

# Positive periodic solutions to nonlinear ODEs with indefinite weight: an overview

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## Abstract

We discuss the periodic problem associated with the second order differential equation

$$u'' + (\lambda a^+(t) - \mu a^-(t))g(u) = 0, \quad (1)$$

where  $\lambda, \mu$  are positive parameters,  $a(t)$  is a sign-changing periodic function and  $g(u)$  is a nonlinear function having superlinear growth at zero and sub-linear growth at infinity. More precisely, we show how various tools from Nonlinear Analysis and Dynamical Systems can be used to provide results about existence, multiplicity and chaotic dynamics of positive solutions to (1). This survey paper is based on a talk given by the author at the *Bru-To PDE's Conference* (University of Torino, May 2-5, 2016).

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## 1 Introduction

The aim of this brief note is to collect together some recent results, obtained in collaboration with Guglielmo Feltrin, Maurizio Garrione and Fabio Zanolin (see [12, 13, 14, 16]), on the existence of *positive periodic solutions* to nonlinear ODEs with *indefinite weight*.

More precisely, throughout the paper we deal with the second order scalar equation

$$u'' + (\lambda a^+(t) - \mu a^-(t))g(u) = 0, \quad (2)$$

where  $\lambda, \mu$  are positive parameters,  $a : \mathbb{R} \rightarrow \mathbb{R}$  is a locally integrable and  $T$ -periodic sign-changing function and  $g : \mathbb{R}^+ := [0, +\infty) \rightarrow \mathbb{R}$  is a  $\mathcal{C}^1$ -function satisfying the sign condition

$$(g_*) \quad g(0) = 0 \quad \text{and} \quad g(u) > 0, \quad \text{for any } u > 0,$$

as well as the growth conditions (at zero and at infinity)

$$(g_{**}). \quad \lim_{u \rightarrow 0^+} \frac{g(u)}{u} = \lim_{u \rightarrow +\infty} \frac{g(u)}{u} = 0.$$

Due to this assumption, the nonlinear term  $g(u)$  will be referred to as a *super-sublinear* function<sup>1</sup>; in this setting, the study of the periodic problem associated with (2) seems to be an interesting and delicate topic and, maybe unexpectedly, several tools from Nonlinear Analysis and Dynamical Systems (topological degree theory, variational methods, Poincaré-Birkhoff theorem, shooting arguments...) have to be used to try to understand as much as possible about it (existence/nonexistence of solutions, multiplicity, chaotic dynamics...).

Before passing to describe our results, let us spend some words about boundary value problems with indefinite weight, trying to better motivate our investigation (incidentally, it seems that the terminology “indefinite” - simply meaning that the coefficient of the nonlinear term is sign-changing - was first used in [5] in the context of a linear eigenvalue problem, and it has then become very popular in nonlinear problems starting with [35]).

The periodic problem associated with an equation like

$$u'' + q(t)g(u) = 0, \tag{3}$$

with  $q(t)$  sign-changing, was first investigated by Butler in its pioneering papers [23, 24], dealing with cases when  $g(u)$  is defined on the whole real line and has superlinear growth at infinity or sublinear growth at zero, respectively. Later on, along this line of research, several contributions followed (especially in the super-linear case) and a quite complete picture concerning existence and multiplicity of *oscillatory* solutions to various boundary value problems associated with (3) is available since fifteen years ago (see, among others, [25, 36, 37, 39]).

On the other hand, starting with the nineties, the existence of *positive* solutions to boundary value problems associated with the nonlinear PDE

$$\Delta u - V(x)u + q(x)g(u) = 0, \quad x \in \Omega \subset \mathbb{R}^N, \tag{4}$$

with  $q(x)$  sign-changing, has been considered, as well (see, among others, [2, 4, 6, 8, 9]). It is worth mentioning that the elliptic equation (4) naturally arises when searching for steady states of the corresponding evolutionary parabolic problem (see [1] for a recent nice survey on the topic). Such kind of equations has a typical interpretation in the context of population dynamics, with the unknown  $u$  playing the role of density of a species inhabiting the spatially heterogeneous domain  $\Omega$ ; accordingly, the (indefinite) sign of the coefficient  $q$  expresses saturation or autocatalytic behavior of the species  $u$ , when  $q \leq 0$  or  $q \geq 0$  respectively.

Needless to say, equation (3) can be meant as the one-dimensional case of (4) when  $V(x) \equiv 0$ ; moreover, it is not difficult to realize that periodic boundary conditions for (3) exhibit strong analogies with Neumann conditions for the elliptic

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<sup>1</sup>This terminology could be a bit misleading, since any function satisfying  $(g_{**})$  is clearly below any line (passing through the origin and having positive slope) both for  $u$  small and for  $u$  large; however, it is quite common and useful when both the behavior at zero and at infinity of a nonlinear function have to be emphasized, since pure power nonlinearities  $g(u) = u^p$  can be referred to as sublinear (i.e., sub-sublinear) when  $p < 1$  and superlinear (i.e., super-superlinear) when  $p > 1$ .

equation (4). To explain why (besides recalling the well-known fact that both these boundary conditions share the same principal eigenvalue  $\lambda_0 = 0$ ) we observe that a mean value condition on the weight function is often necessary for the existence of a positive solution both for  $T$ -periodic and Neumann boundary conditions. Indeed, assuming the existence of a positive  $T$ -periodic solution to (3), we easily obtain - dividing the equation by  $g(u)$  and integrating by parts -

$$\int_0^T q(t) dt = - \int_0^T \left( \frac{u'(t)}{g(u(t))} \right)^2 g'(u(t)) dt.$$

As a consequence, the condition

$$\int_0^T q(t) dt < 0 \tag{5}$$

is necessary for the existence of a positive  $T$ -periodic solution whenever  $g'(u) > 0$  for any  $u > 0$  (we stress that this is not an assumption in our basic setting; however, since there are of course many increasing nonlinearities in the class of the nonlinear functions  $g(u)$  satisfying  $(g_*)$  and  $(g_{**})$ , condition (5) has to be considered unavoidable in general). Essentially the same computation is valid when dealing with positive Neumann solutions to (4) (when  $V(x) \equiv 0$ ); this was indeed first observed by Bandle, Pozio and Tesei in [6], showing that  $\int_{\Omega} q(x) dx < 0$  is actually necessary and sufficient for the existence of a positive solution to the Neumann problem associated with  $\Delta u + q(x)u^p = 0$  in the sublinear case  $0 < p < 1$ . The same result was then proved in the superlinear case  $p > 1$  in [2, 8, 9].

The above discussion should explain how results about the existence of positive periodic solutions to (2) have to be interpreted if compared with the existing literature. In particular, we stress that they seem to be quite new from different points of view: on one hand, indeed, they can be meant as lying somewhat in the middle between results giving oscillatory periodic solutions to ODEs and results giving positive Neumann solutions to elliptic PDEs (it is worth mentioning that the existence of positive periodic solutions to equations like (3) was explicitly raised by Butler as an open problem in [23, p. 477]); on the other hand, growth conditions of super-sublinear type like  $(g_{**})$  seem to represent a novelty in the indefinite setting (in particular, we are not aware of results dealing with positive Neumann solutions to  $\Delta u + q(x)g(u) = 0$  when  $g(u)$  is super-sublinear).

The rest of this paper will be devoted to the description of several results in this direction (existence/nonexistence, multiplicity, subharmonic solutions, chaotic dynamics..). From now on, we impose the following technical condition on the weight function:

$(a_*)$  *there exist  $m \geq 1$  intervals  $I_1^+, \dots, I_m^+$ , closed and pairwise disjoint in the quotient space  $\mathbb{R}/T\mathbb{Z}$ , such that*

$$a(t) \geq 0, a(t) \not\equiv 0, \quad \text{on } I_i^+, \quad \text{for } i = 1, \dots, m;$$

$$a(t) \leq 0, a(t) \not\equiv 0, \quad \text{on each connected component of } (\mathbb{R}/T\mathbb{Z}) \setminus \bigcup_{i=1}^m I_i^+,$$

we also define the value

$$\mu^\#(\lambda) := \lambda \frac{\int_0^T a^+(t) dt}{\int_0^T a^-(t) dt} \quad (6)$$

which is going to play an important role in what follows.

## 2 The $T$ -periodic problem

We first focus on the existence of positive  $T$ -periodic solutions to (2), stating the following result.

**Theorem 2.1.** *Let  $a : \mathbb{R} \rightarrow \mathbb{R}$  be a locally integrable  $T$ -periodic function satisfying  $(a_*)$  and let  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$  be a continuously differentiable function satisfying  $(g_*)$  and  $(g_{**})$ . Then, there exists  $\lambda^* > 0$  such that, for any  $\lambda > \lambda^*$  and for any  $\mu > \mu^\#(\lambda)$ , equation (2) has at least two positive  $T$ -periodic solutions.*

According to this statement, the solvability of (2) (together with periodic conditions) seems to be ensured only when imposing some restrictions on the range of the parameters  $\lambda, \mu$  (see also the discussion in Section 5); on the other hand, it is remarkable that *two* solutions can be obtained. The underlying idea behind this can be traced back to a classical result by Rabinowitz [38], proving indeed, for  $\lambda > 0$  large enough, the existence of a pair of positive solutions for the Dirichlet problem associated with an equation like

$$\Delta u + \lambda f(x, u) = 0, \quad x \in \Omega \subset \mathbb{R}^N,$$

when  $f(x, \cdot)$  is (roughly speaking) super-sublinear (see also [3] for previous results in this direction, from a more abstract point of view). In the indefinite periodic case, the situation is however more subtle, since the restriction  $\mu > \mu^\#(\lambda)$  also needs to be imposed. This is actually unavoidable in general, since

$$\int_0^T (\lambda a^+(t) - \mu a^-(t)) dt < 0 \iff \mu > \mu^\#(\lambda)$$

and, recalling the discussion leading to (5), the above condition is necessary for the existence of a positive  $T$ -periodic solution when  $g'(u) > 0$  for any  $u > 0$ . It can be interesting to observe that, from a functional analytic point of view, this average condition plays the role of pushing the super-sublinear function  $g(u)$  below the principal eigenvalue  $\lambda_0 = 0$  of the periodic problem, both at the origin and at the infinity (notice that this is not needed when Dirichlet boundary conditions are taken into account, since the principal eigenvalue is strictly positive).

Theorem 2.1 is proved in [13] using a topological degree argument (see also [28, 29]); more precisely, it is shown therein that the coincidence degree (for a suitable operator associated with (2)) is equal to 1 both on small balls and on large balls centered at the origin, being instead equal to 0 on balls with intermediate radius. The existence of two positive  $T$ -periodic solutions then follows from the

excision property of the degree and maximum principle arguments. It is worth mentioning that the same strategy works for the damped equation

$$u'' + cu' + (\lambda a^+(t) - \mu a^-(t))g(u) = 0, \quad (7)$$

where  $c \in \mathbb{R}$  is an arbitrary constant. In the conservative case  $c = 0$ , a slightly less general version of Theorem 2.1 was previously obtained in [18] using variational arguments. We mention this here, since the proof can be easily understood: the condition  $\mu > \mu^\#(\lambda)$  is used, together with the super-sublinearity of  $g(u)$ , to show that the action functional is bounded from below and has a strict local minimum at the origin; on the other hand, the largeness of  $\lambda$  ensures that the functional attains negative values. Two  $T$ -periodic solutions are then obtained via global minimization and a mountain pass procedure, respectively (by standard arguments, they can be shown to be positive).

### 3 Subharmonic solutions

Having proved the existence of positive  $T$ -periodic (i.e., harmonic) solutions, a further question which naturally arises is the existence of positive periodic solutions with larger minimal period: say,  $kT$ -periodic solutions, with  $k \geq 2$  an integer number (i.e., subharmonic solutions). This issue, which is peculiar of the periodic setting, is typically quite delicate, the most difficult point consisting of course in the proof of the minimality of the period for the periodic solutions found (we refer to [20] for several remarks about the topic, as well as for an extensive bibliography). As for equation (2), we propose the following result.

**Theorem 3.1.** *In the same setting of Theorem 2.1, let us suppose further that  $g(u)$  is of class  $\mathcal{C}^2$  on  $[0, \rho)$  for some  $\rho > 0$ , with  $g''(u) > 0$  for any  $u \in (0, \rho)$ . Then, for  $\lambda > \lambda^*$  and  $\mu > \mu^\#(\lambda)$ , equation (2) has two positive  $T$ -periodic solutions, as well as positive subharmonic solutions of order  $k$  for any sufficiently large integer  $k$  (moreover, the number of positive subharmonics of order  $k$  goes at infinity for  $k \rightarrow +\infty$ ).*

Let us clarify that by a subharmonic solution of order  $k$  we mean a  $kT$ -periodic solution which is not  $lT$ -periodic for any integer  $l = 1, \dots, k - 1$  (this is the most general definition of subharmonic solution, and is the natural one when just the  $T$ -periodicity of  $a(t)$  is assumed; whenever  $T$  is the *minimal* period of  $a(t)$ , it is easy to see that subharmonic solutions of order  $k$  actually have  $kT$  as the minimal period).

The proof of Theorem 3.1 is given in [12] and it consists in a non-standard application of the Poincaré-Birkhoff fixed point theorem to a suitable Poincaré operator associated with (2). More precisely, the crucial steps are the following: first, the local convexity assumption on  $g(u)$  is used to prove (via a clever algebraic trick first used, in a slightly different context, by Brown and Hess [22]) that one of the  $T$ -periodic solutions given by Theorem 2.1, say  $u^*(t)$ , has Morse index different from

zero; then, following ideas developed in [17, 20], the Poincaré-Birkhoff theorem is applied to give  $kT$ -periodic solutions  $u_k(t)$  oscillating around  $u^*(t)$ : the information on the number of zeros of  $u_k(t) - u^*(t)$  (which is an intrinsic feature of the periodic solutions constructed with this technique) is then the key point in showing that  $kT$  is the minimal period of  $u_k(t)$ .

It is worth recalling that the possibility of applying the Poincaré-Birkhoff theorem strongly relies on the Hamiltonian structure of the equation: accordingly, this technique does not work for the damped equation (7). The other typical way to search for subharmonic solutions is the use of variational methods, which also require  $c = 0$ . Therefore, investigating the existence of positive subharmonic solutions to (7) seems to be a challenging open problem.

## 4 When $\mu \rightarrow +\infty$ : high multiplicity and chaotic dynamics

In this section, we focus on a different aspect of the dynamics of (2): roughly speaking, we show how it becomes extremely rich when the parameter  $\mu$  is very large. More precisely, we state the following result (with obvious notation, we name  $I_i^+$  all the intervals of positivity of the weight function  $a(t)$  on the real line, by letting the index  $i$  to vary on  $\mathbb{Z}$ ).

**Theorem 4.1.** *Let  $g(u)$  and  $a(t)$  be as in Theorem 2.1. Then, fixed an arbitrary constant  $\rho > 0$  there exists  $\lambda^* = \lambda^*(\rho) > 0$  such that for each  $\lambda > \lambda^*$  there exist two constants  $r, R$  with  $0 < r < \rho < R$  and  $\mu^*(\lambda) = \mu^*(\lambda, r, R) > 0$  such that for any  $\mu > \mu^*(\lambda)$  the following holds: given any two-sided sequence  $\mathcal{S} = (\mathcal{S}_i)_{i \in \mathbb{Z}}$  in the alphabet  $\mathcal{A} := \{0, 1, 2\}$  which is not identically zero, there exists at least one positive solution  $u(t)$  of (2) such that*

- $\max_{t \in I_i^+} u(t) < r$ , if  $\mathcal{S}_i = 0$ ;
- $r < \max_{t \in I_i^+} u(t) < \rho$ , if  $\mathcal{S}_i = 1$ ;
- $\rho < \max_{t \in I_i^+} u(t) < R$ , if  $\mathcal{S}_i = 2$ .

*Moreover, whenever the two-sided sequence  $\mathcal{S}$  is  $k$ -periodic for some integer  $k$ , the corresponding positive solution  $u(t)$  of (2) can be chosen to be a  $kT$ -periodic function (hence, a positive  $kT$ -periodic solution to (2)).*

This result describes a typical picture of *symbolic dynamics*: globally defined positive solutions to (2) are constructed, having a multibump chaotic-like behavior coded by a double sequence  $\mathcal{S} = (\mathcal{S}_i)_{i \in \mathbb{Z}}$  in an alphabet of three symbols. A remarkable feature, moreover, is that periodic sequences of symbols can be realized through periodic solutions of the equation. As a consequence, multiple positive  $kT$ -periodic solutions to (2) can be obtained for any  $k \geq 2$ : simply by checking the minimality of the period for the corresponding sequence  $\mathcal{S} \in \{0, 1, 2\}^{\mathbb{Z}}$ , many of

these positive  $kT$ -periodic solutions can be shown to be positive subharmonics of order  $k$ .

Also, assuming  $m \geq 2$  in  $(a_*)$ , we easily obtain from Theorem 4.1 that equation (2) has at least  $3^m - 1$  positive  $T$ -periodic solutions for  $\lambda > \lambda^*$  and  $\mu$  very large (typically, much larger than the sharp value  $\mu^\#(\lambda)$  given in Theorem 2.1). We can interpret this *high multiplicity* result in a singular perturbation spirit. Indeed, it is possible to show that the solutions constructed in Theorem 4.1 converge, for  $\mu \rightarrow +\infty$ , to solutions of the Dirichlet problem associated with  $u'' + \lambda a^+(t)g(u) = 0$  on each  $I_i^+$  (and to zero elsewhere); since three non-negative solutions for this boundary value problem are always available (the trivial one, and two positive solutions - a small one and a large one - given by Rabinowitz's theorem [38], compare with the discussion after Theorem 2.1), the above Theorem 4.1 shows that, on the converse, positive  $T$ -periodic solutions of (2) can be obtained, when  $\mu$  is very large, being either "very small" on  $I_i^+$  (if  $\mathcal{S}_i = 0$ ), "small" (if  $\mathcal{S}_i = 1$ ) or "large" (if  $\mathcal{S}_i = 2$ ).

The proof of Theorem 4.1 is given in [14], using in a very delicate way coincidence degree theory (a previous result about chaotic dynamics - on two symbols only - for (2) was given in [19] using a completely different technique based on topological horseshoes theory). We are confident that the same technique works also for the damped equation (7), though this is not formally proved yet.

We like to mention that the possibility of finding multiple positive solutions of indefinite nonlinear problems by playing with the nodal behavior of the weight function was initially suggested in a paper by Gómez-Reñasco and López-Gómez [34] (therein, an interesting analogy is proposed with the celebrated papers by Dancer [26, 27] providing multiplicity of solutions to elliptic BVPs by playing with the shape of the domain). The first complete result in this direction was then given by Gaudenzi, Habets and Zanolin [31, 32] for the Dirichlet boundary value problem associated with the superlinear indefinite equation

$$u'' + (a^+(t) - \mu a^-(t))u^p = 0, \quad \text{with } p > 1;$$

later on, along this line of research, several contributions followed [7, 10, 11, 29, 30, 33]), dealing both with ODEs and PDEs, with various boundary conditions, always in the superlinear case. Theorem 4.1 thus extends these ideas to the super-sublinear setting, showing that the corresponding dynamics is even richer.

## 5 Some complementary results

To conclude, we observe that the solvability picture described in Theorem 2.1 naturally suggests a couple of (quite subtle) questions:

- (Q1) is the existence of positive  $T$ -periodic solutions still possible when  $\lambda > 0$  is small?
- (Q2) is the existence of positive  $T$ -periodic solutions still possible - for a non-monotone  $g(u)$  - when  $0 < \mu \leq \mu^\#(\lambda)$ , that is, when the average of the weight function is non-negative?

In this final section we try to give partial answers, by imposing some further conditions on  $a(t)$  and  $g(u)$ .

More precisely, as for (Q1) we propose the following result.

**Theorem 5.1.** *Let us assume that  $a(t)$  is even-symmetric, with  $m = 1$  in  $(a_*)$ ; moreover, suppose that  $g'(u) \neq 0$  for  $u \notin [\eta, \frac{1}{\eta}]$  (with  $\eta > 0$ ) and that*

$$\lim_{u \rightarrow +\infty} g'(u) = 0.$$

*Then, for any  $\lambda > 0$  there exists  $\mu^+(\lambda) > \mu^\#(\lambda)$  such that equation has at least two positive  $T$ -periodic solutions for any  $\mu \in (\mu^\#(\lambda), \mu^+(\lambda))$ .*

The meaning of the above result is, roughly speaking, the following: the existence of positive  $T$ -periodic solutions to (2) is certainly ensured for *any*  $\lambda > 0$  provided that  $\mu$  is not too large (from this point of view, Theorem 2.1 thus ensures that  $\mu^+(\lambda) = +\infty$  for  $\lambda > \lambda^*$ ). We mention however that, for the one-parameter equation  $u'' + \lambda p(t)g(u) = 0$ , with  $\int_0^T p(t) dt < 0$  and  $g(u)$  super-sublinear, non-existence of positive  $T$ -periodic solutions can often be proved when  $\lambda > 0$  is small, see [13] (be careful, this is not a contradiction: why?)

We finally turn to the question (Q2).

**Theorem 5.2.** *In the setting of Theorem 5.1, assume further that  $g'(u) < 0$  for  $u > \frac{1}{\eta}$  (implying that  $g(u)$  is non-monotone). Then, for any  $\lambda > 0$  there exist  $\mu^-(\lambda) \in (0, \mu^\#(\lambda))$  such that equation (2) has at least one positive  $T$ -periodic solution for  $\mu = \mu^\#(\lambda)$  and at least two positive  $T$ -periodic solutions for any  $\mu \in (\mu^-(\lambda), \mu^\#(\lambda))$ .*

That is: the existence of positive  $T$ -periodic solutions to (2) is still ensured (for any  $\lambda > 0$ ) provided that  $\mu$  is not too small (namely, when the average of the weight function is non-negative but not too large). Notice in particular that equation (2) with  $\mu = \mu^\#(\lambda)$  (that is, when the weight function has zero average) is always solvable (under the assumptions of Theorem 5.2, of course).

Both Theorem 5.1 and Theorem 5.2 are proved in [16], taking advantage of an ingenious change of variable introduced in [21] (see also [15]), which transforms the sign-indefinite equation (2) into a forced perturbation of an autonomous equation, that is

$$x'' = h(x)(x')^2 + (\lambda a^+(t) - \mu a^-(t)), \tag{8}$$

where  $h(x)$  is a suitable function (obtained from  $g(u)$ ) defined on the whole real line. A shooting argument is then used to find positive solutions to (8) satisfying Neumann boundary conditions on  $[0, \frac{T}{2}]$  and  $T$ -periodic solutions are finally obtained using the symmetry assumption on  $a(t)$ . This is of course a serious restriction, but it seems quite hard to prove the above results using functional analytic techniques.

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## References

- [1] N. Ackermann, *Long-time dynamics in semilinear parabolic problems with autocatalysis*, Recent progress on reaction-diffusion systems and viscosity solutions, 1–30, World Sci. Publ., Hackensack, NJ, 2009.
- [2] S. Alama and G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities*, Calc. Var. Partial Differential Equations **1** (1993), 439–475.
- [3] H. Amann, *On the number of solutions of nonlinear equations in ordered Banach spaces*, J. Functional Analysis **11** (1972), 346–384.
- [4] H. Amann and J. López-Gómez, *A priori bounds and multiple solutions for superlinear indefinite elliptic problems*, J. Differential Equations **146** (1998), 336–374.
- [5] F.V. Atkinson, W.N. Everitt and K.S. Ong, *On the  $m$ -coefficient of Weyl for a differential equation with an indefinite weight function*, Proc. London Math. Soc. (3) **29** (1974), 368–384.
- [6] C. Bandle, M.A. Pozio and A. Tesei, *Existence and uniqueness of solutions of nonlinear Neumann problems*, Math. Z. **199** (1988), 257–278.
- [7] V. Barutello, A. Boscaggin and G. Verzini, *Positive solutions with a complex behavior for superlinear indefinite ODEs on the real line*, J. Differential Equations **259** (2015), 3448–3489.
- [8] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems*, Topol. Methods Nonlinear Anal. **4** (1994) 59–78.
- [9] H. Berestycki, I. Capuzzo-Dolcetta and L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA Nonlinear Differential Equations Appl. **2** (1995), 553–572.
- [10] D. Bonheure, J.M. Gomes and P. Habets, *Multiple positive solutions of superlinear elliptic problems with sign-changing weight*, J. Differential Equations **214** (2005), 36–64.
- [11] A. Boscaggin, *A note on a superlinear indefinite Neumann problem with multiple positive solutions*, J. Math. Anal. Appl. **377** (2011), 259–268.
- [12] A. Boscaggin and G. Feltrin, *Positive subharmonic solutions to nonlinear ODEs with indefinite weight*, submitted, preprint available online at <http://arxiv.org/pdf/1605.02500.pdf>

- [13] A. Boscaggin, G. Feltrin and F. Zanolin, *Pairs of positive periodic solutions of nonlinear ODEs with indefinite weight: a topological degree approach for the super-sublinear case*, Proc. Roy. Soc. Edinburgh Sect. A **146** (2016), 449–474.
- [14] A. Boscaggin, G. Feltrin and F. Zanolin, *Positive solutions for super-sublinear indefinite problems: high multiplicity results via coincidence degree*, to appear on Trans. Amer. Math. Soc., preprint available online at <http://arxiv.org/pdf/1512.07138.pdf>
- [15] A. Boscaggin and M. Garrione, *Multiple solutions to Neumann problems with indefinite weight and bounded nonlinearities*, J. Dynam. Differential Equations **28** (2016), 167–187.
- [16] A. Boscaggin and M. Garrione, *Positive solutions to indefinite Neumann problems when the weight has positive average*, Discrete Contin. Dyn. Syst. **36** (2016), 5231–5244.
- [17] A. Boscaggin, R. Ortega and F. Zanolin, *Subharmonic solutions of the forced pendulum equation: a symplectic approach*, Arch. Math. (Basel) **102** (2014), 459–468.
- [18] A. Boscaggin and F. Zanolin, *Pairs of positive periodic solutions of second order nonlinear equations with indefinite weight*, J. Differential Equations **252** (2012), 2900–2921.
- [19] A. Boscaggin and F. Zanolin, *Positive periodic solutions of second order nonlinear equations with indefinite weight: multiplicity results and complex dynamics*, J. Differential Equations **252** (2012), 2922–2950.
- [20] A. Boscaggin and F. Zanolin, *Subharmonic solutions for nonlinear second order equations in presence of lower and upper solutions*, Discrete Contin. Dyn. Syst. **33** (2013), 89–110.
- [21] A. Boscaggin and F. Zanolin, *Second order ordinary differential equations with indefinite weight: the Neumann boundary value problem*, Ann. Mat. Pura Appl. (4) **194** (2015), 451–478.
- [22] K.J. Brown and P. Hess, *Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem*, Differential Integral Equations **3** (1990), 201–207.
- [23] G.J. Butler, *Rapid oscillation, nonextendability, and the existence of periodic solutions to second order nonlinear ordinary differential equations*, J. Differential Equations **22** (1976), 467–477.
- [24] G.J. Butler, *Periodic solutions of sublinear second order differential equations*, J. Math. Anal. Appl. **62** (1978), 676–690.
- [25] A. Capietto, W. Dambrosio and D. Papini, *Superlinear indefinite equations on the real line and chaotic dynamics*, J. Differential Equations **181** (2002), 419–438.
- [26] E.N. Dancer, *The effect of domain shape on the number of positive solutions of certain nonlinear equations I*, J. Differential Equations **74** (1988), 120–156.
- [27] E.N. Dancer, *The effect of domain shape on the number of positive solutions of certain nonlinear equations II*, J. Differential Equations **87** (1990), 316–339.
- [28] G. Feltrin and F. Zanolin, *Existence of positive solutions in the superlinear case via coincidence degree: the Neumann and the periodic boundary value problems*, Adv. Differential Equations **20** (2015), 937–982.
- [29] G. Feltrin and F. Zanolin, *Multiple positive solutions for a superlinear problem: a topological approach*, J. Differential Equations **259** (2015), 925–963.

- [30] G. Feltrin and F. Zanolin, *Multiplicity of positive periodic solutions in the superlinear indefinite case via coincidence degree*, submitted, preprint available online at <https://arxiv.org/pdf/1508.01867.pdf>
- [31] M. Gaudenzi, P. Habets and F. Zanolin, *An example of a superlinear problem with multiple positive solutions*, Atti Sem. Mat. Fis. Univ. Modena **51** (2003), 259–272.
- [32] M. Gaudenzi, P. Habets and F. Zanolin, *Positive solutions of superlinear boundary value problems with singular indefinite weight*, Commun. Pure Appl. Anal. **2** (2003), 411–423.
- [33] P.M. Girão and J.M. Gomes, *Multibump nodal solutions for an indefinite superlinear elliptic problem*, J. Differential Equations **247** (2009), 1001–1012.
- [34] R. Gómez-Reñasco and J. López-Gómez, *The effect of varying coefficients on the dynamics of a class of superlinear indefinite reaction-diffusion equations*, J. Differential Equations **167** (2000), 36–72.
- [35] P. Hess and T. Kato, *On some linear and nonlinear eigenvalue problems with an indefinite weight function*, Comm. Partial Differential Equations **5** (1980), 999–1030.
- [36] D. Papini and F. Zanolin, *A topological approach to superlinear indefinite boundary value problems*, Topol. Methods Nonlinear Anal. **15** (2000), 203–233.
- [37] D. Papini and F. Zanolin, *On the periodic boundary value problem and chaotic-like dynamics for nonlinear Hill's equations*, Adv. Nonlinear Stud. **4** (2004), 71–91.
- [38] P.H. Rabinowitz, *Pairs of positive solutions of nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **23** (1973/74), 173–186.
- [39] S. Terracini and G. Verzini, *Oscillating solutions to second order ODEs with indefinite superlinear nonlinearities*, Nonlinearity **13** (2000), 1501–1514.

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