma_{(0, 1)}$ in $\omega$; the remaining components of $\Gamma_{(0, 1)}$ will be called \textit{internal}. By abuse of language, we shall confuse $\omega$ with $\Gamma_3$. For each ordered triple $(i, j, k)$ of distinct colours in $\Delta_3$, $n$ vertices of a connected component of $\Gamma_{(i,j)}$ are said to be $k$-consecutive if there exists an ordering $(v_1, v_2, \ldots, v_n)$ of these vertices such that the $k$-coloured edges $e_s, e_{s+1}$ containing $v_s, v_{s+1}$ ($s \in N_{n-1}$) respectively belong to a connected component of $\Gamma_{(i,k)}$ (or $\Gamma_{(j,k)}$) of order four. The vertices $v_1, v_n$ are said to be the \textit{end-points} of the $n$-tuple. Two vertices of $V(\Gamma)$ are said to be $k$-corresponding if there is a $k$-coloured edge containing both of them. A $(0, 1, 2)$ $p$-normal graph $(\Gamma, \gamma)$ with base component $C_{01}$ is said to be $(0, 1, 2)$-normal if there exist an ordering $C^1, C^2, \ldots, C^s$ of the internal components of $\Gamma_{(0, 1)}$ and an ordering $(\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{2h_1}; \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{2h_2}; \ldots; \bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{2h_s})$ of $V(C_{01})$ inducing one of the orientations of the cycle $C_{01}$ such that:

1. the vertices of $C^s$ are 2-consecutive ($s \in N_k$);
2. $\#V(C^s) = \#V(C^{s+1})$ ($s \in N_{k-1}$);
(3) \( v_i^t \in V(C^s) \) \((s \in N_g; t \in N_{2g})\), \( v_i^t \) being the 2-correspondent of \( v_i^s \).

Each closed connected 3-manifold \( M \) admits a \((0, 1, 2)\)-normal crystallization (see [2]).

2. Edge-augmented graphs and dipoles

Let \((\Gamma, \gamma)\) be a 4-coloured graph. If \((i, j, h, k)\) is an arbitrary permutation of \(\Delta_3\), let \(\alpha, \beta\) be two edges of a connected component \(C_{ij}\) of \(\Gamma\) such that \(\gamma(\alpha) = i, \gamma(\beta) = j\). A 4-coloured graph \((\Gamma'', \gamma'')\) is said to be obtained from \((\Gamma, \gamma)\) by adding a dipole \(\theta\) through \(\alpha\) and \(\beta\) if it is obtained from \((\Gamma, \gamma)\) in the following way:

(a) Delete \(\alpha\) and \(\beta\); if \(C', C''\) are the two components into which \(C_{ij}\) splits, let \(\alpha(0), \beta(0)\) be the vertices of \(C'\) and \(\alpha(1), \beta(1)\) those of \(C''\).

(b) Add two distinct vertices \(x, y\) not belonging to \(V(\Gamma)\) and join them by two distinct edges coloured by the colours \(h, k\).

(c) Join \(x\) with \(\alpha(0), y\) with \(\alpha(1)\) by an \(i\)-coloured edge and join \(x\) with \(\beta(0), y\) with \(\beta(1)\) by a \(j\)-coloured edge. Note that, if \((\Gamma, \gamma)\) is a crystallization of a 3-manifold \(M\), the graph \((\Gamma'', \gamma'')\) previously defined is again a crystallization of \(M\) since it is obtained from \((\Gamma, \gamma)\) by adding a dipole of type 2 (see [4]). This definition, introduced in [2], is included to make the reading of the next propositions clear. We shall also need a new move on \((\Gamma, \gamma)\) which is not defined in the quoted papers. If \((i, j, h, k)\) is an arbitrary permutation of \(\Delta_3\), let \(\alpha, \beta\) be two \(i\)-coloured edges of a connected component \(C_{ij}\) of \(\Gamma\) and let \(\delta\) be the \(h\)-coloured edge incident to \(\beta(0)\). Suppose that \(\alpha, \beta\) do not belong to the same \(\{i, k\}\)-coloured connected component of \(\Gamma\). We define a 4-coloured graph \((\Gamma'', \gamma'')\), called the edge-augmented graph of \((\Gamma, \gamma)\) through the triple \((\alpha, \beta, \delta)\), by the following rules (see Fig. 1.):

(1) Delete \(\alpha, \beta, \delta\). Let \(C_0, C_1\) be the two components into which \(C_{ij}\) splits such that \(\alpha(0), \beta(0) = \delta(0)\) (resp. \(\alpha(1), \beta(1)\)) are the end-points of \(C_0\) (resp. \(C_1\)).

(2) Add four distinct vertices \(x_0, x_1, y_0, y_1\) not belonging to \(V(\Gamma)\) and join \(x, x_1\) with \(y, y_1\) with \(y_0\) by a \(k\)-coloured edge.

(3) Join \(x_0\) with \(x_1\) by two distinct edges coloured by the colours \(j, h\) and join \(y_0\) with \(y_1\) by a \(j\)-coloured edge.

(4) Join \(\alpha(r)\) with \(x_r, \beta(r)\) with \(y_r\) (\(r \in \Delta_3\)) by an \(i\)-coloured edge and join \(y_0\) with \(\beta(0) = \delta(0), y_1\) with \(\delta(1)\) by a \(h\)-coloured edge.

**Proposition 1.** With the above notation, let \((\Gamma, \gamma)\) be a crystallization of a closed 3-manifold \(M\). The edge-augmented graph \((\Gamma'', \gamma'')\) of \((\Gamma, \gamma)\) is again a crystallization of \(M\).

**Proof.** The partial subgraph \(\theta\) of \(\Gamma''\), spanned by the vertices \(x_0, x_1\), is a (non-degenerate) dipole of type 2. The two edges of \(\theta\) are coloured by the colours

\[1\) If \(\alpha\) is any edge of \(\Gamma\), we also denote its vertices by \(\alpha(0)\) and \(\alpha(1)\).]
Fig. 1. The edge-augmented graph \((\Gamma', \gamma')\) of \((\Gamma, \gamma)\) through the triple \((\alpha, \beta, \delta)\).
3. 3-manifold representations

Recall that each closed connected 3-manifold $M$ admits a $(0, 1, 2)$-normal crystallization ([2, Proposition 10]). This allows a representation of $M$ by a triple $(n, \lambda, \tau)$, where $n$ is a positive integer, $\lambda = \{h_1 \geq \cdots \geq h_k\}$ a partition of $n$ and $\tau$ a fixed-point-free involutory permutation on $\mathbb{N}_{4n}$ ([2, Proposition 12]). In the present section we improve these statements by making use of the concept of edge-augmented graph previously introduced.

**Proposition 2.** Let $M$ be a closed connected 3-manifold.

(a) $M$ admits a $(0, 1, 2)$-normal crystallization $(\Gamma, \gamma)$ such that $\#V(C^i) = \#V(C') = 2h_j$, for each $i, j \in \mathbb{N}_k$ ($i \neq j$), $C^i$ (or $C'$) and $g$ respectively being an internal component of $\Gamma_{(0,1)}$ and the Heegaard genus of $M$.

(b) $M$ admits a $(0, 1, 2)$-normal crystallization $(\bar{\Gamma}, \bar{\gamma})$ such that $\#V(\bar{C}^i) = 4$, for each $i \in \mathbb{N}_k$, $\bar{C}^i$ and $k$ being an internal component of $\bar{\Gamma}_{(0,1)}$ and a suitable positive integer respectively.

Let $\mathcal{G}_n$ be the set of the $(0, 1, 2)$-normal graphs of order $4n$ satisfying the property (b) of proposition 2 and let $\mathcal{P}_k$ be the set of pairs $(k, \tau)$, where $k$ is a positive integer and $\tau$ is a fixed-point-free involutory permutation on $\mathbb{N}_{8k}$.

**Proposition 3.** For every positive integer $k$, there exists a bijection $\Omega_k$ between $\mathcal{G}_{2k}$ and $\mathcal{P}_k$.

We can restate Proposition 3 in the following way:

**Proposition 3'.** For every closed connected 3-manifold $M$, there exists a pair $(k, \tau)$ representing $M$, where $k$ is a positive integer and $\tau$ is a fixed-point-free involutory permutation on $\mathbb{N}_{8k}$.

**Proof of Proposition 2.** (a) Let $(\Xi, \xi)$ be a $(0, 1, 2)$-normal crystallization of $M$ and let $\bar{T}_{01}$ be the external component of $\Xi_{(0,1)}$. By definition, there exists an ordering $T^1, T^2, \ldots, T^k$ ($g$ being the Heegaard genus of $M$) of the internal components of $\Xi_{(0,1)}$ and an ordering $(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_{2h_1}; \bar{w}_1^2, \bar{w}_2^2, \ldots, \bar{w}_{2h_2}^2; \ldots; \bar{w}_1^k, \bar{w}_2^k, \ldots, \bar{w}_{2h_k}^k)$ of $V(\bar{T}_{01})$ inducing one of the orientations of the cycle $\bar{T}_{01}$ such
that:

1. The vertices of $T^r$ are 2-consecutive ($r \in N_8$);
2. $\# V(T^r) \geq \# V(T^{r+1})$ ($r \in N_{k-1}$);
3. $w_i^r \in V(T^r)$ ($r \in N_8; i \in N_{2h}$), $w_i^r$ being the 2-correspondent of $\bar{w}_i^r$.

Suppose that $\# V(T^r) = 2h < 2h_1 = \# V(T^1)$, for some $s \in N_8 \cup \{1\}$. Let $\delta$ be the 3-coloured edge which contains $w_i^r$ and let $(\Xi', \xi')$ be the edge-augmented graph of $(\Xi, \xi)$ through the triple $(w_i^r, w_j^r, \overline{w}_k^r, \delta)$. In $(\Xi', \xi')$, each component $T^r$ ($r \neq s$) is left unchanged while $T^s$ is transformed into a new component of $\Xi_{(0,1)}$, say $\tilde{T}_s$, such that $\# V(\tilde{T}_s) = 2h_2 + 2$ (see Fig. 2). Let $y_0^s, y_1^s, y_2^s, \ldots, w_i^{2h}$ be the four distinct vertices used to construct the graph $(\Xi', \xi')$. The $(2h_2 + 2)$-tuple of 2-consecutive vertices of $\tilde{T}_s$ is $(w_i^s, y_0^s, y_1^s, w_2^s, \ldots, w_i^{2h})$ and their 2-correspondent vertices are respectively $\bar{w}_i^s, y_0^s, y_1^s, \bar{w}_2^s, \ldots, \bar{w}_i^{2h}$. If $2h_2 + 2 < 2h_1$, repeat the process to obtain the edge-augmented graph $(\Xi'', \xi'')$ of $(\Xi', \xi')$ through the triple $(y_0^s, y_1^s, y_2^s, \delta')$, $\delta'$ being the 3-coloured edge incident to $y_0^s$ (see Fig. 2). Going on like this, a crystallization $(\Gamma^*, \gamma^*)$ is derived. In $(\Gamma^*, \gamma^*)$, each component $T^r$ ($r \neq s$) is left unchanged and the above component $\tilde{T}_s$ is successively transformed into a new one, say $R_1^s$, such that $\# V(R_1^s) = 2h_3 + 2q = 2h_1$, for a suitable positive integer $q$. By repeating the process for all connected components $T^r$ ($r \neq s, r > 1$), we obtain a $(0, 1, 2)$-normal crystallization $(\tilde{F}, \tilde{\gamma})$ of $M$ such that $\# V(C_i) = 2h_i$ for each $i \in N_8$, $C_i$ being an internal component of $\Gamma_{(0,1)}$.

(b) Let $(\Xi, \xi)$ be the $(0, 1, 2)$-normal crystallization of $M$ considered in (a) and let $T^p$ ($p \in N_8$) be the first component of $\Xi_{(0,1)}$ in the above ordering such that $\# V(T^p) = 2h_p > 4$. Add a dipole through the edges $w_{2h}^p, w_i^p, w_j^p$ and let $\sigma$ be the obtained 2-coloured edge joining the two components $T_0^p, T_1^p$ into which $T^p$ splits. By adding a dipole through $\sigma$ and the edge $\bar{w}_i^p, \bar{w}_j^p \in E(\tilde{T}_{(0,1)})$, we get a $(0, 1, 2)$-normal crystallization $(\tilde{F}, \tilde{\gamma})$ of $M$.

In $(\tilde{F}, \tilde{\gamma})$ each component $T^r$ ($r \neq p$) is left unchanged and $T^p$ splits into two components $T_0^p, T_1^p$ such that $\# V(T_0^p) = 4$, $\# V(T_1^p) = 2h_p - 2$. If $2h_p - 2 > 4$, repeat the above construction on $T_1^p$ and so on. In this way a new $(0, 1, 2)$-normal crystallization $(\tilde{F}, \tilde{\gamma})$ of $M$ can be obtained.

In $(\tilde{F}, \tilde{\gamma})$, each component $T^r$ ($r \neq p$) is left unchanged and $T^p$ splits into $(h_p - 1)$ connected components of $\Xi_{(0,1)}$ with order four. By repeating this process for all components $T^r$ ($r > p$) with $\# V(T^r) = 2h_i > 4$, the statement follows. □

Remark. With the notation used in the proof of Proposition 2, the positive integer $k$ (compare (b) in the statement of Proposition 2) can be assumed to be $(\# V(\Xi)/4) - g$, where $g$ is the Heegaard genus of $M$ and $(\Xi, \xi)$ is the $(0, 1, 2)$-normal crystallization of $M$ named in (a) and (b).

We show in Fig. 3 a $(0, 1, 2)$-normal crystallization of the lens space $L(5, 1)$, which satisfies the property (b) of Proposition 2. (For the lens space see [10].) A standard crystallization of the lens space $L(p, q)$ (see also [3]) can be constructed
Fig. 2. Edge-augmented graph constructions.
as follows: take two cycles of length $2p$ and let $u_1, \ldots, u_{2p}$ and $v_1, \ldots, v_{2p}$ be their vertices cyclically ordered in the plane and with indices in $\mathbb{Z}_{2p}$. Colour the edges alternatively with 0 and 1 such that the indices correspond. Then, put edges of colour 2 (resp. 3) between $u_i$ and $v_i$ (resp. $v_{i+2p}$).

**Proof of Proposition 3 (3').** Let $M$ be a closed connected 3-manifold and let $(\tilde{G}, \tilde{\gamma})$ be a $(0, 1, 2)$-normal crystallization of $M$ which satisfies property (b) of Proposition 2. By $\mathcal{C}_{01}$ we denote the set of all the internal components of $\tilde{G}_{(0,1)}$. Let $V(\tilde{G}) = \{v_{1}^{1}, v_{2}^{1}, v_{3}^{1}, \ldots; v_{1}^{k}, v_{2}^{k}, v_{3}^{k}; \tilde{v}_{1}^{1}, \tilde{v}_{2}^{1}, \tilde{v}_{3}^{1}, \tilde{v}_{4}^{1}; \ldots; \tilde{v}_{1}^{k}, \tilde{v}_{2}^{k}, \tilde{v}_{3}^{k}, \tilde{v}_{4}^{k}\}$, $k$ being the cardinality of $\mathcal{C}_{01}$. By $\mu: N_{sk} \rightarrow V(\tilde{G})$ we indicate the one-to-one map which induces the above ordering. Now define the involutory permutation $\tau_i$ ($i \in \Delta_3$) on $V(\tilde{G})$ as follows: $\tau_i(u) = v$, for each pair $u, v \in V(\tilde{G})$, iff there exists an $i$-coloured edge ($i \in \Delta_3$) containing $u, v$. The permutations of the set $\{\tau_i | i \in \Delta_3\}$ are the fundamental permutations of the colour group of $(\tilde{G}, \tilde{\gamma})$ (see [3], [14], [1]). Let $\tau$ be the fixed-point-free involution on $N_{sk}$ given by the composition $\mu^{-1} \circ \tau_3 \circ \mu$. Finally the graph $(\tilde{G}, \tilde{\gamma})$ can be completely represented by the positive integer $k = \#\mathcal{C}_{01}$ and by the involutory permutation $\tau$. $\square$
4. Lins–Mandel crystallizations

Recently Lins and Mandel have defined a 4-parameter family of connected closed 3-manifolds, which includes the lens spaces (see [12]). Initially they introduce a 4-coloured graph, written $\mathcal{S}(b, l, t, c)$, whose vertices are the elements of $\mathbb{Z}_b \times \mathbb{Z}_{2l}$. To define the coloured edges they use four fixed-point-free involutions on $\mathbb{Z}_b \times \mathbb{Z}_{2l}$: 

$$
\begin{align*}
\pi(i, j) &= (i, j + (-1)^i) \\
\varepsilon(i, j) &= (i, j + (-1)^j) \\
\nu(i, j) &= (i + \mu(j), 1 - j) \\
\alpha(i, j) &= (i + c\mu(j - t), 1 - j + 2t),
\end{align*}
$$

where the arithmetics is mod $b$ (resp. mod $2l$) in the first (resp. second) coordinate. Also the arguments of $\mu$ are between 1 and $2l$: $\mu(j) = 1$ if $1 \leq j \leq l$ or $\mu(j) = -1$ if $l + 1 \leq j \leq 2l$. To define a 4-coloured graph just interpret the involutions also as coloured edges, that is, if $\chi \in \{\pi, \varepsilon, \nu, \alpha\}$, link vertices $a$ and $a'$ with an edge of colour $\chi$ if $a' = \chi(a)$. Each $\mathcal{S}(b, l, t, c)$ is proved to be a crystallization of a closed connected 3-manifold, whenever $(b, c) = 1$, $(t, l) = 1$ and $l$ odd implies $c = (-1)^t$ (see [12]). The lens spaces are obtained when $b = 2$. In this section we prove that each closed connected 3-manifold admits a crystallization in which three colours are induced by the involutions $\pi, \varepsilon, \nu$ just defined. Thus such a crystallization will be called a Lins–Mandel crystallization. This again allows a representation of all 3-manifolds with Heegaard genus $g$ by means of a positive integer $l$ and a fixed-point-free involutory permutation on $\mathbb{Z}_b \times \mathbb{Z}_{2l}$ (where $b = g + 1$). In order to state our theorems, we need the following definition.

**Definition 1.** A crystallization $(\Gamma, \gamma)$ of a closed connected 3-manifold $M$ is said to be a **Lins–Mandel crystallization of $M$** iff

1. $V(\Gamma) = \mathbb{Z}_b \times \mathbb{Z}_{2l}$, for some positive integers $b, l$.
2. For each edge $e \in E(\Gamma)$ with vertices $a, a'$, then $\gamma(e) = 0$ iff $\pi(a) = a'$; $\gamma(e) = 1$ iff $\varepsilon(a) = a'$; or $\gamma(e) = 2$ iff $\nu(a) = a'$ (where $\pi, \varepsilon, \nu$ are just the fixed-point-free involutory permutations previously introduced).
3. The set of all edges coloured 3 in $\Gamma$ is described by some fixed-point-free involution $\alpha^*$ on $\mathbb{Z}_b \times \mathbb{Z}_{2l}$.

In this case $(\Gamma, \gamma)$ will be denoted by $\mathcal{S}(b, l, \alpha^*)$.

**Proposition 4.** Every closed connected 3-manifold $M$ admits a Lins–Mandel crystallization $\mathcal{S}(b, l, \alpha^*)$, where $b = g + 1$ and $g$ is the Heegaard genus of $M$.

**Proof.** Let $(\Xi, \xi)$ be a $(0, 1, 2)$-normal crystallization with base component $\tilde{C}_{01}$ representing $M$. Since the standard crystallization of a lens space [3] is just a Lins–Mandel crystallization, we can assume the Heegaard genus of $M$ greater than one. By definition, there exists an ordering $C^1, C^2, \ldots, C^g$ (g being the Heegaard genus of $M$) of the internal components of $\Xi_{(0,1)}$ and an ordering $(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{2h_1}; \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{2h_2}; \ldots; \tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_{2h_g})$ of $V(\tilde{C}_{01})$ inducing one of the orientations of the cycle $\tilde{C}_{01}$ such that:

1. the vertices of $C^s$ are 2-consecutive ($s \in N_g$);
Fig. 4. Adding a dipole of type two.
Lins–Mandel crystallizations

(2) \( \#V(C^s) \geq \#V(C^{s+1}) \) \((s \in N_{g-1})\);
(3) \( v_i^s \in V(C^s) \) \((s \in N_g; \ t \in N_{2h})\), \( v_i^s \) being the 2-correspondent of \( \bar{v}_i^s \).

Add a dipole \( \theta_0 \) through the edges \( v_{2h_1}^1 \bar{v}_{2h_2}^1, v_{2h_2}^2 \bar{v}_{2h_1}^2 \) and let \( y_0^0, y_0^1 \) be the vertices spanning \( \theta_0 \) (see Fig. 4). In the new crystallization \( (\Xi^0, \xi^0) \), each component \( C^s \) \((s \neq 2)\) is left unchanged and \( C^2 \) is transformed into a component, say \( \bar{C}^2 \), of \( \Xi_{(0,1)}^0 \). There are exactly \( 2h_2+1 \) 2-coloured edges incident to \( \bar{C}^2 \), that is \( v_{2h_1}^1 y_0^0, x_0^2 \bar{v}_{2h_1}^2, v_0^1 \bar{v}_1^2 \) (i \( \in N_{2h_2} \)). The vertices of \( \bar{C}^2 \) are \( y_0^0, x_0^2, v_1^2 \) (i \( \in N_{2h_2} \)), where \( x_0^2, v_1^2, v_2^2, \ldots, v_{2h_2}^2 \) are 2-consecutive and their 2-corresponding vertices are \( \bar{x}_1^2, \bar{v}_1^2, \bar{v}_2^2, \ldots, \bar{v}_{2h_2}^2 \) respectively. There is exactly one 2-coloured edge, \( v_{2h_1}^1 y_0^1 \), joining \( C^1 \) with \( \bar{C}^2 \) (see Fig. 4). Let \( \mathcal{A}_2 \) be the set of all 2-coloured edges joining \( \bar{C}^2 \) with \( \bar{C}_{01} \). Add a dipole \( \theta_1 \) through the edges \( v_{2h_1}^2 \bar{v}_{2h_2}^2, v_{2h_2}^3 \bar{v}_1^3 \) and let \( y_1^3, x_1^3 \) be the vertices spanning \( \theta_1 \). Then repeat this process to obtain dipoles \( \theta_2, \ldots, \theta_{2h_2}, \theta_{2h_2+1} \) respectively through the pairs of edges \( (y_2^3, x_2^3), (y_3^3, x_3^3), \ldots, (y_{2h_2}^3, x_{2h_2}^3) \) until \( \mathcal{A}_2 \) is void (see Fig. 5). In the new crystallization \( (\Xi^1, \xi^1) \), each component \( C^s \) \((s \in N_g-\{2, 3\})\) and \( \bar{C}^2 \) are left unchanged while \( C^3 \) is transformed into a component, \( \bar{C}^3 \) say, of \( \Xi_{(0,1)}^1 \). There are no 2-coloured edges joining \( \bar{C}^2 \) with \( \bar{C}_{01} \). The component \( \bar{C}^3 \) is joined with \( \bar{C}^2 \) (resp. \( \bar{C}_{01} \)) by means of \( 2h_2+1 \) (resp. \( 2h_2+2h_3+1 \)) 2-coloured edges \( y_1^3 \bar{v}_{2h_1}^2, y_2^3 \bar{v}_{2h_2}^2, \ldots, y_{2h_2}^3 \bar{v}_{2h_2+1}^2 \) (resp. \( x_{2h_1}^3, x_{2h_2}^3, \ldots, x_{2h_2+1}^3 \)) until \( \mathcal{A}_2 \) is void (see Fig. 5). In the new crystallization \( (\Xi^2, \xi^2) \), each component \( C^s \) \((s \in N_g-\{2, 3\})\) and \( \bar{C}^2 \) are left unchanged while \( C^3 \) is transformed into a component, \( \bar{C}^3 \) say, of \( \Xi_{(0,1)}^2 \). There are no 2-coloured edges joining \( \bar{C}^2 \) with \( \bar{C}_{01} \). The component \( \bar{C}^3 \) is joined with \( \bar{C}^2 \) (resp. \( \bar{C}_{02} \)) by means of \( 2h_2 \) (resp. \( 2h_2+1 \)) 2-coloured edges between \( T^r, T^{r-1} \) (resp. \( T^r, T^{r+1} \) \((r = 2, \ldots, g)\));

(1) \( T^1 \) is joined with \( T^2 \) (resp. \( T^{g+1} \)) by exactly one 2-coloured edge \( v_{2h_1}^1 y_0^2 \) (resp. \( v_{2h_1}^1 \bar{v}_{2h_1}^2, \ldots, v_{2h_1-1}^1 \bar{v}_{2h_1-1}^2 \));
(2) there are exactly \( 2 \Sigma_{i-1}^r h_i + 1 \) (resp. \( 2 \Sigma_{i-2}^r h_i + 1 \)) 2-coloured edges between \( T^r, T^{r-1} \) (resp. \( T^r, T^{r+1} \) \((r = 2, \ldots, g)\));
(3) there are no 2-coloured edges between \( T^0, T^a \) with \( |p-q| \neq 1 \).

Starting from \( (\bar{F}, \bar{\gamma}) \) and by making use of the edge-augmented graph construction (compare Proposition 2), we can easily obtain a final crystallization \( (\Gamma, \gamma) \) of \( M \) whose shape can be geometrically described as follows (see Fig. 6). It consists of \( g+1 \) connected components coloured 0 and 1, cyclically set in the plane and numbered 1, 2, \ldots, \( g+1 \). There are \( l = 2 \Sigma_{i=2}^g h_i + 1 \) 2-coloured links between two successive connected components of \( \Gamma_{(0,1)} \). There are no 2-coloured edges between two non-consecutive components of \( \Gamma_{(0,1)} \). Obviously we have \( \#V(\Gamma) = 2l \times b \), where \( b = g+1 \). Let now \( \sigma: Z_2 \times Z_{2l} \rightarrow V(\Gamma) \) be an arbitrary bijection such that \( \sigma(i, j) \) belongs to the ith component of \( \Gamma_{(0,1)} \) and the sequence \( \sigma(i, 1), \ldots, \sigma(i, 2l) \) induces one of the orientations of the same ith component. Define \( \tau_i \) to be the fundamental permutations of the colour group of \( (\Gamma, \gamma) \). Then set \( \pi = \sigma^{-1} \circ \tau_0 \circ \sigma, \ \varepsilon = \sigma^{-1} \circ \tau_1 \circ \sigma \) and \( \nu = \sigma^{-1} \circ \tau_2 \circ \sigma \). By construction, these
involutory permutations $\tau$, $\epsilon$, $\nu$ are just expressed by means of the formulas listed in definition 1. Further $\alpha^* = $ is a fixed-point-free involutory permutation on $\mathbb{Z}_2 \times \mathbb{Z}_2$ induced by the colour 3. Since $(\Gamma, \gamma)$ is colour-isomorphic with a Lins–Mandel crystallization of $M$, the proof is completed. 

**Remark 1.** In the above proof, the number $l = 2 \sum_{i=2}^{k} h_i + 1$ is odd. However we can construct a Lins–Mandel crystallization $(\Gamma, \gamma)$ of $M$ with $l = 2 \sum_{i=2}^{k} h_i + 2$ even. This is obtained by first adding a dipole through the edges $v_{2i-1}^1 v_{2i-1}^1, y_0^2 x_0^2$. Then use edge-augmented arguments to make $l$ equal to the number of 2-coloured links between two consecutive components of $\Gamma_{(0,1)}$. 

Fig. 5. An iterated construction of dipoles.
Fig. 6. General form of a Lins–Mandel crystallization.
Remark 2. For each closed connected 3-manifold $M$, the fixed-point-free involutory permutation $\alpha^*$ on $\mathbb{Z}_b \times \mathbb{Z}_{2t}$ can be written as follows:

$$\alpha^*(i, j) = (i + c(i, j), 1 - j + 2t(i, j)).$$

The unknown maps $c, t: \mathbb{Z}_b \times \mathbb{Z}_{2t} \to \mathbb{Z}$ are respectively solutions of these func-
tional equations
\[ c(i, j) + c(i + c(i, j), 1 - j + 2t(i, j)) = 0, \]
\[ -t(i, j) + t(i + c(i, j), 1 - j + 2t(i, j)) = 0. \]

This expression for \( \alpha^* \) extends the formula which defines \( \alpha \) in \( \mathcal{S}(b, l, t, c) \).

A Lins–Mandel crystallization \( \mathcal{S}(3, 9, \alpha^*) \) of the spherical dodecahedral space is illustrated in Fig. 7.

5. Normalizing a Lins–Mandel crystallization

In [4] two equivalent sets of crystallization moves are described and it is proved that any two crystallizations of the same manifold are related by a finite sequence of such moves. In this section we get Lins–Mandel crystallizations of a 3-manifold with particular properties by only using one of these move types. This allows a representation of all 3-manifolds by a positive integer \( b \) and a fixed-point-free involutory permutation on \( \mathbb{Z}_b \times \mathbb{Z}_b \).

**Proposition 5.** Every closed connected 3-manifold \( M \) admits a Lins–Mandel crystallization \( \mathcal{S}(b, 3, \alpha^*) \).

**Proof.** Let \( (\Gamma, \gamma) \) be a Lins–Mandel crystallization \( \mathcal{S}(d, l, \beta^*) \) of \( M \), where \( l \) (\( l > 3 \)) is odd. By definition, there exist an ordering \( C^1, C^2, \ldots, C^d \) of the connected components of \( \Gamma_{(0,1)} \) and an ordering \( (w_i^1, w_i^1, \ldots, w_i^1, v_i^1, v_i^2, \ldots, v_i^2) \) of \( V(C^s) \) inducing one of the orientations of the cycle \( C^s \) such that each 2-coloured edge \( v_i^1w_i^{s+1} \) (resp. \( v_i^{s-1}w_i^s \)) joins \( C^s \) with \( C^{s+1} \) (resp. \( C^{s-1} \)) for every \( i \in \mathbb{N}_d \). For \( i = d + 1 \) or \( i = 0 \), we mean \( i = d \). Let \( \mathcal{A} \) (resp. \( \mathcal{B} \)) be the subgraph spanned by \( V(C^s) \cup V(C^{s+1}) \) (resp. by \( V(\Gamma) - V(\mathcal{A}) \)). There are exactly \( 2l \) 2-coloured edges \( v_i^{s+1}w_i^{s+2}, \ldots, v_i^1w_i^{1+2}, v_i^{s-1}w_i^1, \ldots, v_i^1w_i^{1+s} \) joining \( \mathcal{A} \) with \( \mathcal{B} \). Denote by \( \mathcal{E}(\mathcal{A}, \mathcal{B}) \) the set of all these edges. Add a dipole \( \Lambda_1 \) through the edges \( w_i^{s+1}w_i^{s+2} \) and \( w_i^1v_i^{s+1} \) of \( E(C^{s+1}) \) and let \( x_{s+1}^1, x_{s+1}^2 \) be the vertices spanning \( \Lambda_1 \). In the obtained crystallization \( (\Gamma^1, \gamma^1) \), each connected component \( C^r \) (\( r \neq s+1 \)) is left unchanged and \( C^{s+1} \) splits into two components, say \( C_1^{s+1} \) and \( C_2^{s+1} \), of \( \Gamma_{(0,1)} \). Let \( \sigma \) be the 2-coloured edge joining \( C_1^{s+1} \) with \( C_2^{s+1} \). There are exactly \( l - 3 \) (resp. 3) 2-coloured edges \( v_i^1w_i^{s+1}, \ldots, v_i^{s-3}w_i^{s+1} \) (resp. \( v_i^{s-2}w_i^{s+1}, v_i^{s-1}w_i^{s+1}, v_i^1w_i^{s+1} \)) joining \( C^s \) with \( C_1^{s+1} \) (resp. with \( C_2^{s+1} \)). Add a dipole \( \Lambda_2 \) through the edges \( \sigma \) and \( v_i^{s-3}v_i^{s-2} \) (where \( v_i^{s-3}v_i^{s-2} \in E(C^s) \)) and let \( x_1^s, x_2^s \) be the vertices spanning \( \Lambda_2 \). In the derived crystallization \( (\Gamma^2, \gamma^2) \), each component of \( \Gamma_{(0,1)} - \{ C^s \} \) is left unchanged while \( C^s \) is transformed into a connected component, say \( \tilde{C}^s \), of \( \Gamma^2_{(0,1)} \). There exists an ordering \( (w_i^1, w_i^1, \ldots, w_i^1, v_i^1, \ldots, v_i^{s-3}, x_1^s, x_2^s, v_i^{s-2}, v_i^{s-1}, v_i^1) \) of \( V(\tilde{C}^s) \) inducing one of the orientations of the cycle \( \tilde{C}^s \). The 2-coloured edges \( v_i^1w_i^{s+1} (i \in \mathbb{N}_{l-3}) \) and \( x_1^s x_i^{s+1} \) join \( C^s \) with \( C_1^{s+1} \).
There are exactly four 2-coloured edges $x_2 x_2^{+1}$, $v_1^{i-2} w_1^{-1}$, $v_1^{-1} w_1^{i+1}$, $v_1^i w_1^{i+1}$ joining $\tilde{C}$ with $C_2^{s+1}$. Add a dipole $\Lambda_3$ through the edges $w_1^i v_1^i$, $v_1^{-2} v_1^{-1}$ which belong to $V(\tilde{C})$. Let $\tilde{x}_1$, $\tilde{x}_2$ be the vertices spanning $\Lambda_3$ and let $\beta$ be the 2-coloured edge joining the two connected components, say $\tilde{C}_1$ and $\tilde{C}_2$, into which $\tilde{C}$ splits. Thus we construct a new crystallization $(\Gamma^3, \gamma^3)$ of $M$ in which each component of $\Gamma^3_{(0,1)} \setminus \{\tilde{C}_1\}$ is left unchanged. There are exactly two 2-coloured edges $v_1^{-2} w_1^{i+2}$, $x_2 x_2^{+1}$ (resp. $v_1^{-1} w_1^{i+1}$, $v_1^i w_1^{i+1}$) joining $C_2^{s+1}$ with $\tilde{C}_1$ (resp. with $\tilde{C}_2$). The two connected components $\tilde{C}_1$, $C_2^{s+1}$ are linked in $\Gamma^3$ by exactly $l - 2$ 2-coloured edges $v_1^i w_1^{i+1}$, $\ldots$, $v_1^{-3} w_1^{i+3}$, $x_1^i x_1^{i+1}$. Add a dipole $\Lambda_4$ through the edges $\beta$ and $w_1^{i+1} w_1^{-1}$ (where $w_1^{i+1} w_1^{-1} \in E(C_2^{s+1})$) and let $y_1^{i+1}$, $y_1^{i-1}$ be the vertices spanning $\Lambda_4$. A new crystallization $(\Gamma^4, \gamma^4)$ of $M$ is obtained in which each component of $\Gamma^4_{(0,1)} \setminus \{C_2^{s+1}\}$ is left unchanged and $C_2^{s+1}$ is transformed into a component, say $\tilde{C}_2^{s+1}$, of $\Gamma^4_{(0,1)}$. There are exactly three 2-coloured edges $v_1^{i+1} w_1^{-1}$, $v_1^{-2} w_1^{i+3}$, $x_2 y_1^{i+1}$ (resp. $x_2 y_2^{i+1}$, $v_1^{-2} w_1^{i+2}$, $x_2 x_2^{+1}$) joining $C_2^{s+1}$ with $C_2^{s}$ (resp. with $\tilde{C}_1$). The two components $\tilde{C}_1$ and $C_2^{s+1}$ are joined by exactly $l - 2$ 2-coloured edges $v_1^{i+1}$, $\ldots$, $v_1^{-3} w_1^{i+3}$, $x_1^i x_1^{i+1}$. There exists an ordering $C_1, \ldots, C_{s-1}, \tilde{C}_2^{s}, C_2^{s+1}, C_1^{s+1}, C_{s+2}, \ldots, C_d$ of the connected components of $\Gamma^4_{(0,1)}$ such that:

1. $C'$ and $C'^{+1}$ (resp. $C'$ and $C'^{-1}$) are linked by exactly $l$ 2-coloured edges for each $r \in N_d \setminus \{s - 1, s\}$. For $r = d + 1$ or $r = 0$, we mean $r = d$;
2. the components $\tilde{C}_2^{s}$, $\tilde{C}_2^{s+1}$ and $C_2^{s+1}$, $\tilde{C}_1$ are linked by exactly three 2-coloured edges;
3. there are exactly $l$ (resp. $l - 2$) 2-coloured edges between $C^{s-1}$ and $\tilde{C}_2^{s}$ (resp. $\tilde{C}_1$ and $C_1^{s+1}$).

There exists a subgraph $\mathcal{B}'$ of $\Gamma^4$ colour-isomorphic with $\mathcal{B}$. If $\mathcal{A}'$ is the subgraph of $\Gamma^4$ spanned by $V(\mathcal{A}') \setminus V(\mathcal{B}')$, we can assume $\mathcal{G}(\mathcal{A}, \mathcal{B}) = \mathcal{G}(\mathcal{A}', \mathcal{B}')$ by construction. Set $\Gamma' = (V(\mathcal{A}) \cup V(\mathcal{B}), E(\mathcal{A}) \cup E(\mathcal{B}) \cup \mathcal{G}(\mathcal{A}, \mathcal{B}))$, then $\Gamma^4 = (V(\mathcal{A}') \cup V(\mathcal{B}'), E(\mathcal{B}') \cup E(\mathcal{A}) \cup \mathcal{G}(\mathcal{A}', \mathcal{B}'))$. Further, $\mathcal{A}'_{(0,1)}$ is formed by four consecutive connected components $\tilde{C}_2^{s}$, $\tilde{C}_2^{s+1}$, $\tilde{C}_1$, $C_1^{s+1}$ pairwise linked by at most $p$ ($p \leq \max \{3, l - 2\}$) 2-coloured edges. Thus an inductive argument can be used to obtain a Lins–Mandel crystallization $\tilde{\mathcal{S}}(b, 3, s^*)$ of $M$. In such a crystallization two $\{0, 1\}$-coloured components, consecutive in a suitable ordering of them, are linked by exactly three 2-coloured edges.

In Fig. 8 a Lins–Mandel crystallization $\tilde{\mathcal{S}}(3, 3, s^*)$ of the connected sum $L(2, 1) \# L(2, 1)$, $L(2, 1)$ being the real projective 3-space, is shown.

6. A family of closed connected 3-manifolds

In this section we investigate in more detail the family of 4-coloured graphs $\mathcal{S}(b, l, t, \gamma)$ constructed by using the involutions $\pi$, $e$, $\nu$ of Section 4 and the following fixed-point-free involution on $Z_b \times Z_{2l}$:

$$\beta(i, j) = (\gamma^{i(j - t)}(i), 1 - j + 2t),$$
where \( t \) is a positive integer and \( \gamma : N_b \rightarrow N_b \) is a cycle of order \( b \). If the cycle \( \gamma \) is defined by \( \gamma(i) = i + c \) (where \( c \) is an integer constant), then \( \mathcal{I}(b, l, t, \gamma) \) is exactly the 4-coloured graph \( \mathcal{I}(b, l, t, c) \) of [12].

The geometrical shape for \( \mathcal{I}(b, l, t, \gamma) \), where \( \gamma = \langle a_1, a_2, \ldots, a_b \rangle \), can be described by the following proposition.

**Proposition 6.** The 4-coloured graph \( \mathcal{I}(b, l, t, \gamma) \) consists of \( b \) \( \{\pi, \varepsilon\}\)-coloured connected components cyclically set in the plane and numbered as \( \xi = (1, 2, \ldots, b) \) (resp. \( \eta = (a_1, a_2, \ldots, a_b) \)). There are \( l \) \( \nu \)-coloured (resp. \( \beta \)-coloured) links between two \( \{\pi, \varepsilon\}\)-coloured connected components which are consecutive in \( \xi \) (resp. \( \eta \)).

The proof is straightforward.

**Proposition 7.** Assume that \( (t, l) = 1 \). Then \( \mathcal{I}(b, l, t, \gamma) \) is a crystallization of a closed connected 3-manifold iff \( \gamma \) satisfies the following formula:

\[
\gamma^{(1-j-(2l-1)t)}(\gamma^{(1-j-(2l-3)t)}(\cdots(\gamma^{(1-j-5t)}(\gamma^{(1-j-3t)}(\gamma^{(1-j-t)}(i + \mu(j)) + \mu(j+2t)))) + \mu(j + 4t) + \mu(j + 6t)\cdots) + \mu(j + (2l-2)t)) = i.
\]

\((*)\)
Note that $\mathcal{S}(b, l, t, \gamma)$ is the Lins–Mandel crystallization named $\tilde{\mathcal{S}}(b, l, \beta)$, whenever $(t, l) = 1$ and $\gamma$ satisfy $(\ast)$.

We need the following Lemmas

**Lemma 1.** Assume that $(t, l) = 1$. The cycle $\gamma$ satisfies $(\ast)$ iff $(\beta \nu)^m \neq Identity$ for each $m < l$ and $(\beta \nu)^l = Identity$.

**Lemma 2.** If $(t, l) = 1$ and $\gamma$ satisfies $(\ast)$, then $\mathcal{S}(b, l, t, \gamma)$ is a contracted graph.

**Proof of Lemma 1.** We have

$$(\beta \nu)(i, j) = \beta(i + \mu(j), 1 - j) = (\gamma^{\mu(1-i-t)}(i + \mu(j)), j + 2t).$$

By induction ($m > 1$), it follows

$$(\beta \nu)^m(i, j) = (\gamma^{\mu(1-i-(2m-1)t)}(\gamma^{\mu(1-i-(2m-3)t)}(\cdots (\gamma^{\mu(1-i-3t)}(\gamma^{\mu(1-i-t)}(i + \mu(j)) + \mu(j + 2t))

+ \mu(j + 4t)) + \mu(j + 6t)) \cdots) + \mu(j + (2m - 2)t)), j + 2mt)$$

Since $(l, t) = 1$, the second coordinates of
give all the $l$ distinct values of the same parity as $j$ in the interval $[1, 2l]$. Thus $(\beta \nu)^m \neq Identity$ if $m < l$. Further, the second coordinate of $(\beta \nu)^l(i, j)$ is $j$. A necessary and sufficient condition for the first coordinate of $(\beta \nu)^l(i, j)$ to be $i$ is that $\gamma$ satisfies $(\ast)$.

**Proof of Lemma 2.** Obviously $g_b = 1$. Since $\gamma: N_b \rightarrow N_b$ is a cycle of order $b$, then $g_\nu = 1$. Now we prove that $g_m = 1$. Vertex representatives of the $\{\beta, \nu\}$-coloured connected components of $\mathcal{S}(b, l, t, \gamma)$ can be chosen to be $(1, 1), (2, 1), \ldots, (b, 1)$, because each cycle of $\beta \nu$ has length $l$ and the second coordinates in this are distinct (see the proof of Lemma 1). The pairs $(i, 1)$ and $(i + 1, 1)$ are in the same connected component of $\mathcal{S}(b, l, t, \gamma)_c$ because

$$(\epsilon \nu)(i, 1) = \epsilon(i + 1, 2l) = (i + 1, 1).$$

We consider two cases to prove that $\mathcal{S}(b, l, t, \gamma)_c$ is connected.

**Case 1.** $t$ odd.

Take $(1, t), (2, t), \ldots, (b, t)$ to represent the $\{\beta, \nu\}$-coloured connected components of $\mathcal{S}(b, l, t, \gamma)$. The vertices $(i, t)$ and $(\gamma(i), t)$ are in the same connected component of $\mathcal{S}(b, l, t, \gamma)_c$ since

$$(\beta \pi)(i, t) = \beta(i, t + 1) = (\gamma^{\mu(1)}(i), t) = (\gamma(i), t).$$

**Case 2.** $t$ even.

Since $(t, l) = 1$, then $l$ is odd. Take now $(1, l), (2, l), \ldots, (b, l)$ to represent the
Fig. 9. The 3-manifold crystallization \( \mathcal{F}(4, 3, 1, \gamma = (0, 1, 3, 2)) \).

\{\beta, \nu\}-coloured connected components of \( \mathcal{F}(b, l, t, \gamma) \). We have

\[(\pi \nu)(i, l) = \pi(i + 1, 1 - l) = (i + 1, 1 - l - (-1)^{l-1}) = (i + 1, l) \]

Proof of Proposition 7. Necessary condition: Let \( \mathcal{F}(b, l, t, \gamma) \) be a crystallization of a closed connected 3-manifold \( M \). By definition, we have \( g_{(\pi, e)} = g_{(\nu, \beta)} = b \). Since \((\beta \nu)^m \neq \text{Identity}\) for each \( m < l \), each \{\beta, \nu\}-coloured connected component of \( \mathcal{F}(b, l, t, \gamma) \) has at least \( 2l \) vertices.
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Since $\# V(\mathcal{F}(b, l, t, \gamma)) = 2lb$, then $(\beta \nu)^l = \text{Identity}$. Now Lemma 1 implies the statement.

**Sufficient condition:** Let formula (*) be true. Then by Lemmas 1 and 2 $\mathcal{F}(b, l, t, \gamma)$ is a contracted graph and $g_\pi = g_e = g_\beta = g_\nu = 1$. By construction, we have

$$g_{(\pi, \nu)} + g_{(e, \nu)} + g_{(\nu, \beta)} + g_{(e, \beta)} = (2l - 2)b + 4.$$

If $\gamma$ satisfies (*) and $(t, l) = 1$, then $(\beta \nu)^l = \text{Identity}$ and $(\beta \nu)^m \neq \text{Identity}$ for each $m < l$ by Lemma 1. Thus $\beta \nu$ has $2b$ cycles of length $l$; therefore $g_{(\beta, \nu)} = b$. We have

$$g_{(e, e)} + g_{(\pi, \nu)} + g_{(e, \beta)} + g_{(e, \beta)} + g_{(\nu, \beta)} = b + (2l - 2)b + 4 + b = 2lb + 4$$

$$= \# V(\mathcal{F}(b, l, t, \gamma)) + g_\pi + g_e + g_\nu + g_\beta.$$

Then $\mathcal{F}(b, l, t, \gamma)$ is a crystallization of a closed connected 3-manifold (see [7]).

In Fig. 9 the 3-manifold crystallization $\mathcal{F}(4, 3, 1, \gamma = (0, 1, 3, 2))$ is illustrated. This graph is not a member of the 4-parameter family defined in [12]. The fundamental group of the closed connected 3-manifold $M$ represented by $\mathcal{F}(4, 3, 1, \gamma = (0, 1, 3, 2))$ is the $ZS$-metacyclic group $(-2, 2, 3)$

$$\pi_1(M) = \langle X, Y \mid X^2 = Y^{-3}, Y^2 = XYX \rangle = \langle U, V \mid U^2 = V^2 = (UV)^3 \rangle$$

and the first Homology group $H_1(M)$ is $Z_8$.

**References**