Period-m Motions to Chaos in a Periodically Forced, Duffing Oscillator with a Time-Delayed Displacement

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In this paper, periodic motions in a periodically excited, Duffing oscillator with a time-delayed displacement are investigated through the Fourier series, and the stability and bifurcation of such periodic motions are discussed through eigenvalue analysis. The time-delayed displacement is from the feedback control of displacement. The analytical bifurcation trees of period-1 motions to chaos in the time-delayed Duffing oscillator are presented through asymmetric period-1 to period-4 motions. Stable and unstable periodic motions are illustrated through numerical and analytical solutions. From numerical illustrations, the analytical solutions of stable and unstable period-m motions are relatively accurate with $A_{N/m} < 10^{-6}$ compared to numerical solutions. From such analytical solutions, any complicated solutions of period-m motions can be obtained for any prescribed accuracy. Because time-delay may cause discontinuity, the appropriate time-delay inputs (or initial conditions) in the initial time-delay interval should satisfy the analytical solution of periodic motions in the time-delayed dynamical systems. Otherwise, periodic motions in such a time-delayed system cannot be obtained directly.

Keywords: Time-delayed Duffing oscillator; period-m motions; nonlinear dynamical systems; generalized harmonic balance; Hopf bifurcation; bifurcation trees.

1. Introduction

The Duffing oscillator is extensively used to investigate structural vibrations in engineering. In order to reduce structural vibration, the feedback control is often used. Because the feedback states in the structural vibration are introduced and such a feedback state is time-delayed, the time-delayed Duffing oscillator in structural dynamical systems is obtained, which is used to investigate structural vibrations with feedback control. In [Luo & Jin, 2014a], complex period-1 motions of the periodically forced Duffing oscillator with a time-delayed displacement were investigated, and the complicated period-1 motions cannot be obtained from the traditional harmonic balance and perturbation methods. Herein, the period-m motions of the time-delayed Duffing oscillator will be investigated analytically in order to understand how the feedback stabilizes periodic motions in the periodically forced Duffing oscillator, and complex period-m motions will be obtained in such time-delayed Duffing oscillator. The bifurcation trees of period-m motions to chaos will be discussed to understand the global characteristics of periodic motion in such a time-delayed oscillator.

For periodic motions in dynamical systems, Lagrange [1788] used the method of averaging for the periodic motions of three-body problems.

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In the 19th century, Poincaré [1899] developed the perturbation theory to determine the periodic motions of celestial bodies. van der Pol [1920] employed the method of averaging for the periodic solutions of oscillation systems in circuits. Fatou [1928] used the existing theorems of solutions of differential equations to give the first proof of asymptotic validity of the method of averaging. Krylov and Bogoliubov [1935] further developed the method of averaging. Bogoliubov and Mitropolsky [1961] presented the asymptotic perturbation methods for nonlinear oscillations. Hayashi [1964] studied the approximate periodic solutions of nonlinear oscillators, and the corresponding stability of the approximate periodic solutions were determined by the improved Mathieu equation. Nayfeh [1973] applied multiscale methods for approximate solutions of periodic motions in nonlinear structural dynamics (see also [Nayfeh & Mook, 1979]). Coppola and Rand [1990] employed the method of averaging with elliptic functions for the approximation of limit cycle. Luo [2012] developed a methodology for analytical solutions of periodic motions in nonlinear dynamical systems. Luo and Huang [2012a] applied such a method for the analytical bifurcation trees of period-m motions to chaos in the Duffing oscillator. Luo and Huang [2012b, 2012c] investigated the analytical routes of period-1 to chaos and stable and unstable period-m motions in the Duffing oscillator. In recent years, time-delayed systems are of great interest because of extensive applications (e.g. [Tlusty, 2000; Hu & Wang, 2002]). One tried to work on numerical methods for the corresponding complicated behaviors. The stability and bifurcation of equilibriums of the time-delayed systems were investigated (e.g. [Stepan, 1989; Sun, 2009; Insperger & Stepan, 2011]). Periodic solutions in time-delayed dynamical systems were investigated by perturbation methods (e.g. [Hu et al., 1998; Wang & Hu, 2006]). The harmonic balance method was also employed to determine approximate periodic solutions for delayed nonlinear oscillators (e.g. [MacDonald, 1995; Liu & Kalmar-Nagy, 2010]), but such approximate solutions of periodic motions in the time-delayed oscillators are not accurate enough. Luo [2013] systematically proposed a methodology for periodic motions in time-delayed, nonlinear dynamical systems. Luo and Jin [2014a] used such a methodology to investigate the bifurcation tree of period-1 motion to chaos in a periodically forced, quadratic nonlinear oscillator with time-delay. To further understand the properties of periodic solutions in time-delayed nonlinear systems, the periodically forced, time-delayed, Duffing oscillator should be investigated analytically, and in [Luo & Jin, 2014b], symmetric and asymmetric period-1 motions of the periodically forced Duffing oscillator with a time-delayed displacement were discussed. Herein, the analytical bifurcation trees of asymmetric period-1 motion to chaos will be investigated to the global behaviors of periodic motions in such a time-delayed oscillator.

In this paper, the analytical solutions of period-m motions in the time-delayed Duffing oscillator will be developed. The stability and bifurcation analysis of period-m motions in the time-delayed Duffing oscillator will be carried out through the eigenvalue analysis. The bifurcation trees of period-1 motion to chaos will be illustrated by the period-1 motion to period-4 motion. Numerical simulations will be performed to compare analytical and numerical solutions of period-m motions, and harmonic effects on periodic motions will be discussed through harmonic amplitude spectrums.

2. Analytical Solutions

Consider a time-delayed Duffing oscillator as

\[ \dot{x} + \delta \dot{x} + \alpha_2 x - \alpha_2 x^2 + \gamma x^3 = Q_0 \cos(\Omega t) \]  

where \( x^\tau = x(t - \tau) \). Coefficients are \( \delta \) for linear damping, \( \alpha_1 \) and \( \alpha_2 \) for linear spring and linear time-delay, \( \gamma \) for cubic nonlinearity, \( Q_0 \) and \( \Omega \) for excitation amplitude and frequency, respectively. In Luo [2012, 2013], the standard form of Eq. (1) can be written as

\[ \ddot{x} + f(x, \dot{x}, x^\tau, \dot{x}^\tau, t) = 0 \]  

where

\[ f(x, \dot{x}, x^\tau, \dot{x}^\tau, t) = \delta \ddot{x} + \alpha_1 x - \alpha_2 x^2 + \gamma x^3 - Q_0 \cos(\Omega t). \]  

In [Luo, 2012, 2013], the analytical solution of period-m motion is assumed as

\[ x^{(m)}(t) = a_0^{(m)}(t) + \sum_{k=1}^{N} b_k/\omega(t) \cos \left( \frac{k}{\omega} t \right) + c_k/\omega(t) \sin \left( \frac{k}{\omega} t \right). \]
where \( \alpha_0^{(m)}(t) = \alpha_0^{(m)}(t - \tau) \), \( \beta_{k/m}^{(m)}(t) = \beta_{k/m}(t - \tau) \), \( \gamma_{k/m}^{(m)}(t) = \gamma_{k/m}(t - \tau) \), \( \theta = \Omega t \) and \( \theta' = \Omega \tau \). The first and second order derivatives of \( x^{(m)}(t) \) and \( x^{(m)}(t) \) are

\[
\dot{x}^{(m)} = \dot{a}_0^{(m)}(t) + \sum_{k=1}^{N} \left[ \left( \frac{\Omega k}{m} \right)^2 \beta_{k/m}(t) + \frac{\Omega k}{m} \beta_{k/m}(t) \cos \left( \frac{k m \theta}{m} \right) \right] \cos \left( \frac{k m \theta}{m} \right) + \left[ \frac{\Omega k}{m} \phi_{k/m}(t) - \frac{\Omega k}{m} \beta_{k/m}(t) \sin \left( \frac{k m \theta}{m} \right) \right] \sin \left( \frac{k m \theta}{m} \right)
\]

\[
\ddot{x}^{(m)} = \ddot{a}_0^{(m)}(t) + \sum_{k=1}^{N} \left[ \left( \frac{\Omega k}{m} \right)^2 \beta_{k/m}(t) + \frac{\Omega k}{m} \beta_{k/m}(t) \cos \left( \frac{k m \theta}{m} \right) \right] \cos \left( \frac{k m \theta}{m} \right) + \left[ \frac{\Omega k}{m} \phi_{k/m}(t) - \frac{\Omega k}{m} \beta_{k/m}(t) \sin \left( \frac{k m \theta}{m} \right) \right] \sin \left( \frac{k m \theta}{m} \right)
\]

Substitution of Eqs. (4)-(6) into Eq. (2) and averaging all terms of constant, \( \cos(k \theta/m) \) and \( \sin(k \theta/m) \) give

\[
\ddot{a}_0^{(m)} + \frac{\Omega^2}{m^2} \beta_0^{(m)} = 0,
\]

\[
\dot{b}_{k/m}(t) + 2 \frac{\Omega k}{m} \beta_{k/m}(t) - \frac{\Omega^2}{m^2} \beta_{k/m}(t) \cos \left( \frac{k m \theta}{m} \right) \cos \left( \frac{k m \theta}{m} \right) + \frac{\Omega k}{m} \phi_{k/m}(t) - \frac{\Omega k}{m} \beta_{k/m}(t) \sin \left( \frac{k m \theta}{m} \right) \sin \left( \frac{k m \theta}{m} \right) = 0, \quad k = 1, 2, \ldots, N
\]

where

\[
x^{(m)} = (x_0^{(m)}, x_1^{(m)}, \ldots, x_N^{(m)})^T \quad \text{and} \quad \dot{x}^{(m)} = (\dot{x}_0^{(m)}, \dot{x}_1^{(m)}, \ldots, \dot{x}_N^{(m)})^T,
\]

\[
\ddot{x}^{(m)} = (\ddot{x}_0^{(m)}, \ddot{x}_1^{(m)}, \ldots, \ddot{x}_N^{(m)})^T \quad \text{and} \quad \dot{\ddot{x}}^{(m)} = (\dot{\ddot{x}}_0^{(m)}, \dot{\ddot{x}}_1^{(m)}, \ldots, \dot{\ddot{x}}_N^{(m)})^T;
\]

\[
b^{(m)} = (b_0^{(m)}, b_1^{(m)}, \ldots, b_N^{(m)})^T \quad \text{and} \quad \dot{b}^{(m)} = (\dot{b}_0^{(m)}, \dot{b}_1^{(m)}, \ldots, \dot{b}_N^{(m)})^T,
\]

\[
c^{(m)} = (c_0^{(m)}, c_1^{(m)}, \ldots, c_N^{(m)})^T \quad \text{and} \quad \dot{c}^{(m)} = (\dot{c}_0^{(m)}, \dot{c}_1^{(m)}, \ldots, \dot{c}_N^{(m)})^T;
\]

\[
\dot{b}^{(m)} = (\dot{b}_0^{(m)}, \dot{b}_1^{(m)}, \ldots, \dot{b}_N^{(m)})^T \quad \text{and} \quad \dot{\ddot{b}}^{(m)} = (\dot{\ddot{b}}_0^{(m)}, \dot{\ddot{b}}_1^{(m)}, \ldots, \dot{\ddot{b}}_N^{(m)})^T,
\]

\[
\ddot{c}^{(m)} = (\ddot{c}_0^{(m)}, \ddot{c}_1^{(m)}, \ldots, \ddot{c}_N^{(m)})^T \quad \text{and} \quad \ddot{\ddot{c}}^{(m)} = (\ddot{\ddot{c}}_0^{(m)}, \ddot{\ddot{c}}_1^{(m)}, \ldots, \ddot{\ddot{c}}_N^{(m)})^T;
\]
and the Fourier series coefficients of function $f(x^{(m)})$, $\dot{x}^{(m)}$, $\ddot{x}^{(m)}$, $\ldots$, $\ddots$, $x^{(m)}$, $x^{(m)}, x^{(m)}, \ldots$, are

\[
F_0^{(m)}(\mathbf{x}^{(m)}, \mathbf{\dot{x}}^{(m)}, \mathbf{\ddot{x}}^{(m)}, \mathbf{\ldots}),
\]

where $k = 1, 2, \ldots, N$.

Therefore, the constant term in the Fourier series of $f(x^{(m)}, \dot{x}^{(m)}, \ddot{x}^{(m)}, \ldots, x^{(m)})$, $f(\mathbf{x})$, is given by

\[
F_0^{(m)}(\mathbf{x}^{(m)}, \mathbf{\dot{x}}^{(m)}, \mathbf{\ddot{x}}^{(m)}) = \delta \omega_0^{(m)} + \alpha_1 \omega_0^{(m)} - \alpha_2 \omega_0^{(m)} + \gamma f^{(0)}
\]

where the constants caused by cubic nonlinearity are

\[
f^{(0)} = \omega_0^{(m)} + \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} f^{(0)}(i, j, l, q)
\]

The cosine term in the Fourier series of $f(x^{(m)}, \dot{x}^{(m)}, \ddot{x}^{(m)}, \ldots, x^{(m)})$, $f^{(c)}(\mathbf{x})$, is given by

\[
F_{1k}^{(m)}(\mathbf{x}^{(m)}, \mathbf{\dot{x}}^{(m)}, \mathbf{\ddot{x}}^{(m)}) = \delta \omega_k^{(m)} + \alpha_1 \omega_k^{(m)} - \alpha_2 \omega_k^{(m)} + \gamma f^{(c)}
\]

where $k = 1, 2, \ldots, N$.

The sine term in the Fourier series of $f(x^{(m)}, \dot{x}^{(m)}, \ddot{x}^{(m)}, \ldots, x^{(m)})$, $f^{(s)}(\mathbf{x})$, is given by

\[
F_{2k}^{(m)}(\mathbf{x}^{(m)}, \mathbf{\dot{x}}^{(m)}, \mathbf{\ddot{x}}^{(m)}) = \delta \omega_k^{(m)} + \alpha_1 \omega_k^{(m)} - \alpha_2 \omega_k^{(m)} + \gamma f^{(s)}
\]

where $k = 1, 2, \ldots, N$. 
where Eq. (22) becomes
\[
\Delta_k = (k_{i,j,l,q}^{(3)}(s_i - \delta_i z_j^2 + \sum_{m} c_{i,j,l,q}^m \Delta_k^{(s)})
\]
with
\[
\begin{align*}
\Delta_k^{(s)} &= \delta_{i+j} - \delta_{i-j} \\
\Delta_k^{(2)} &= (\delta_{i+j} + \delta_{i-j} - \delta_{i+j} - \delta_{i-j}) \\
\Delta_k^{(3)} &= (\delta_{i+j} - \delta_{i-j} + \delta_{i+j} - \delta_{i-j}) \\
&+ \delta_{i+j} - \delta_{i-j} - \delta_{i+j} - \delta_{i-j},
\end{align*}
\]

Define a set of new variables as
\[
\begin{align*}
\mathbf{z}^{(m)} &= (a_0^{(m)}, b_1^{(m)}, \ldots, b_N^{(m)}, c_1^{(m)}, \ldots, c_N^{(m)})^T \\
&\equiv (z_0^{(m)}, z_1^{(m)}, \ldots, z_N^{(m)})^T \\
\mathbf{z}^{(r)} &= (a_0^{(r)}, b_1^{(r)}, \ldots, b_N^{(r)}, c_1^{(r)}, \ldots, c_N^{(r)})^T \\
&\equiv (z_0^{(r)}, z_1^{(r)}, \ldots, z_N^{(r)})^T
\end{align*}
\]

Equation (7) is rewritten as
\[
\begin{align*}
\dot{\mathbf{z}}^{(m)} &= \mathbf{g}^{(m)}(\mathbf{z}^{(r)}, z_1^{(r)}, z_1^{(m)}), \\
&\text{and} \\
\dot{\mathbf{z}}^{(r)} &= \mathbf{g}^{(r)}(\mathbf{z}^{(m)}, z_1^{(m)}, z_1^{(r)}),
\end{align*}
\]

where
\[
\begin{align*}
\mathbf{g}^{(m)}(\mathbf{z}^{(m)}, z_1^{(m)}, z_1^{(r)}) &= \begin{bmatrix}
-F_0^{(m)}(\mathbf{z}^{(m)}, z_1^{(m)}, z_1^{(r)}) \\
-F_1^{(m)}(\mathbf{z}^{(m)}, z_1^{(m)}, z_1^{(r)}) - 2 k_1 \Omega (\mathbf{z}^{(m)})^2 + k_2 (\Omega / m)^2 \mathbf{b}^{(m)} \\
-F_2^{(m)}(\mathbf{z}^{(m)}, z_1^{(m)}, z_1^{(r)}) + 2 k_1 \Omega (\mathbf{z}^{(m)})^2 + k_2 (\Omega / m)^2 \mathbf{c}^{(m)}
\end{bmatrix}
\end{align*}
\]

where
\[
k_1 = \text{diag}(1, 2, \ldots, N) \quad \text{and} \quad k_2 = \text{diag}(1, 2^2, \ldots, N^2)
\]

for \( N = 1, 2, \ldots, \infty \). Setting
\[
\begin{align*}
\mathbf{y}^{(m)} &= (\mathbf{z}^{(m)}, z_1^{(m)}), \\
\mathbf{y}^{(r)} &= (\mathbf{z}^{(r)}, z_1^{(r)}) \quad \text{and} \\
\mathbf{f}^{(m)} &= (\mathbf{z}^{(m)}, \mathbf{g}^{(m)})^T,
\end{align*}
\]

Eq. (22) becomes
\[
\begin{align*}
\dot{\mathbf{y}}^{(m)} &= \mathbf{f}^{(m)}(\mathbf{y}^{(m)}, \mathbf{y}^{(r)}),
\end{align*}
\]
The Jacobian matrices relative to the non-delay and delay derivatives for the non-delay and delay terms can be obtained by setting $\dot{y}^{(m)} = 0$ and $\dot{y}^{(m)} = z^{(m)}$, i.e.,

\[
\begin{align*}
 F_0^{(m)} (0, 0) \dot{y}^{(m)} + c^{(m)} = 0, \\
 F_1^{(m)} (0, 0) \dot{y}^{(m)} + c^{(m)} = 0, \\
 F_2^{(m)} (0, 0) \dot{y}^{(m)} + c^{(m)} = 0, \\
 b^{(m)} (0, 0) \dot{y}^{(m)} + c^{(m)} = 0.
\end{align*}
\]

where

\[
\begin{align*}
 b^{(m)} (0, 0) \dot{y}^{(m)} + c^{(m)} &= 0, \\
 b^{(m)} (0, 0) \dot{y}^{(m)} + c^{(m)} &= 0, \\
 b^{(m)} (0, 0) \dot{y}^{(m)} + c^{(m)} &= 0.
\end{align*}
\]

In [Luo, 2012, 2013], the $(2N + 1)$ nonlinear equations in Eq. (27) are solved by the Newton–Raphson method. The linearized equation at the equilibrium point $y^{(m)} = (0, 0)^T$, $y^{(m)} = (z^{(m)}, 0)^T$ in

\[
\Delta y^{(m)} = A \Delta y^{(m)} + B \Delta y^{(m)}
\]

where

\[
\begin{align*}
 A &= \frac{\partial (y^{(m)})}{\partial (y^{(m)})} \bigg|_{y^{(m)}, y^{(m)}} \\
 B &= \frac{\partial (y^{(m)})}{\partial (y^{(m)})} \bigg|_{y^{(m)}, y^{(m)}}
\end{align*}
\]

The corresponding eigenvalues are determined by

\[
\begin{align*}
 &|A + bc^{(m)} \lambda - M_1^{(m)}(2N + 1)\lambda^{(m)}(2N + 1)| = 0. \\
 &\text{The Jacobian matrices relative to the non-delay and delay terms are given by}
\end{align*}
\]

\[
\begin{align*}
 A &= \begin{bmatrix} 0 & (2N + 1) \times (2N + 1) \end{bmatrix}, \\
 G &= \begin{bmatrix} I & (2N + 1) \times (2N + 1) \end{bmatrix}, \\
 B &= \begin{bmatrix} 0 & (2N + 1) \times (2N + 1) \end{bmatrix}, \\
 G^* &= \begin{bmatrix} 0 & (2N + 1) \times (2N + 1) \end{bmatrix}
\end{align*}
\]

where the derivatives for the non-delay and delay terms are

\[
\begin{align*}
 &G = \frac{\partial (y^{(m)})}{\partial (y^{(m)})} = (G^{(0)}, G^{(c)}, G^{(s)})^T, \\
 &G^* = \frac{\partial (y^{(m)})}{\partial (y^{(m)})} = (G^{(0)}, G^{(c)}, G^{(s)})^T
\end{align*}
\]

and

\[
\begin{align*}
 &G^{(0)} = \{c^{(0)}, G_1^{(0)}, G_2^{(0)}, \ldots, G_{2N}^{(0)}\}, \\
 &G^{(c)} = \{G_{1}^{(c)}, G_2^{(c)}, \ldots, G_N^{(c)}\}^T, \\
 &G^{(s)} = \{G_{1}^{(s)}, G_2^{(s)}, \ldots, G_N^{(s)}\}^T, \\
 &G^{(0)} = \{c^{(0)}, G_1^{(0)}, G_2^{(0)}, \ldots, G_{2N}^{(0)}\}, \\
 &G^{(c)} = \{G_{1}^{(c)}, G_2^{(c)}, \ldots, G_N^{(c)}\}^T, \\
 &G^{(s)} = \{G_{1}^{(s)}, G_2^{(s)}, \ldots, G_N^{(s)}\}^T
\end{align*}
\]

for $i = 1, 2, \ldots, N$ and $N = 1, 2, \ldots, \infty$ with

\[
\begin{align*}
 &G_k^{(c)} = \{G_{k,1}^{(c)}, G_{k,2}^{(c)}, \ldots, G_{k,N}^{(c)}\}^T, \\
 &G_k^{(s)} = \{G_{k,1}^{(s)}, G_{k,2}^{(s)}, \ldots, G_{k,N}^{(s)}\}^T
\end{align*}
\]

for $k = 1, 2, \ldots, N$. The corresponding components are

\[
\begin{align*}
 &G_k^{(0)} = -\alpha_1 G_k^{(0)} - y_G^{(0)}, \\
 &G_k^{(c)} = \begin{bmatrix} \frac{\Omega_m^2}{m} \delta_k^c - \frac{\Omega_m}{m} \delta_{k+1}^c - \alpha_1 \delta_k^c - \gamma_1 \delta_{k}^c \end{bmatrix}, \\
 &G_k^{(s)} = \begin{bmatrix} \frac{\Omega_m^2}{m} \delta_k^s + \frac{\Omega_m}{m} \delta_{k+1}^s - \alpha_1 \delta_k^s - \gamma_1 \delta_{k}^s \end{bmatrix}, \\
 &G_k^{(0)} = \alpha_2 G_k^{(0)}, \\
 &G_k^{(c)} = \alpha_2 \begin{bmatrix} \sin \left( \frac{\Omega_m}{m} \omega \right) \delta_k^c \cos \left( \frac{\Omega_m}{m} \omega \right) \delta_{k+1}^c \sin \left( \frac{\Omega_m}{m} \omega \right) \end{bmatrix}, \\
 &G_k^{(s)} = \alpha_2 \begin{bmatrix} \sin \left( \frac{\Omega_m}{m} \omega \right) \delta_k^s \cos \left( \frac{\Omega_m}{m} \omega \right) \delta_{k+1}^s \sin \left( \frac{\Omega_m}{m} \omega \right) \end{bmatrix}
\end{align*}
\]

For the constant term in the Fourier series of $f(x^{(m)} r, z^{(m)} r, z^{(m)} r, t)$, we have

\[
\begin{align*}
 g_0^{(0)} &= \{c^{(0)}, G_1^{(0)}, G_2^{(0)} + \sum_{i=1}^{N} \frac{2}{\delta_k^0} \delta_k^0 + c_1^0 \delta_k^0, \\
 &+ 2a_0 \delta_k^0 \delta_k^c + 2 \delta_k^0 \delta_k^s \delta_k^s \delta_k^s, \\
 &+ \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} \frac{3}{\delta_k^0} \delta_k^0 \delta_k^c \delta_k^c \delta_k^s \delta_k^s \delta_k^s\}
\end{align*}
\]
for \( k = 1, 2, \ldots, N \). The corresponding components are

\[ H_r^{(0)} = -\delta_0 \delta_r, \]
\[ H_r^{(c)} = -2\frac{\Omega}{m} \delta_r + \delta_0 \delta_r, \]
\[ H_r^{(s)} = \frac{k \Omega}{m} \delta_r - \delta_0 \delta_r \]

for \( r = 0, 1, \ldots, 2N \). From [Luo, 2012], the eigenvalues of Eq. (28) are classified as

\[ (n_1, n_2, n_3 | \mu_1, \mu_2, \mu_3) \]

where \( n_1 \) is the total number of negative real eigenvalues, \( n_2 \) is the total number of positive real eigenvalues, \( n_3 \) is the total number of zero eigenvalues; \( n_4 \) is the total pair number of complex eigenvalues with negative real parts, \( n_5 \) is the total pair number of complex eigenvalues with positive real parts, \( n_6 \) is the total pair number of complex eigenvalues with zero real parts. The corresponding boundary between the stable and unstable solutions is given by the saddle-node bifurcation and Hopf bifurcation. To compute eigenvalue in Eq. (30), the initial guess can be adopted from the eigenvalues of the dynamical system without time-delay, and the Newton-Raphson method will be employed in Eq. (30) to compute the eigenvalue of the time-delayed system. This way is much easier than the eigenvalues assumed as \( \lambda = \pm \omega \cdot 1 \) (i = \( \sqrt{-1} \)). Such a way with \( \lambda = \pm \omega \cdot 1 \) makes the computation of eigenvalues very complicated or almost impossible in higher-dimensional systems.

The eigenvalues computed in this paper were verified by other techniques. For instance, the discrete mapping techniques were used, which is not presented in this paper. For lower-dimensional systems, the discrete mapping techniques give the results of eigenvalues fast. However, for higher-dimensional systems, the method used in this paper can generate eigenvalues fast.

3. Bifurcation Trees

The harmonic amplitudes varying with excitation frequency \( \Omega \) are illustrated herein. The corresponding solution in Eq. (4) can be rewritten as

\[ x^{(n)}(t) = a_0^{(m)} + \sum_{k=1}^{N} A_{k/m} \]
\[ \times \cos \left( \frac{k}{m} \Omega - \varphi_{k/m} \right), \]
The harmonic amplitude and phase are

\[ A_{k/m} = \sqrt{b_{k/m}^2 + c_{k/m}^2} \quad \text{and} \quad \varphi_{k/m} = \arctan \frac{c_{k/m}}{b_{k/m}}, \]

where the harmonic amplitude and phase are

\[ A_{k/m} = \sqrt{b_{k/m}^2 + c_{k/m}^2} \quad \text{and} \quad \varphi_{k/m} = \arctan \frac{c_{k/m}}{b_{k/m}}. \]

For periodic motion, we have \( a_0^{(m)} = a_0, b_{k/m} = b_{k/m} \) and \( c_{k/m} = c_{k/m} \), so \( A_{k/m} = A_{k/m} \) and \( \varphi_{k/m} = \varphi_{k/m} \). The system parameters are considered as randomly selected by

\[ \delta = 0.5, \quad \alpha_1 = 10.0, \quad \alpha_2 = 5.0, \quad \gamma = 10.0, \quad Q_0 = 200, \quad \tau = \frac{T}{4}. \]

The acronyms “SN” and “HB” represent the saddle-node bifurcation, and Hopf bifurcation, respectively. Solid and dashed curves represent stable and unstable periodic motions, respectively. The two saddle-node bifurcations are traditionally called the jumping points at the jumping phenomena.

In Fig. 1, the harmonic amplitude varying with excitation frequency is presented in the range of excitation frequency \( \Omega \in (0, 30) \) for a global view of periodic motions relative to the period-1 motion of the time-delayed Duffing oscillator. In Fig. 1(a), constant \( a_0^{(m)} \) versus excitation frequency is presented. Symmetric and asymmetric period-1 motions possess \( a_0^{(m)} = 0 \) and \( a_0^{(m)} \neq 0 \), respectively. Because the plot in the global view of period-1 motion is very crowded, the excitation frequency is broken from \( \Omega \in (9, 9, 28) \) in the total range \( \Omega \in (0, 30) \) for a clear illustration of the asymmetric motion branches. The harmonic amplitude of periodic motions associated with asymmetric period-1 motions will not be labeled herein, which will be zoomed into later. The saddle-node bifurcations occur from the symmetric to asymmetric period-1 motion. For this time-delayed Duffing oscillator, such saddle-node points are \( \Omega \approx 6.61, 3.47 \) (first branch, B1), \( \Omega \approx 2.98, 2.26 \) (second branch, B2), \( \Omega \approx 2.04, 1.63 \) (third branch, B3), \( \Omega \approx 1.49, 1.29 \) (fourth branch, B4), \( \Omega \approx 1.16, 1.08 \) (fifth branch, B5) and \( \Omega \approx 1.00, 0.88 \) (sixth branch, B6). In Fig. 1(b), the primary harmonic amplitude of the time-delayed, Duffing oscillator is presented. To illustrate symmetric motions, the excitation frequency range will not be broken. The primary harmonic amplitude of symmetric period-1 motion is similar in the traditional analysis, but the traditional analysis cannot provide such accurate frequency-amplitude curves. For this time-delayed Duffing oscillator, the saddle-node bifurcation of symmetric period-1 motion is connected with stable and unstable symmetric period-1 motions. The saddle-node bifurcation between the upper stable symmetric period-1 motion and middle unstable symmetric period-1 motions is at \( \Omega \approx 28.85 \), and the saddle-node bifurcation between the lower stable symmetric period-1 motion and middle unstable symmetric period-1 motions is at \( \Omega \approx 11.68 \). The two saddle-node points are traditionally called the jumping points at the jumping phenomena.

To avoid abundant illustrations, harmonic amplitude \( A_0 \) versus excitation frequency is presented in Fig. 1(e). For \( \Omega > 10 \), we have \( A_0 < 10^{-10} \).
Fig. 1. A global view for frequency-amplitude curves of period-1 to period-4 motions based on 80 harmonic terms (HB80) in the time-delayed Duffing oscillator: (a) $a_0^{(m)}$ ($m = 1, 2, 4$), (b)-(f) $A_k^{(m)}$ ($k = 4, 8, 12, 76, 80$), ($\delta = 0.5, \alpha_1 = 10, \alpha_2 = 5$, $\beta = 10, Q_0 = 200, \tau = T/4$).
For Ω near zero, $A_{19} \sim (0.01, 0.1)$. Thus more harmonic terms in the Fourier series solution should be considered. In Fig. 1(f), harmonic amplitude $A_{20}$ varying with excitation frequency is presented. For symmetric period-1 motion, we have $A_{20} = 0$, and for asymmetric period-1 motion, we have $A_{20} \neq 0$.

For asymmetric motion, the harmonic amplitude $A_{20}$ is zoomed for $\Omega \in (0, 0.7)$. The asymmetric period-1 motions lie in the range of $A_{20} \in (10^{-7}, 10^{-2})$.

For a better understanding of the bifurcation trees of period-1 motions to chaos, the zoomed views for harmonic amplitudes varying with excitation frequency are arranged in Fig. 2. Acronyms “B1” to “B6” are also employed for branch-1 to branch-6 of asymmetric periodic motions. From the asymmetric period-1 motion, one can find the bifurcation tree of period-1 motion to chaos. Herein, the bifurcation trees are presented through the asymmetric period-1 motion to period-4 motion. In fact, the six branches of bifurcation trees of period-1 motion to chaos experience similar structures. In Fig. 2(a), the constant terms $a_0^m$ ($m = 1, 2, 4$) varying with excitation amplitude is presented. The bifurcation tree of asymmetric period-1 motion to period-4 motion is presented through the constant versus excitation frequency. The local area is also further zoomed. The saddle-node bifurcation from symmetric period-1 to asymmetric period-1 motion is already discussed as before. The

![Image](c)
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Fig. 2. (Continued)
saddle-node bifurcations for asymmetric period-1 motions are \( \Omega \approx 6.04, 5.78 \) (first branch, B1), \( \Omega \approx 2.77, 2.76 \) (second branch, B2), \( \Omega \approx 1.84, 1.64 \) (third branch, B3), \( \Omega \approx 1.38, 1.30 \) (fourth branch, B4), \( \Omega \approx 1.13, 1.08 \) (fifth branch, B5) and \( \Omega \approx 0.97, 0.90 \) (sixth branch, B6). For lower frequency, the results of harmonic amplitudes may not be accurate enough. Thus more harmonic terms should be included in the Fourier series solution of the time-delayed Duffing oscillator. The Hopf bifurcations of asymmetric period-1 motion are at \( \Omega \approx 3.61, 4.89 \) for the first branch (B1) of bifurcation trees and \( \Omega \approx 2.31, 2.56 \) for the second branch (B2) of bifurcation trees. For other branches of bifurcation trees (branch 3 to branch 6; B3 to B6), harmonic terms may not be enough in the Fourier series. Thus the Hopf bifurcation of asymmetric period-1 motion may not be accurate enough to get the period-2 motion. In Fig. 2(h), the harmonic amplitude \( A_{1/4} \) varying with excitation frequency is presented for branch-1 (B1) and branch-2 (B2) of bifurcation trees of period-1 motions to chaos. The harmonic amplitude \( A_{1/4} \neq 0 \) is for period-4 motion only, but \( A_{1/4} = 0 \) is for period-1 and period-2 motions. The saddle-node bifurcations for period-4 motion are at \( \Omega \approx 2.33, 2.48 \) (first branch, B1) and \( \Omega \approx 3.63, 3.91 \) (second branch, B2). The Hopf bifurcations of the period-4 motion are at \( \Omega \approx 2.34, 2.44 \) (first branch, B1) and \( \Omega \approx 3.64, 3.89 \) (second branch, B2), from which period-8 motions will appear. In Fig. 2(c), harmonic amplitude \( A_{1/2} \) versus excitation frequency \( \Omega \) is presented for branch-1 (B1) and branch-2 (B2) of periodic motions. \( A_{1/2} \neq 0 \) is for period-4 motion and period-2 motion, but \( A_{1/2} = 0 \) is for the period-1 motion. The saddle-node bifurcations of period-2 motion.
are at \(\Omega \approx 3.61, 4.89\) (first branch, B1) and \(\Omega \approx 2.31, 2.56\) (second branch, B2), which are the Hopf bifurcations of period-1 motions. The Hopf bifurcations of period-2 motion are at \(\Omega \approx 3.63, 3.91\) (first branch, B1) and \(\Omega \approx 2.33, 2.48\) (second branch, B2), which are the saddle-node bifurcations for the period-1 motion. In Fig. 2(d), harmonic amplitude \(A_{4/4}\) varying with excitation frequency is presented for branch-1 (B1) and branch-2 (B2) of bifurcation trees of period-1 motions to chaos, which is similar to harmonic amplitude \(A_{1/4}\). The bifurcation locations of excitation frequency are the same as in the harmonic amplitude \(A_{0/4}\). The primary harmonic amplitude \(A_1\) varying with excitation frequency is presented in Fig. 2(c). The excitation frequencies for saddle-node and Hopf bifurcations are the same as in the constant \(\alpha_0^{(m)} (m = 1, 2, 4)\). Such primary harmonic amplitude exists for period-1, period-2 and period-4 motions because of \(A_1 = A_{2/2} = A_{4/4}\). In addition, the harmonic amplitude for the symmetric period-1 motion is nonzero (\(A_1 \neq 0\)). However, the constant term for the symmetric period-1 motion is zero (\(a_0 = a_0^{(1)} = 0\)). To avoid abundant illustrations, harmonic amplitudes \(A_{k/4} (k = 5, 7, 9, \ldots, 35)\) versus excitation frequency will not be presented. For comparison, harmonic amplitude \(A_{1/2}\) varying with excitation frequency is presented in Fig. 2(f) for branch-1 (B1) and branch-2 (B2) of bifurcation trees of period-1 motion to chaos, which is similar to the harmonic amplitude \(A_{1/4}\). The bifurcation locations of excitation frequency are the same as in the harmonic amplitude \(A_{1/2}\). In Figs. 2(g)-(2(i), harmonic amplitudes \(A_2, A_4, A_4\) are presented, respectively. The bifurcation patterns in the bifurcation tree of period-1 motion to chaos are the same as discussed before. But the quantity levels of harmonic amplitudes for \(A_2, A_4, A_4\) are very high. For \(\Omega \in (3.0, 7.0)\), we have \(A_2 \approx 3.0, A_4 \approx 3.5, A_3 \approx 0.2\). However, for \(\Omega < 3.0\), we have \(A_2 \approx 0.5, A_4 \approx 1\) and \(A_3 \approx 1\). To avoid abundant illustrations, harmonic amplitude \(A_{13}\) are presented in Fig. 2(j). The quantity level of harmonic amplitude \(A_{13}\) decays with excitation frequency, which is labeled by an arrow. To look into the effects on period-2 and period-4 motions, harmonic amplitudes \(A_{7/4}, A_{3/2}, A_{9/4}\) are presented for branch-1 (B1) and branch-2 (B2) of periodic motions in Figs. 2(k)-(2(m), respectively. The harmonic amplitudes \(A_{7/4}\) and \(A_{9/4}\) are similar to harmonic amplitudes \(A_{1/4}\) and \(A_{3/4}\). However, their quantity levels are different: \(A_{7/4} \approx 3 \times 10^{-4}\) and \(10^{-5}\) and \(A_{9/4} \approx 3 \times 10^{-5}\) and \(10^{-5}\) for two branches of bifurcation trees. The harmonic amplitude \(A_{10/2}\) is similar to harmonic amplitude \(A_{1/2}\) but the corresponding quantity level is \(A_{10/2} \approx 3 \times 10^{-4}\) and \(5 \times 10^{-5}\) for two branches of bifurcation trees. Finally, harmonic amplitude \(A_{20}\) for \(\Omega \in (2.0, 5.2)\) is presented in Fig. 2(n) for two branches of bifurcation trees. The quantity levels are \(A_{20} \approx 5 \times 10^{-4}\) and \(10^{-4}\) for two branches of bifurcation trees. For \(a_0^{(m)}L = -a_0^{(m)}R\), there is a set of asymmetric solutions and bifurcation trees, and all the harmonic amplitudes will not be changed but the harmonic phases have a relation of \(\psi_k^{(m)} = \text{mod} \left[\psi_k^{(R)} \left(\frac{m}{L} + 1\right) + \pi, 2\pi\right]\). So the harmonic phases will not be presented herein.

4. Numerical Illustrations

To illustrate period-m motions in the time-delayed Duffing oscillator, numerical and analytical solutions of periodic motions will be presented. The stable symmetric period-1 motion, stable asymmetric period-1 motion, stable period-2 motions and stable period-4 motions are illustrated for first and second branches of bifurcation trees. The initial conditions for numerical simulations are computed from approximate analytical solutions of periodic solutions. In all plots, circular symbols give approximate solutions, and solid curves give numerical simulation results. The time-delay, initial starting and final points are represented by D.I.S. and D.I.F., respectively. The green circular symbols are for the initial delay. The D.I.F. point is also the starting point for dynamical systems without delay. The numerical solutions of periodic motions are generated through the midpoint discrete scheme.

In Fig. 3, a symmetric period-1 motion of the time-delayed Duffing oscillator, based on 20 harmonic terms (HB20), is presented for \(\Omega = 7.0\) with other parameters in Eq. (46). The displacement and velocity of the symmetric period-1 motion are presented in Figs. 3(a) and 3(b), respectively. One period \((T)\) for the symmetric period-1 motion response is labeled in the two plots. The values for the initial time-delay are depicted by green circles. The analytical and numerical solutions match very well. The symmetric displacement and velocity are observed. The corresponding trajectory is presented for over 40 periods in Fig. 3(c). For a better understanding of harmonic contributions, the harmonic amplitude spectrum is presented in Fig. 3(d). The
A symmetric period-1 motion of the time-delayed Duffing oscillator ($\Omega = 7.0$, HB20): (a) displacement, (b) velocity; (c) trajectory and (d) harmonic amplitude. Initial condition $(x_0, \dot{x}_0) = (3.718872, 7.736050)$. ($\delta = 0.5$, $\alpha_1 = 10$, $\alpha_2 = 5$, $\beta = 10$, $Q_0 = 200$, $\tau = T/4$).

harmonic amplitude spectrum is computed from analytical solutions. The main harmonic amplitudes are

$a_0 = 0$, $A_1 \approx 3.374485$, $A_3 \approx 0.385393$, and $A_5 \approx 0.035536$.

The other harmonic amplitudes are

$A_7 \sim 3.39 \times 10^{-3}, \quad A_9 \sim 3.24 \times 10^{-4}, \quad A_k \in (10^{-5}, 10^{-9})$ for $(k = 11, 13, \ldots, 19)$.

In Fig. 4, a stable asymmetric period-1 motion of the time-delayed Duffing oscillator is presented for $\Omega = 6.0$ with other parameters in Eq. (46). The displacement and velocity responses are presented in Figs. 4(a) and 4(b), respectively. One period ($T$) for the symmetric period-1 motion response is also labeled in the two plots. The values for the initial time-delay are also depicted by green circles. The analytical and numerical solutions match very well. Compared to the symmetric period-1 motion, the asymmetry of displacement and velocity are observed. The corresponding trajectory is presented for over 40 periods in Fig. 4(c). This asymmetric period-1 motion possesses two cycles. For a better understanding of harmonic contributions, the harmonic amplitude spectrum is presented in Fig. 4(d).

For the symmetric period-1 motion, we have $a_0 \neq 0$ and $A_l \neq 0$ with $A_{l+1} \neq 0$ ($l = 1, 2, \ldots$). However, for the asymmetric period-1 motion, we have $a_0 = 0$ and $A_l \neq 0$ with $A_{l+1} \neq 0$ ($l = 1, 2, \ldots$). The harmonic amplitude spectrum of...
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![Graphs showing time vs. displacement and velocity for a periodic oscillator.](image)

Fig. 4. An asymmetric period-1 motion of the time-delayed Duffing oscillator ($\Omega = 6.0, \text{HB20}$): (a) displacement, (b) velocity, (c) trajectory and (d) harmonic amplitude. Initial condition $(x_0, \dot{x}_0) = (1.474888, -9.627825)$. ($\delta = 0.5, \alpha_1 = 10, \alpha_2 = 5, \beta = 10, Q_0 = 200, \tau = T/4$).

The asymmetric period-1 motion is computed from the analytical solution. The main harmonic amplitudes are

- $a_0 \approx 0.715833$, $A_1 \approx 2.558856$,
- $A_2 \approx 2.172754$, $A_3 \approx 0.187586$,
- $A_4 \approx 0.180075$, $A_5 \approx 0.144299$,
- $A_6 \approx 0.044487$, $A_7 \approx 0.016212$, and
- $A_8 \approx 0.010501$.

The other harmonic amplitudes are

- $A_9 \approx 4.56 \times 10^{-3}$, $A_{10} \approx 1.72 \times 10^{-3}$, and
- $A_k \in (10^{-9}, 10^{-3})$ for $k = 11, 12, \ldots, 20$.

Compared to the symmetric period-1 motion, more harmonic terms effect on the asymmetric period-1 motions.

From such a branch of asymmetric period-1 motion, the corresponding Hopf bifurcation of the asymmetric period-1 motion will generate a bifurcation tree to chaos. Thus, the trajectories and harmonic amplitude spectrums of period-2 and period-4 motions are presented in Fig. 5 for $\Omega = 4.10$ and $3.90$, respectively, the initial condition $(x_0, y_0) = (3.644607, 4.389778)$ is computed for $\Omega = 4.10$ from the analytical solution, and the initial condition for $\Omega = 3.90$ is $(x_0, y_0) = (1.768009, 7.827822)$. In Fig. 5(a) the trajectory of the stable period-2 motion is presented, and the initial time-delay inputs are depicted by the green circular symbols. The period-doubling motion of period-1 motion is observed, i.e. the trajectory of period-2 motion is observed. The trajectory of period-2 motion becomes more complicated than the...
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Fig. 5. A time-delayed Duffing oscillator: Period-2 motion ($\Omega = 4.10$, HB40): (a) trajectory and (b) harmonic amplitude with initial condition ($x_0, \dot{x}_0$) = (3.644607, 4.589778). Period-4 motion ($\Omega = 3.90$, HB88): (c) trajectory and (d) harmonic amplitude with initial condition ($x_0, \dot{x}_0$) = (3.768009, 7.827822). ($\delta = 0.5, \alpha_1 = 10, \alpha_2 = 5, \beta = 10, Q_0 = 200, \tau = T/4$).

The corresponding amplitude spectrum for such a period-2 motion is presented, and the time-delay is also period-1 motion. The main harmonic amplitudes are:

- $A_{1/2} \approx 0.158739$,
- $A_1 \approx 2.350267$,
- $A_{3/2} \approx 9.27 \times 10^{-3}$,
- $A_2 \approx 0.651487$,
- $A_{5/2} \approx 0.279721$,
- $A_3 \approx 1.127657$,
- $A_{7/2} \approx 0.020717$,
- $A_4 \approx 0.134459$,
- $A_{9/2} \approx 0.076399$,
- $A_5 \approx 0.241658$,
- $A_{11/2} \approx 0.013314$,
- $A_6 \approx 0.057105$,
- $A_{13/2} \approx 0.031616$,
- $A_7 \approx 0.065797$, $A_{15/2} \approx 8.48 \times 10^{-3}$,
- $A_8 \approx 0.018915$, $A_{15/2} \approx 0.011012$, and
- $A_9 \approx 0.017613$.

The other harmonic amplitudes are:

- $A_{19/2} \approx 3.58 \times 10^{-3}$, $A_{21/2} \approx 5.85 \times 10^{-3}$,
- $A_{23/2} \approx 3.70 \times 10^{-3}$, $A_{25/2} \approx 4.47 \times 10^{-3}$,
- $A_{27/2} \approx 1.40 \times 10^{-3}$, $A_{29/2} \approx 1.73 \times 10^{-3}$,
- $A_{31/2} \approx 1.21 \times 10^{-3}$, and $A_{33/2} \approx 1.27 \times 10^{-3}$,
- $A_{35/2} \in (10^{-3}, 10^{-5})$ for ($k = 27, 29, \ldots, 40$).

In Fig. 5(c) the trajectory of the stable period-4 motion is presented, and the time-delay is also
The other harmonic amplitudes are measured experimentally. The main harmonic amplitudes are

\[
A_0 \approx 0.483676, \quad A_{1/4} \approx 0.049026, \\
A_{1/2} \approx 0.183148, \quad A_{3/4} \approx 3.96 \times 10^{-3}, \\
A_1 \approx 2.14645, \quad A_{5/4} \approx 2.58 \times 10^{-3}, \\
A_{3/2} \approx 0.031395, \quad A_{7/4} \approx 0.016412, \\
A_2 \approx 0.558204, \quad A_{9/4} \approx 0.043561, \\
A_{5/2} \approx 0.260733, \quad A_{11/4} \approx 0.010099, \\
A_3 \approx 1.367006, \quad A_{13/4} \approx 2.48 \times 10^{-3}, \\
A_{7/2} \approx 0.053118, \quad A_{15/4} \approx 2.30 \times 10^{-3}, \\
A_4 \approx 0.097138, \quad A_{17/4} \approx 0.010051, \\
A_{9/2} \approx 0.069351, \quad A_{19/4} \approx 2.85 \times 10^{-3}, \\
A_5 \approx 0.305399, \quad A_{21/4} \approx 1.99 \times 10^{-4}, \\
A_{11/2} \approx 0.015748, \quad A_{23/4} \approx 1.26 \times 10^{-3}, \\
A_6 \approx 0.049756, \quad A_{25/4} \approx 4.92 \times 10^{-3}, \\
A_{13/2} \approx 0.032860, \quad A_{27/4} \approx 1.43 \times 10^{-3}, \\
A_7 \approx 0.093100, \quad A_{29/4} \approx 6.52 \times 10^{-4}, \\
A_{15/2} \approx 8.56 \times 10^{-3}, \quad A_{31/4} \approx 9.36 \times 10^{-4}, \\
A_8 \approx 0.200045, \quad A_{33/4} \approx 1.94 \times 10^{-3}, \\
A_{17/2} \approx 0.013059, \quad A_{35/4} \approx 5.78 \times 10^{-4}, \quad \text{and} \\
A_9 \approx 0.027340.
\]

The other harmonic amplitudes are

\[
A_{17/4} \approx 2.89 \times 10^{-4}, \quad A_{29/2} \approx 3.61 \times 10^{-3}, \\
A_{39/4} \approx 4.12 \times 10^{-4}, \quad A_{10} \approx 6.95 \times 10^{-3}, \\
A_{41/4} \approx 6.93 \times 10^{-4}, \quad A_{21/2} \approx 4.74 \times 10^{-3}, \\
A_{43/4} \approx 2.33 \times 10^{-4}, \quad A_{11} \approx 8.06 \times 10^{-3}, \\
A_{45/4} \approx 1.36 \times 10^{-4}, \quad A_{23/2} \approx 1.47 \times 10^{-3}, \\
A_{47/4} \approx 1.83 \times 10^{-4}, \quad A_{12} \approx 2.36 \times 10^{-3}, \\
A_{49/4} \approx 2.43 \times 10^{-4}, \quad A_{25/2} \approx 1.71 \times 10^{-3}, \\
A_{51/4} \approx 9.52 \times 10^{-5}, \quad A_{13} \approx 2.39 \times 10^{-3}, \quad \text{and} \\
A_{k/2} \in (10^{-6}, 10^{-3}) \quad \text{for} \quad (k = 27, 28, \ldots, 40).
\]

The effects of harmonic amplitudes on periodic motions are clearly presented. This bifurcation tree is asymmetric with the center on the right-hand side of y-axis.

From the second bifurcation trees of asymmetric period-1 motion, the trajectories and harmonic amplitude spectrums of period-1, period-2 and period-4 motions are presented in Fig. 6 for \(\Omega = 2.761\), \(\Omega = 2.49\), and 2.45, respectively. From the analytical solution, the initial conditions for \(\Omega = 2.761\), \(\Omega = 2.49\) and \(\Omega = 2.45\) are \((x_0, y_0) = (1.00140, -12.815478), (x_0, y_0) = (1.005765, -3.858392)\) and \((x_0, y_0) = (0.964546, -3.971407)\), respectively. The trajectory of the asymmetric period-1 motion is presented in Fig. 6(a) for \(\Omega = 2.761\) with over 40 periods, and the corresponding harmonic amplitude spectrum is presented in Fig. 6(b) from the analytical solution. The main harmonic amplitudes are

\[
a_0 = -0.376459, \quad A_1 \approx 2.678145, \\
A_2 \approx 0.470034, \quad A_3 \approx 0.617403, \\
A_4 \approx 1.168688, \quad A_5 \approx 0.235772, \\
A_6 \approx 0.339210, \quad A_7 \approx 0.016870, \\
A_8 \approx 0.068067, \quad A_9 \approx 0.091164, \\
A_{10} \approx 0.033509, \quad \text{and} \quad A_{11} \approx 0.020739.
\]

The other harmonic amplitudes are

\[
A_k \in (10^{-6}, 10^{-3}) \quad \text{for} \quad (k = 12, \ldots, 20).
\]

The trajectory of the asymmetric period-1 motion at \(\Omega = 2.761\) is more complex than the asymmetric period-1 motion at \(\Omega = 4.10\). With decreasing excitation frequency, the trajectory of asymmetric period-1 motion will become more complex, as discussed in [Luo et al., 2014b]. Since the trajectory of asymmetric period-1 motion becomes more complex, the corresponding period-2 motions will become much more complicated. In Fig. 6(c), the trajectory of the stable period-2 motion is presented for \(\Omega = 2.49\), and the corresponding amplitude spectrum for such a period-2 motion is presented.
Fig. 6. A time-delayed Duffing oscillator: Period-1 motion ($\Omega = 2.761$, HB20): (a) trajectory, and (b) harmonic amplitude with initial condition $(x_0, \dot{x}_0) = (1.100410, -12.815478)$. Period-2 motion ($\Omega = 2.49$, HB40): (c) trajectory, and (d) harmonic amplitude with initial condition $(x_0, \dot{x}_0) = (1.005765, -3.858302)$. Period-4 motion ($\Omega = 2.45$, HB80): (e) trajectory, and (f) harmonic amplitude with initial condition $(x_0, \dot{x}_0) = (0.964546, -3.971407)$. ($\delta = 0.5$, $\alpha_1 = 10$, $\alpha_2 = 5$, $\beta = 10$, $Q_0 = 200$, $\tau = T/4$).
in Fig. 6(d). The main harmonic amplitudes are
\[ a^{(2)}_0 = -0.289859, \quad A_{1/2} \approx 0.058344, \]
\[ A_1 \approx 2.715865, \quad A_{3/2} \approx 0.017884, \]
\[ A_2 \approx 0.298006, \quad A_{5/2} \approx 0.088402, \]
\[ A_3 \approx 0.348520, \quad A_{7/2} \approx 0.031612, \]
\[ A_4 \approx 0.439704, \quad A_{9/2} \approx 0.211388, \]
\[ A_5 \approx 0.761699, \quad A_{11/2} \approx 0.030376, \]
\[ A_6 \approx 0.226714, \quad A_{13/2} \approx 0.093883, \]
\[ A_7 \approx 0.187311, \quad A_{15/2} \approx 6.35 \times 10^{-3}, \]
\[ A_8 \approx 4.63 \times 10^{-3}, \quad A_{16/2} \approx 8.01 \times 10^{-3}, \]
\[ A_9 \approx 0.044368, \quad A_{17/2} \approx 0.013287, \]
\[ A_{10} \approx 0.044474, \quad A_{19/2} \approx 0.023180, \]
\[ A_{11} \approx 0.037844, \quad A_{21/2} \approx 6.24 \times 10^{-3}, \text{ and} \]
\[ A_{12} \approx 0.015490. \]

The other harmonic amplitudes are
\[ A_{k/2} \in (10^{-5}, 10^{-2}) \text{ for } (k = 25, 26, \ldots, 40). \]

The trajectory of such a period-2 motion at \( \Omega = 2.49 \) is much more complicated than the trajectory of the period-2 motion at \( \Omega = 2.60 \). Further, the complexity of period-4 motions in this bifurcation can be discussed. In Fig. 6(e) the trajectory of the stable period-4 motion is presented, and the corresponding amplitude spectrum for such a period-4 motion is presented in Fig. 6(f) to show the harmonic effects on the period-4 motion. The main harmonic amplitudes are
\[ a^{(4)}_0 = -0.269947, \quad A_{1/4} \approx 0.029364, \]
\[ A_{1/2} \approx 0.066763, \quad A_{1/4} \approx 2.08 \times 10^{-3}, \]
\[ A_1 \approx 2.670734, \quad A_{5/4} \approx 2.24 \times 10^{-3}, \]
\[ A_{3/2} \approx 0.014820, \quad A_{7/4} \approx 9.13 \times 10^{-3}, \]
\[ A_2 \approx 0.269595, \quad A_{9/4} \approx 0.021630, \]
\[ A_{5/2} \approx 0.088391, \quad A_{11/4} \approx 7.42 \times 10^{-3}, \]
\[ A_3 \approx 0.309230, \quad A_{13/4} \approx 7.81 \times 10^{-3}, \]
\[ A_{7/2} \approx 0.022944, \quad A_{15/4} \approx 0.010991, \]
\[ A_4 \approx 0.377505, \quad A_{17/4} \approx 0.038591, \]
\[ A_{9/2} \approx 0.192696, \quad A_{19/4} \approx 0.014457, \]
\[ A_5 \approx 0.856472, \quad A_{21/4} \approx 0.013025, \]
\[ A_{11/2} \approx 0.043338, \quad A_{23/4} \approx 5.09 \times 10^{-3}, \]
\[ A_6 \approx 0.191938, \quad A_{25/4} \approx 0.018989, \]
\[ A_{13/2} \approx 0.090741, \quad A_{27/4} \approx 7.16 \times 10^{-3}, \]
\[ A_7 \approx 0.238037, \quad A_{29/4} \approx 4.45 \times 10^{-4}, \]
\[ A_{15/4} \approx 0.010862, \quad A_{31/4} \approx 1.25 \times 10^{-3}, \]
\[ A_8 \approx 9.62 \times 10^{-3}, \quad A_{33/4} \approx 5.63 \times 10^{-4}, \]
\[ A_{15/2} \approx 5.23 \times 10^{-3}, \quad A_{35/4} \approx 6.82 \times 10^{-3}, \]
\[ A_9 \approx 0.041367, \quad A_{37/4} \approx 1.91 \times 10^{-3}, \]
\[ A_{19/2} \approx 0.010460, \quad A_{39/4} \approx 1.21 \times 10^{-3}, \]
\[ A_{10} \approx 0.041938, \quad A_{41/4} \approx 5.05 \times 10^{-3}, \]
\[ A_{21/2} \approx 0.023810, \quad A_{43/4} \approx 1.18 \times 10^{-3}, \]
\[ A_{11} \approx 0.046781, \quad A_{45/4} \approx 7.36 \times 10^{-4}, \]
\[ A_{23/2} \approx 5.02 \times 10^{-3}, \quad A_{47/4} \approx 5.57 \times 10^{-3}, \text{ and} \]
\[ A_{12} \approx 0.016738. \]

The other harmonic amplitudes are
\[ A_{k/2} \in (10^{-6}, 10^{-2}) \text{ for } (k = 25, 26, \ldots, 80). \]

This bifurcation tree is asymmetric with the center on the left-hand side of y-axis. The complex trajectory of the period-4 motion at \( \Omega = 2.49 \) is observed. Similarly, the other complex trajectory of period-m motion in other bifurcation trees can be illustrated. For complex period-1 motion, one can refer to [Luo & Jin, 2014b].

5. Conclusions

In this paper, the analytical solutions of period-m motions in a periodically excited, Duffing oscillator with a time-delayed displacement were obtained through the Fourier series, and the corresponding stability and bifurcation of such period-m motions were discussed through eigenvalue analysis. The analytical bifurcation trees from asymmetric period-1 motions to chaos in such a time-delayed Duffing oscillator were presented through period-1 to period-4 motions. Stable and unstable period-m
motions were illustrated by numerical and analytical solutions. From this study, to obtain periodic motions in the time-delayed dynamical systems directly, the appropriate time-delay inputs (or initial conditions) in the initial time-delay interval should be computed from the analytical solutions of periodic motions.

References


