Periodic Flows to Chaos Based on Discrete Implicit Mappings of Continuous Nonlinear Systems

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This paper presents a semi-analytical method for periodic flows in continuous nonlinear dynamical systems. For the semi-analytical approach, differential equations of nonlinear dynamical systems are discretized to obtain implicit maps, and a mapping structure based on the implicit maps is employed for a periodic flow. From mapping structures, periodic flows in nonlinear dynamical systems are predicted analytically and the corresponding stability and bifurcations of the periodic flows are determined through the eigenvalue analysis. The periodic flows predicted by the single-step implicit maps are discussed first, and the periodic flows predicted by the multistep implicit maps are also presented. Periodic flows in time-delay nonlinear dynamical systems are discussed by the single-step and multistep implicit maps. The time-delay nodes in discretization of time-delay nonlinear systems were treated by both an interpolation and a direct integration. Based on the discrete nodes of periodic flows in nonlinear dynamical systems with/without time-delay, the discrete Fourier series responses of periodic flows are presented. To demonstrate the methodology, the bifurcation tree of period-1 motion to chaos in a Duffing oscillator is presented as a sampled problem. The method presented in this paper can be applied to nonlinear dynamical systems, which cannot be solved directly by analytical methods.

Keywords: Discrete implicit maps; bifurcation trees; periodic flows to chaos; nonlinear dynamical systems; time-delay nonlinear dynamical systems.

1. Introduction

For solutions of periodic motions in nonlinear dynamical systems, analytical and numerical techniques have been adopted. The analytical methods include the method of averaging, perturbation methods, harmonic balance method, and generalized harmonic balance method. Through the analytical methods, one can obtain the analytical expressions of approximate solutions of periodic motions in dynamical systems. The numerical methods are based on discrete maps obtained by discretization of differential equations for dynamical systems. The discrete maps include explicit and implicit maps. The explicit maps can be directly used to obtain numerical solutions of differential equations for dynamical systems, but the computational errors for the recurrence iteration of explicit maps will be accumulated in numerical results. Once the recurrence iteration times become large, the numerical results may not be adequate for numerical solutions of dynamical systems. Herein, implicit maps will be used to develop mapping structures for periodic motions. The implicit maps cannot be done simply by the recurrence iteration. For periodic flows in nonlinear dynamics, mapping structures based on implicit maps can be developed. Of course, an explicit mapping can be expressed by an implicit map as a special case. Based on the mapping
structures, analytical prediction of periodic flows in nonlinear dynamical systems can be completed. The mapping structure gives a set of nonlinear algebraic equations, which can be solved. Without the recurrence iteration, the solution errors of node points of periodic flows are fixed without computational errors caused by iterations. The purpose of this paper is to develop a semi-analytical method for periodic flows to chaos in nonlinear dynamical systems with/without time-delay through implicit mapping structures.

To determine periodic flows in nonlinear dynamical systems, existing techniques for periodic motions in nonlinear systems are reviewed briefly. The analytical methods for periodic motions are discussed first. Lagrange [1788] developed the method of averaging for periodic motions in the three-body problem as a perturbation of the two-body problem. The idea is based on the solutions of linear systems. Such an idea was further extended by Poincaré in the end of the 19th century. Thus, Poincaré [1899] developed the perturbation theory for motions of celestial bodies. van der Pol [1920] used the method of averaging for the periodic solutions of oscillation systems in circuits. Such an application caused great interest in the perturbation theory for the approximate analytical solution of periodic motions in nonlinear systems. Until 1928, the asymptotic validity of the method of averaging was not proved. Fatou [1928] gave the proof of the asymptotic validity through the solution existence theorems of differential equations. Krylov and Bogoliubov [1935] further developed the method of averaging, and the detailed presentation was given in [Bogoliubov & Mitropolsky, 1961]. Hayashi [1964] presented the perturbation methods including averaging method and the principle of harmonic balance. Barkham and Soudack [1969] extended the Krylov–Bogoliubov method for the approximate solutions of nonlinear autonomous second-order differential equations (also see, [Barkham & Soudack, 1970]). Nayfeh [1975] employed the multiple-scale perturbation method to develop approximate solutions of periodic motions in the Duffing oscillators. Holmes and Rand [1976] discussed the stability and bifurcation of periodic motions in the Duffing oscillator. Nayfeh and Mook [1977] used the perturbation method to investigate nonlinear structural vibrations, and Holmes [1979] demonstrated chaotic motions in nonlinear oscillators through the Duffing oscillator with a twin-well potential. Ueda [1980] numerically simulated chaos by period-doubling of periodic motions of Duffing oscillators. A generalized harmonic balance approach was used by Garcia-Margallo and Bejarano [1987] to determine approximate solutions of nonlinear oscillations with strong nonlinearity. Rand and Armbruster [1987] determined the stability of periodic solutions by the perturbation method and bifurcation theory. Yuste and Bejarano [1989] employed the elliptic functions instead of trigonometric functions to extend the Krylov–Bogoliubov method. Coppola and Rand [1990] used the averaging method with elliptic functions for approximation of limit cycle. Wang et al. [1992] used the harmonic balance method and the Floquet theory to investigate the nonlinear behaviors of the Duffing oscillator with a bounded potential well (also see, [Kao et al., 1992]). Luo and Han [1997] determined the stability and bifurcation conditions of periodic motions of the Duffing oscillator. However, only symmetric periodic motions of the Duffing oscillators were investigated. Luo and Han [1999] investigated the analytical prediction of chaos in nonlinear rods through the Duffing oscillator. Peng et al. [2008] presented the approximate symmetric solution of period-1 motions in the Duffing oscillator by the harmonic balance method with three harmonic terms. Luo [2012a] developed a generalized harmonic balance method for the approximate analytical solutions of periodic motions and chaos in nonlinear dynamical systems. This method used the finite-term Fourier series to approximately express periodic motions and the coefficients are time-varying. With averaging, a dynamical system of coefficients is obtained, and from such a dynamical system, the approximate solutions of periodic motions are achieved and the corresponding stability and bifurcation analysis are completed. Luo and Huang [2012b] also employed a generalized harmonic balance method to find analytical solutions of period-3 motions in such a Duffing oscillator. The analytical bifurcation trees of periodic motions in the Duffing oscillator to chaos were obtained (also see, [Luo & Huang, 2012c, 2012d, 2013a, 2013b, 2013c, 2013d]). Such analytical bifurcation trees show the connection from periodic solution to chaos analytically. For a better understanding of nonlinear behaviors in nonlinear dynamical systems, analytical bifurcation
trees of period-1 motions to chaos in a periodically forced oscillator with quadratic nonlinearity were presented in [Luo & Yu, 2013a, 2013b, 2013c], and period-m motions in the periodically forced, van der Pol equation was presented in [Luo & Laken, 2013]. The analytical solutions of periodic oscillations in the van der Pol oscillator can be used to verify the conclusions in [Cartwright & Littlewood, 1947] and [Levinson, 1948]. The results for the quadratic nonlinear oscillator in [Luo & Yu, 2013a, 2013b, 2013c] analytically show the complicated period-1 motions and the corresponding bifurcation structures. The detailed presentation for analytical methods for periodic flows in nonlinear dynamical systems can be found in Luo [2014a, 2014b].

In recent years, time-delay systems are of great interest since such systems extensively exist in engineering (e.g. [Trusly, 2000; Hu and Wang, 2002]). The infinite-dimensional state space causes significant difficulty to solve such time-delay problems. Thus, one used numerical methods for the corresponding complicated behaviors. On the other hand, one is interested in the stability and bifurcation of equilibriums of the time-delay systems (e.g. [Stepan, 1989; Sun, 2009; Insperger & Stepan, 2011]). In addition, one is also interested in analytical solutions of periodic motions in time-delay dynamical systems. Perturbation methods have been used for such periodic motions in delay dynamical systems. For instance, the approximate solutions of the time-delay nonlinear oscillator were investigated by the method of multiple scales (e.g. [Hu et al., 1998; Wang & Hu, 2006]). The harmonic balance method was also used to determine time-delay approximate solutions of periodic motions in nonlinear oscillators (e.g. [MacDonald, 1995; Liu & Kalmar-Nagy, 2010; Leung & Guo, 2014]). However, such approximate solutions of periodic motions in the time-delay oscillators are based on one or two harmonic terms, which are not accurate enough. In addition, the corresponding stability and bifurcation analysis of such approximate solutions of periodic motions may not be adequate. Luo [2013] presented an alternative way for the accurate analytical solutions of periodic flows in time-delay dynamical systems (see also, [Luo, 2014c]). This method is without any small-parameter requirement. In addition, this approach can also be applicable to the coefficient varying with time. Luo and Jin [2014a] analytically presented the bifurcation tree of period-1 motions to chaos in a periodically forced, time-delay quadratic nonlinear oscillator. Luo and Jin [2014b, 2014c, 2014d] discussed the bifurcation trees of period-m motions to chaos in the periodically forced Duffing oscillator with a linear time-delay displacement.

From the literature survey, for some simple nonlinear systems, the approximate analytical solutions of periodic motions can be obtained. However, for most nonlinear dynamical systems, it is very difficult to obtain analytical solutions of periodic motions. Thus, numerical results of periodic motions in complicated nonlinear dynamical systems become very significant in engineering. In fact, humans have a long history almost as old as human civilization to use numerical algorithms to get approximate numerical results instead of exact results. For instance, the Rhind papyrus of ancient Egypt describes a root-finding method for solving a simple equation in about 1650 BC, and Archimedes of Syracuse (287–212 BC) used numerical algorithms to approximately compute lengths, areas, volumes of geometric figures. Based on the ideas and spirits of numerical approximations, Isaac Newton and Gottfried Leibnitz developed the calculus by infinitesimal elements to linear approximation and infinitesimal summarization to integration. Because of calculus development, one can describe more complicated mathematical models for real physical problems, but it is very difficult to solve such accurate mathematical models explicitly. This is an important impetus for one to develop numerical methods to approximate solutions of the accurate mathematical models. Thus, Newton developed several numerical methods to find approximate solutions. For instance, numerical methods for root-finding and polynomial interpolation were developed by Newton. Since then, Euler (1707–1783), Lagrange (1736–1813), and Gauss (1777–1855) further developed numerical methods for approximate results, such as Euler method for differential equations, Lagrange interpolation method, and Gauss interpolation. The more detailed information about numerical methods can be found in [Goldstine, 1977].

This paper will focus on numerical methods of nonlinear dynamical systems. For this issue, Euler developed an explicit method to achieve approximate solutions numerically. Such Euler method is a one-step discrete method. This method is still used in numerical computations but its computational
accuracy is very low, and numerical solutions are not accurate. Bashforth and Adams [1885] presented a multistep discrete method to numerical solutions for differential equations. Moulton [1926] extended such a method to the Adams-Moulton method. The Adams-Bashforth method is the explicit method as a predictor, and the Adams-Moulton method is the implicit method as a corrector. In addition, the Adams-Bashforth method can be extended for the practical application of the Taylor series method as presented in [Nordsieck, 1962]. Milne [1949] used the entire interval for integration based on Newton-Cotes quadrature formulas. The recent theory of linear multistep method was systematically discussed by Dahlquist [1956, 1959]. The general formulas were presented and the corresponding consistency, stability and convergence were discussed by the linear stability theory. Runge [1895] started modern one-step methods with order of two and three for numerical solutions of differential equations. Heun [1900] raised the order of the method from two and three to four. Kutta [1901] gave the formulation of the method with the order conditions. Nystrom [1925] made the correction of the fifth-order method of Kutta and showed how to apply the Runge-Kutta method to the second-order differential equations. Butcher [1963] discussed the coefficients of Runge-Kutta method, and the implicit Runge-Kutta methods were presented in [Butcher, 1964, 1975].

With extensive application of computers, numerical computation becomes very popular to obtain numerical results for differential equations through discretization. Once the discrete maps are obtained for dynamical systems, discrete dynamical systems can be used to investigate nonlinear dynamics of dynamical systems. Based on nonlinear maps, the existence of chaotic motions was discovered in nonlinear dynamical systems through the iteration of discrete maps.

In 2005, Luo [2005a, 2005b] presented a mapping dynamics of discrete dynamical systems which is a more generalized symbolic dynamics. The systematical description of mapping dynamics in discontinuous dynamical systems was presented in [Luo, 2009]. The discrete maps can be any implicit and/or explicit functions rather than explicit maps in numerical iterative methods only. From discrete mapping structures, periodic motions in discrete dynamical systems can be predicted analytically, and the stability and bifurcation analysis of periodic motions in nonlinear dynamical systems can be completed. Such an idea was applied to discontinuous dynamical systems in [Luo, 2009, 2012b, 2012c].

This paper will develop discrete implicit maps to determine periodic flows in nonlinear dynamical systems through mapping structures, and the analytical prediction of periodic flows can be completed. The periodic flows in nonlinear dynamical systems with/without time-delay will be discussed. The time-delay terms in nonlinear time-delay systems are treated by interpolation and direct integration. From the discrete implicit mappings, the discrete node points of periodic flows are determined. From the discrete Fourier series expansion, the harmonic amplitudes can be achieved, which will provide very useful information for analytical expressions, physical applications and measurements. In [Luo & Guo, 2014], this new method was used to analytically predict the bifurcation trees of periodic motions in the Duffing oscillator as a sample problem. The periodic motions in the Duffing oscillator were presented, and the analytical prediction of the bifurcation trees of periodic motions in nonlinear dynamical systems were also presented through the mapping structures rather than the recurrence iteration. The bifurcation trees of periodic motions to chaos were presented through harmonic amplitudes to show the significance of harmonic terms of the Fourier series in periodic motions. The numerical results and analytically predicted results were compared.

2. Periodic Flows in Nonlinear Dynamical Systems

Periodic flows in dynamical systems will be presented herein. If a nonlinear system has a periodic flow with a period of $T = 2\pi/\Omega$, then such a periodic flow can be expressed by discrete points through discrete mappings of continuous dynamical systems. The method is stated through the following theorem.

Theorem 1. Consider a nonlinear dynamical system as

$$\dot{x} = f(x, t, p) \in \mathbb{R}^n$$

(1)

where $f(x, t, p)$ is a $C^r$-continuous nonlinear vector function ($r \geq 1$). If such a dynamical system has a periodic flow $x(t)$ with finite norm $\|x\|$ and period $T = 2\pi/\Omega$, there is a set of discrete time
are

The resultant Jacobian matrices of the periodic flow under \( \| x(t_k) - \mathbf{X}_k \| \leq \varepsilon_k \) with a small \( \varepsilon_k \geq 0 \) and

\[
\| \mathbf{f}(x(t_k), t_k, p) - f(x_k, t_k, p) \| \leq \delta_k
\]  

(2)

with a small \( \delta_k \geq 0 \). During a time interval \( t \in [t_{k-1}, t_k] \), there is a mapping \( \mathbf{P}_k : x_{k-1} \rightarrow x_k \) \( (k = 1, 2, \ldots, N) \) as

\[ x_k = \mathbf{P}_k x_{k-1} \quad \text{with} \quad \mathbf{g}_k(x_{k-1}, x_k, p) = 0, \quad k = 1, 2, \ldots, N, \]  

(3)

where \( \mathbf{g}_k \) is an implicit vector function. Consider a mapping structure as

\[ \mathbf{P} = \mathbf{P}_N \circ \cdots \circ \mathbf{P}_2 \circ \mathbf{P}_1 : \mathbf{x}_0 \rightarrow \mathbf{x}_N; \]  

with \( \mathbf{P}_k : \mathbf{x}_{k-1} \rightarrow \mathbf{x}_k \) \( (k = 1, 2, \ldots, N) \).

(4)

For \( \mathbf{x}_N = \mathbf{P} \mathbf{x}_0 \), if there is a set of points \( \mathbf{x}_k^* \) \( (k = 0, 1, \ldots, N) \) computed by

\[ \mathbf{g}_k(\mathbf{x}_{k-1}^*, \mathbf{x}_k^*, p) = 0, \quad (k = 1, 2, \ldots, N), \]

(5)

then the points \( \mathbf{x}_k^* \) \( (k = 0, 1, \ldots, N) \) are approximations of points \( \mathbf{x}_k \) of the periodic solution. In the neighborhood of \( \mathbf{x}_k^* \), with \( \mathbf{x}_k = \mathbf{x}_k^* + \Delta \mathbf{x}_k \), the linearized equation is given by

\[ \Delta \mathbf{x}_k = \mathbf{D}_\mathbf{P}_k \cdot \Delta \mathbf{x}_{k-1} \]

with \( \mathbf{g}_k(\mathbf{x}_{k-1}^* + \Delta \mathbf{x}_{k-1}, \mathbf{x}_k^* + \Delta \mathbf{x}_k, p) = 0 \)

\( (k = 1, 2, \ldots, N). \)

(6)

The resultant Jacobian matrices of the periodic flow are

\[ \mathbf{D} \mathbf{P}_{k(k-1)} = \mathbf{D} \mathbf{P}_k \cdot \mathbf{D} \mathbf{P}_{k-1} \cdot \cdots \cdot \mathbf{D} \mathbf{P}_1, \]

\( (k = 1, 2, \ldots, N); \)

\[ \mathbf{D} \mathbf{P} = \mathbf{D} \mathbf{P}_N \cdot \mathbf{D} \mathbf{P}_{N-1} \cdot \cdots \cdot \mathbf{D} \mathbf{P}_1 \]

where

\[ \mathbf{D} \mathbf{P}_k = \left[ \frac{\partial \mathbf{g}_k}{\partial \mathbf{x}_k} \right]_{(\mathbf{x}_{k-1}^*, \mathbf{x}_k^*)}^{-1} \left[ \frac{\partial \mathbf{g}_k}{\partial \mathbf{x}_k} \right]_{(\mathbf{x}_{k-1}^*, \mathbf{x}_k^*)}
\]

\( (k = 1, 2, \ldots, N). \)

(8)

The eigenvalues of \( \mathbf{D} \mathbf{P} \) for such periodic flow are determined by

\[ \left| \mathbf{D} \mathbf{P}_{k(k-1)} - \lambda \mathbf{X}_{k(k-1)} \right| = 0, \quad (k = 1, 2, \ldots, N); \]

\[ \left| \mathbf{D} \mathbf{P} - \lambda \mathbf{X}_{k(k-1)} \right| = 0. \]

(9)

Thus, the eigenvalues of \( \mathbf{D} \mathbf{P}_{k(k-1)} \) are found with \( \mathbf{D} \mathbf{X}_k \) varying with \( \Delta \mathbf{X}_k \). The stability conditions of the periodic solution can be classified by the eigenvalues of \( \mathbf{D} \mathbf{P}(\mathbf{x}_k^*) \) with

\[ \left[ \left| n_1^m, n_1^o \right| ; \left| n_2^m, n_2^o \right| ; \left| n_3^m, n_3^o \right| , \left| n_4^m, n_4^o \right| , n_5^m, n_5^o \right] \]

(10)

where \( n_1^m \) is the number of real eigenvalues with magnitudes less than one \( (n_1 = n_1^m + n_1^o). n_2^o \) is the total number of real eigenvalues with magnitude greater than one \( (n_2 = n_2^m + n_2^o) , n_3 \) is the total number of real eigenvalues equal to \( +1; n_4 \) is the total number of real eigenvalues equal \( -1; n_5 \) is the total pair number of complex eigenvalues with magnitudes greater than one \( n_7^o \) is the total pair number of complex eigenvalues with magnitudes equal to one.

(i) If the magnitudes of all eigenvalues of \( \mathbf{D} \mathbf{P} \) are less than one \( (\varepsilon_1 < 1, i = 1, 2, \ldots, n), \) the approximate periodic solution is stable.

(ii) If at least the magnitude of one eigenvalue of \( \mathbf{D} \mathbf{P} \) is greater than one \( (\varepsilon_1 > 1, i \in [1, 2, \ldots, n]), \) the approximate periodic solution is unstable.

(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof If \( \mathbf{f}(\mathbf{x}, p, t) \) is a \( C^r \)-continuous nonlinear vector function \( (r \geq 1) \), then the velocity \( \mathbf{X} \) should be \( C^r \)-continuous \( (r \geq 1). \) If such a dynamical system has a periodic flow \( \mathbf{x}(t) \) with finite norm \( \| \mathbf{x} \| \) and period \( T = 2\pi/\Omega, \) there is a set of discrete time \( t_k (k = 0, 1, \ldots, N) \) with \( (N \rightarrow \infty) \) during one period \( T. \) The corresponding solution \( \mathbf{x}(t_k) \) and vector fields \( \mathbf{f}(\mathbf{x}(t_k), t_k, p) \) are exact. Consider a time interval \( t \in [t_{k-1}, t_k], \)

\[ \mathbf{x}(t) = \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t} \mathbf{f}(\mathbf{x}(t), t, p)dt. \]

(11)

For the time interval \( [t_{k-1}, t_k] \) divided into \( s \)-nodes \( \mathbf{t}_{k(i)} = t_k + c_i h_{k+1} \) with \( c_i \in [0, 1] \) and \( \mathbf{f}(\mathbf{x}(t_{k(i)})), \)

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Let \( P(t_0), p \) \((i = 1, \ldots, s)\), there is an approximate function \( P(t) \) with unknown \( C = (C_1, \ldots, C_s)^T \) and \( C_i \) \((i = 1, \ldots, s)\), and the following condition is satisfied, i.e.

\[
\frac{\partial P}{\partial C} \neq 0.
\]

(12)

From the foregoing equation, the unknowns \( C(t_k) = (C_1, \ldots, C_s)^T \) with \( t_k = (t_{k-1}, \ldots, t_k)^T = t_k(1, 1, \ldots, 1)^T + \xi_k (c_1, \ldots, c_s)^T \) are determined. For a small \( \delta > 0 \), if there is a relation

\[
|P(t, C(t_k)) - f(x, t, p)| \leq \delta
\]

for \( t \in [t_{k-1}, t_k] \), Eq. (11) can be approximated as

\[
x(t) = x(t_{k-1}) + \int_{t_{k-1}}^{t} P(t, C(t_k)) dt + O(\delta);
\]

(14)

\[
\mathbf{x}(t) = \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t} P(t, C(t_k)) dt
\]

and

\[
\mathbf{x}(t_k) \approx \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} P(t, C(t_k)) dt.
\]

(15)

Let \( \mathbf{x}(t_{k-1}) = x_{k-1} \) and \( \mathbf{x}(t_k) = x_k \). For any small \( \varepsilon_{k-1} > 0 \) and \( \varepsilon_k > 0 \), under \( \|x(t_k) - x_k\| \leq \varepsilon_k \) and \( \|x(t_{k-1}) - x_{k-1}\| \leq \varepsilon_{k-1} \), Eq. (15) gives

\[
x_k = x_{k-1} + \mathbf{g}_k(x_{k-1}, x_k, p),
\]

(16)

Thus, a discrete mapping relation is obtained by

\[
g_k(x_{k-1}, x_k, p) \equiv x_k - x_{k-1} - \mathbf{g}_k(x_{k-1}, x_k, p) = 0.
\]

(17)

From the discrete mapping, two points \( x(t_{k-1}) \) and \( x(t_k) \) for the time interval \( t \in [t_{k-1}, t_k] \) \((k = 1, \ldots, N)\) can be approximated by \( x_{k-1} \) and \( x_k \), respectively. If \( f(x, t, p) \) is a \( C^1 \)-continuous non-linear vector function, we have \( |f| \leq L \) \((L \text{ constant})\). Thus

\[
|f(x(t_{k-1}), t_{k-1}, p) - f(x_{k-1}, t_{k-1}, p)| \leq \frac{L}{2} |x(t_{k-1}) - x_{k-1}| \leq \frac{L}{2} \varepsilon_{k-1},
\]

(18)

\[
|f(x(t_k), t_k, p) - f(x_{k}, t_k, p)| \leq |x(t_k) - x_k| \leq \frac{L}{2} \varepsilon_k.
\]

(19)

Once the mapping \( P_k : x_{k-1} \to x_k (k = 1, 2, \ldots, N) \) with \( g_k(x_{k-1}, x_k, p) = 0 \) exists, the periodic flow can be formed by \( P : x_0 \to x_N \) with \( P = P_N \circ \cdots \circ P_2 \circ P_1 \), i.e.

\[
P_1 : x_0 \to x_1 \Rightarrow g_1(x_0, x_1, p) = 0,
\]

(20)

\[
P_2 : x_1 \to x_2 \Rightarrow g_2(x_1, x_2, p) = 0,
\]

(21)

\[
P_N : x_{N-1} \to x_N \Rightarrow g_N(x_{N-1}, x_N, p) = 0,
\]

(22)

If \( \partial g_k / \partial x_{k-1} \mid_{x_{k-1}, x_k} = 0 \) and \( \partial g_k / \partial x_k \mid_{x_{k-1}, x_k} = 0 \), the deformed equation for Eq. (15) is

\[
\frac{\partial g_k}{\partial x_{k-1}} \mid_{(x_{k-1}, x_k)} = 0
\]

(23)

\[
(1, 2, \ldots, N).
\]
and the linearization of the foregoing equation gives

$$\Delta x_k = D\mathbf{P}_k \cdot \Delta x_{k-1}$$

with

$$D\mathbf{P}_k = \left[ \frac{\partial x_k}{\partial x_{(k-1)\cdot N}} \right]_{x_{(k-1)\cdot N}} \quad (k = 1, 2, \ldots, N).$$

(24)

In other words,

$$\Delta x_k = D\mathbf{P}_{(k-1)\cdot N} \cdot \Delta x_0$$

with

$$D\mathbf{P}_{(k-1)\cdot N} = D\mathbf{P}_k \cdot D\mathbf{P}_{k-1} \cdot \ldots \cdot D\mathbf{P}_1$$

$$= \prod_{j=k}^{1} \left[ \frac{\partial x_j}{\partial x_{(j-1)\cdot N}} \right]_{x_{(j-1)\cdot N}}$$

$$\Delta x_N = D\mathbf{P} \cdot \Delta x_0$$

with

$$D\mathbf{P} = D\mathbf{P}_N \cdot D\mathbf{P}_{N-1} \cdot \ldots \cdot D\mathbf{P}_1$$

$$= \prod_{k=N}^{1} \left[ \frac{\partial x_k}{\partial x_{(k-1)\cdot N}} \right]_{x_{(k-1)\cdot N}}.$$

(25)

Setting $\Delta x_N = \lambda_0 \Delta x_0$ and $\Delta x_k = \lambda_k \Delta x_0$, the foregoing equation becomes

$$\langle D\mathbf{P}_{(k-1)\cdot N} - \lambda_k \mathbf{I} \rangle \Delta x_0 = 0, \quad (k = 1, 2, \ldots, N);$$

$$\langle D\mathbf{P} - \lambda_{\text{max}} \mathbf{I} \rangle \Delta x_0 = 0.$$

(26)

For any nontrivial solution ($\| \Delta x_0 \| \neq 0$), we have

$$\| D\mathbf{P}_{(k-1)\cdot N} - \lambda_k \mathbf{I} \| = 0, \quad (k = 1, 2, \ldots, N);$$

$$\| D\mathbf{P} - \lambda_{\text{max}} \mathbf{I} \| = 0.$$

(27)

Thus, the eigenvalues of $D\mathbf{P}$ and $D\mathbf{P}_{(k-1)\cdot N}$ are computed for the periodic solution. The eigenvalues of $D\mathbf{P}_{(k-1)\cdot N}$ give the properties of $\Delta x_0$ varying with $\Delta x_k$. From the stability and bifurcation theory of dynamical systems at fixed points in discrete nonlinear systems, the stability and bifurcation of the periodic solutions can be classified as in [Luo, 2012b, 2012c]. This theorem is proved. ■

To explain how to approximate the periodic flow in an $n$-dimensional nonlinear dynamical systems, consider an $n_1 \times n_2$ plane ($n_1 + n_2 = n$), as shown in Fig. 1. $N$-nodes of the periodic flow are chosen for an approximate solution with a certain accuracy $\| \mathbf{x}(t_k) - \mathbf{x}_k \| \leq \varepsilon_k$ ($\varepsilon_k > 0$) and $\| \mathbf{f}(\mathbf{x}(t_k), t_k, \mathbf{p}) - \mathbf{f}(\mathbf{x}_k, t_k, \mathbf{p}) \| \leq \delta_k$ ($\delta_k > 0$). Letting $\delta = \max(\delta_k)_{k \in \{1, 2, \ldots, N\}}$ and $\varepsilon = \max(\varepsilon_k)_{k \in \{1, 2, \ldots, N\}}$ be small positive quantities prescribed, the periodic flow can be approximately described by a set of mappings $\mathbf{P}_k$ with $\mathbf{g}_{k}(\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{p}) = 0$ ($k = 1, 2, \ldots, N$) with periodicity condition $\mathbf{x}_N = \mathbf{x}_0$. Based on the approximate mapping functions, the nodes of periodic motions are computed approximately, which is depicted by a solid curve. The exact solution of the periodic flow is described by a dashed curve. The node points on the periodic flows are depicted with short lines. The red symbols are node points on the exact solution of the periodic flow.

The discrete mapping $\mathbf{P}_k$ is developed from the differential equation. With the control of computational accuracy, the nodes of the periodic flow can be obtained with a good approximation.

From the previous methodology, a set of nonlinear discrete mappings $\mathbf{P}_k$ with $\mathbf{g}_k(\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{p}) = 0$ ($k = 1, 2, \ldots, N$) are developed for periodic motion. Such mapping can be used for numerical simulations. For given $\mathbf{x}_{k-1}$, one can compute $\mathbf{x}_k$ through the algebraic equation of $\mathbf{g}_k(\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{p}) = 0$. For the explicit form, the mapping is directly used for computation of $\mathbf{x}_k$. For the implicit form, the mapping iteration or Newton–Raphson method can be adopted to compute $\mathbf{x}_k$. In addition to a one-step mapping of $\mathbf{P}_k$ with $\mathbf{g}_k(\mathbf{x}_{k-1}, \mathbf{x}_k, \mathbf{p}) = 0$, one can develop a multistep (or $l$-steps) mapping of $\overline{\mathbf{P}}_k$. The diagram illustrates the periodic flow with $N$-nodes with short lines.
From the mapping structure, we have

\[ \Delta x_N = D \Delta x_0 \quad \text{and} \quad D P = \left[ \frac{\partial x_N}{\partial x_0} \right]. \]
Letting \( \Delta x_N = \lambda \Delta x_0 \), we have

\[
(DP - \lambda I_{n \times n}) \Delta x_0 = 0. 
\]  
\( (37) \)

The eigenvalues of \( DP \) are given by \( |DP - \lambda I_{n \times n}| = 0 \). In addition, we have

\[
\Delta x_k = DP_{k(k-1)-1} \cdot \Delta x_0 \quad \text{and} \quad DP_{k(k-1)-1} = \left[ \frac{\partial \Delta x_k}{\partial x_0} \right] \quad (k = 1, 2, \ldots, N). 
\]  
\( (38) \)

Letting \( \Delta x_k = \Delta x_0 \), we have

\[
(DP_{k(k-1)-1} - \lambda I_{n \times n}) \Delta x_0 = 0. 
\]  
\( (39) \)

The eigenvalues of \( DP_{k(k-1)-1} \) are given by \( |DP_{k(k-1)-1} - \lambda I_{n \times n}| = 0 \). Such eigenvalues tell effects of variation of \( x_0 \) on nodes points \( x_k \) in the corresponding neighborhood. The neighborhood of \( x_k^* \), \( U_k(x_k^*) \), is presented in Fig. 2 through a large circle. In such a neighborhood, the eigenvalues can be used to measure the effects \( \Delta x_k \) of \( x_k^* \) varying with \( \Delta x_k \) at \( x_k^* \). The eigenvalues of \( DP \) are given by \( |DP - \lambda I_{n \times n}| = 0 \), which implies the stability and bifurcation of the period-1 flow.

(i) If \( l = 1 \), Eq. (34) becomes

\[
\frac{\partial g_k}{\partial x_{k-1}} \frac{\partial x_{k-1}}{\partial x_0} + \frac{\partial g_k}{\partial x_k} \frac{\partial x_k}{\partial x_0} = 0 \quad (k = 1, 2, \ldots, N). 
\]  
\( (40) \)

The deformation of the foregoing equation yields

\[
\frac{\partial g_k}{\partial x_{k-1}} \frac{\partial x_{k-1}}{\partial x_0} + \frac{\partial g_k}{\partial x_k} \frac{\partial x_k}{\partial x_0} = 0 \quad (k = 1, 2, \ldots, N). 
\]  
\( (41) \)

That is,

\[
\frac{\partial g_k}{\partial x_{k-1} x_0} = \left[ \frac{\partial g_k}{\partial x_k} \right]^{-1} \frac{\partial g_k}{\partial x_{k-1}} \quad (k = 1, 2, \ldots, N). 
\]  
\( (42) \)

From Eq. (40), the following matrix form can be formed.

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} \\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & 0_{n \times n} & \cdots & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0_{n \times n} & 0_{n \times n} & \cdots & \frac{\partial g_{k-1}}{\partial x_{k-1}} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} \\
0_{n \times n} & \vdots & \cdots & \frac{\partial g_k}{\partial x_{k-1}} & \frac{\partial g_k}{\partial x_k} & \cdots & 0_{n \times n} & 0_{n \times n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} & \cdots & \frac{\partial g_N}{\partial x_{N-1}} & \frac{\partial g_N}{\partial x_N} \\
0_{n \times n} & 0_{n \times n} & \cdots & 0_{n \times n} & 0_{n \times n} & \cdots & \frac{\partial g_N}{\partial x_{N-1}} & \frac{\partial g_N}{\partial x_N}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} \\
\frac{\partial g_2}{\partial x_1} \\
\vdots \\
\frac{\partial g_k}{\partial x_{k-1}} \\
\vdots \\
\frac{\partial g_N}{\partial x_{N-1}} \\
\frac{\partial g_N}{\partial x_N}
\end{bmatrix}
= \begin{bmatrix}
0_{n \times n} \\
0_{n \times n} \\
\vdots \\
0_{n \times n} \\
\vdots \\
0_{n \times n} \\
0_{n \times n}
\end{bmatrix}
\]  
\( (43) \)

So we have

\[
DP = \begin{bmatrix}
\frac{\partial g_1}{\partial x_1} \\
\frac{\partial g_2}{\partial x_1} \\
\vdots \\
\frac{\partial g_N}{\partial x_{N-1}} \\
\frac{\partial g_N}{\partial x_N}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial g_1}{\partial x_0} \\
\frac{\partial g_2}{\partial x_0} \\
\vdots \\
\frac{\partial g_N}{\partial x_{N-1}} \\
\frac{\partial g_N}{\partial x_N}
\end{bmatrix}. 
\]  
\( (44) \)

(ii) For \( l = k \), Eq. (33) with periodicity condition \( (x_0 = x_N) \) gives node points \( x_k^* \) \( (k = 0, 1, 2, \ldots, N) \). The corresponding stability and bifurcation can be analyzed in the neighborhood of \( x_k^* \) with \( x_k = x_k^* + \Delta x_k \). Equation (34) becomes

\[
\frac{\partial g_k}{\partial x_0} + \frac{\partial g_k}{\partial x_1} \frac{\partial x_1}{\partial x_0} + \cdots + \frac{\partial g_k}{\partial x_{k-1}} \frac{\partial x_{k-1}}{\partial x_0} + \frac{\partial g_k}{\partial x_k} \frac{\partial x_k}{\partial x_0} = 0_{n \times n} \quad (k = 1, 2, \ldots, N). 
\]  
\( (45) \)
In other words, we have

\[
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & 0_{N \times N} & 0_{N \times N} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial g_{N-1}}{\partial x_1} & \cdots & 0_{N \times N} & 0_{N \times N} \\
\frac{\partial g_N}{\partial x_1} & \cdots & \frac{\partial g_N}{\partial x_{N-1}} & \frac{\partial g_N}{\partial x_N}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial ^2 x_1}{\partial t^2} \\
\vdots \\
\frac{\partial ^2 x_{N-1}}{\partial t^2} \\
\frac{\partial ^2 x_N}{\partial t^2}
\end{bmatrix}
= -
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & 0_{N \times N} & 0_{N \times N} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial g_{N-1}}{\partial x_1} & \cdots & 0_{N \times N} & 0_{N \times N} \\
\frac{\partial g_N}{\partial x_1} & \cdots & \frac{\partial g_N}{\partial x_{N-1}} & \frac{\partial g_N}{\partial x_N}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial ^2 x_1}{\partial t^2} \\
\vdots \\
\frac{\partial ^2 x_{N-1}}{\partial t^2} \\
\frac{\partial ^2 x_N}{\partial t^2}
\end{bmatrix}
\]  

(46)

and

\[
\begin{bmatrix}
\frac{\partial x_1}{\partial x_0} \\
\vdots \\
\frac{\partial x_{N-1}}{\partial x_0} \\
\frac{\partial x_N}{\partial x_0}
\end{bmatrix}
= -
\begin{bmatrix}
\frac{\partial g_1}{\partial x_1} & \cdots & 0_{N \times N} & 0_{N \times N} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial g_{N-1}}{\partial x_1} & \cdots & 0_{N \times N} & 0_{N \times N} \\
\frac{\partial g_N}{\partial x_1} & \cdots & \frac{\partial g_N}{\partial x_{N-1}} & \frac{\partial g_N}{\partial x_N}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial ^2 x_1}{\partial t^2} \\
\vdots \\
\frac{\partial ^2 x_{N-1}}{\partial t^2} \\
\frac{\partial ^2 x_N}{\partial t^2}
\end{bmatrix}
\]  

(47)

Using \(\partial x_k/\partial x_0\), the eigenvalues are determined by

\[
|DP_{k(k-1)-1} - \lambda I_{N \times N}| = 0
\]

with \(DP_{k(k-1)-1} = \begin{bmatrix}
\frac{\partial x_k}{\partial x_0} \\
\vdots \\
\frac{\partial x_{N-1}}{\partial x_0} \\
\frac{\partial x_N}{\partial x_0}
\end{bmatrix}\) (48)

which is used to measure the properties of node points on the periodic flow.

The multistep mappings are developed from the previous determined nodes of periodic motion. During time interval \(t \in [t_0, t_0 + T]\), the periodic flow can be determined by

\[x(t) = x(t_0) + \int_{t_0}^t f(x, t, p) dt, \quad t \in \{0, 1, 2, \ldots, k \} \tag{49}\]

For such a periodic flow, at most, all of \(N\)-nodes during the time interval \(t \in [t_0, t_0 + T]\) are selected, and the corresponding points \(x(t_k) \ (k = 1, 2, \ldots, N)\). Under \(\|x(t_k) - x_k\| \leq \varepsilon_k \) with \(\varepsilon = \max \{\varepsilon_k \}_{k \in \{1, 2, \ldots, N\}} \) and \(\varepsilon_k \geq 0\),

\[|f(x(t_k), t_k, p) - f(x_k, t_k, p)| \leq \delta_k. \]  

(50)

Suppose that \(x_0, \ldots, x_N\) are given, \(f(x_k, t_k, p) \ (k = 0, 1, \ldots, N)\) can be determined. An interpolation polynomial \(P(t, x_0, \ldots, x_N, l_0, \ldots, l_N, p)\) is determined, which can be used to approximate \(f(x, t, p)\). That is,

\[f(x, t, p) \approx P(t, x_0, \ldots, x_N, l_0, \ldots, l_N, p) \]  

(51)

and \(x(t) \approx x_k\) can be computed by

\[x_k = x_l + \int_{t_l}^{t_k} P(t, x_0, \ldots, x_N, l_0, \ldots, l_N, p) dt, \quad l \in \{0, 1, 2, \ldots, k - 1\}. \]  

(52)
Therefore, we have
\[ x_k = x_0 + \sum_{i=0}^{k} g_i(x_0, \ldots, x_N, p). \quad (53) \]
The mapping \( P_k \ (k \in \{1, 2, \ldots, N\}) \) for specific \( l \) is
\[ g_i(x_0, \ldots, x_N, p) = 0. \quad (54) \]
The periodic flow is determined by mapping \( P_k \) \((k = 1, 2, \ldots, N)\) and the periodicity condition, i.e.
\[ g_i(x_0, \ldots, x_N, p) = 0 \quad \text{for } k = 1, 2, \ldots, N \]
\[ x_0 = x_N. \quad (55) \]

From the foregoing equation, node points \( x_k^* \ (k = 0, 1, 2, \ldots, N) \) can be determined. The corresponding stability and bifurcation is discussed in the neighborhood of \( x_k^* \) with \( x_0 = x_k^* + \Delta x_k \). The derivative of Eq. (55) with respect to \( x_0 \) gives
\[ \frac{\partial g_k}{\partial x_0} + \frac{\partial g_k}{\partial x_1} \frac{\partial x_1}{\partial x_0} + \cdots + \frac{\partial g_k}{\partial x_{k-1}} \frac{\partial x_{k-1}}{\partial x_0} \]
\[ + \frac{\partial g_k}{\partial x_k} \frac{\partial x_k}{\partial x_0} = \mathbf{0} \quad (k = 1, 2, \ldots, N). \quad (56) \]

In other words, we have
\[ \begin{bmatrix} \frac{\partial g_0}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_0} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_{N-1}}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_N} \end{bmatrix} \begin{bmatrix} \frac{\partial g_0}{\partial x_0} \\ \vdots \\ \frac{\partial g_{N-1}}{\partial x_0} \end{bmatrix} = - \begin{bmatrix} \frac{\partial g_1}{\partial x_0} \\ \vdots \\ \frac{\partial g_N}{\partial x_0} \end{bmatrix}. \quad (57) \]

and
\[ \begin{bmatrix} \frac{\partial x_1}{\partial x_0} \\ \vdots \\ \frac{\partial x_{N-1}}{\partial x_0} \end{bmatrix} = - \begin{bmatrix} \frac{\partial g_1}{\partial x_0} \\ \vdots \\ \frac{\partial g_N}{\partial x_0} \end{bmatrix} \frac{\partial x_0}{\partial x_0} \begin{bmatrix} \frac{\partial x_1}{\partial x_0} \\ \vdots \\ \frac{\partial x_{N-1}}{\partial x_0} \end{bmatrix}^{-1}. \quad (58) \]

From the above discussion, the discrete mapping can be developed through many forward and backward nodes. The periodic flow in a nonlinear dynamical system can be determined through the following theorem.

**Theorem 2.** Consider a nonlinear dynamical system in Eq. (1). If such a dynamical system has a periodic flow \( x(t) \) with finite norm \( |x| \) and period \( T = 2\pi/|\Omega| \), there is a set of discrete time \( t_k \ (k = 0, 1, 2, \ldots, N) \) with \( N \to \infty \) during one period \( T \), and the corresponding solution \( x(t) \) and vector fields \( f(x(t), t, p) \) are exact. Suppose a discrete node \( x_k \) is on the approximate solutions of the periodic flow under \( |x(t) - x_k| \leq \varepsilon_k \) with a small \( \varepsilon_k \geq 0 \) and
\[ \|f(x(t_k), t_k, p) - f(x_k, t_k, p)\| \leq \varepsilon_k. \quad (59) \]

During a time interval \( t \in [t_k, t_{k+1}] \), there is a mapping \( P_k : x_{k+1} \to x_k \ (k = 1, 2, \ldots, N) \) as
\[ x_k = P_k x_{k+1} \quad \text{with} \] \[ g_k(x_{k+1}, \ldots, x_{k+1}, x_{k+1}) \]
\[ x_{k+1} = x_0, \quad g_k(x_{k+1}, \ldots, x_{k+1}, x_{k+1}) = 0, \]
\[ x_{k+1} = x_0 + \varepsilon_k \quad \text{with} \]
\[ P_k : x_{k+1} \to x_k \quad (k = 1, 2, \ldots, N). \quad (61) \]
A. C. J. Luo

For $x_N = P x_0$, if there is a set of points $x_i^* (k = 0, 1, \ldots, N)$ computed by

$$g_k (x_{i1}^*, \ldots, x_i^{*k}, x_{i(k-1)}^*, \ldots) \seteq x_{i(k-1)}^* (p) = 0, \quad (k = 1, 2, \ldots, N) \quad (62)$$

then the points $x_i^* (k = 0, 1, \ldots, N)$ are approximations of points $x(t_k)$ of the periodic solution. In the neighborhood of $x_i^*$, with $x_i^* = x_i^* + \Delta x_i$, the linearized equation is given by

$$\frac{\partial g_k}{\partial x_0} + \cdots + \frac{\partial g_k}{\partial x_{k-1}} + \frac{\partial g_k}{\partial x_k} = 0$$

with

$$\left[ \begin{array}{c} \frac{\partial g_k}{\partial x_0} \\ \frac{\partial g_k}{\partial x_1} \\ \vdots \\ \frac{\partial g_k}{\partial x_{k-1}} \\ \frac{\partial g_k}{\partial x_k} \end{array} \right] = \left[ \begin{array}{c} \frac{\partial g_k}{\partial x_0} \\ \frac{\partial g_k}{\partial x_1} \\ \vdots \\ \frac{\partial g_k}{\partial x_{k-1}} \\ \frac{\partial g_k}{\partial x_k} \end{array} \right] = 0$$

The properties of discrete points $x_i (k = 1, 2, \ldots, N)$ can be estimated by the eigenvalues of $DP_{(k-1)\ldots}$ as

$$|DP_{(k-1)\ldots}| = 0 \quad (k = 1, 2, \ldots, N). \quad (66)$$

The eigenvalues of $DP$ for such periodic flow are determined by

$$|DP - \lambda I_{x_{k\ldots}}| = 0. \quad (67)$$

Thus, the stability and bifurcation of the periodic flow can be classified by the eigenvalues of $DP(x_i^*)$ with

$$([n_{1^*}, n_{2^*}] : [n_{1^*}, n_{2^*}^*] : [n_{3^*}, n_{4^*}] : [n_{4^*}, n_{5^*} : [n_{5^*}, n_{6^*}]) \quad (68)$$

(i) If the magnitudes of all eigenvalues of $DP(\lambda_i) < 1, i = 1, 2, \ldots, n$ are less than one, the approximate periodic solution is stable.

(ii) If at least the magnitude of one eigenvalue of $DP(\lambda_i) > 1, i \in \{1, 2, \ldots, n\}$ is greater than one, the approximate periodic solution is unstable.

(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. The proof is similar to Theorem 1. □

From the stability and bifurcation analysis, the period-1 flow under period $T = 2\pi/\Omega$, on the set of discrete mapping $P_k$ with $g_k(x_{k-1}, x_k, p) = 0 \quad (k = 1, 2, \ldots, N)$, are stable and unstable. If the period-doubling bifurcation occurs, the periodic flow will become a periodic flow under period $T' = 2T$, and such a periodic flow is called the period-2 flow. Due to the period-doubling, $2N$ nodes of the period-2 flow will be employed to describe the period-2 flow. Thus, consider a mapping structure of the period-2 flow with $2N$ mappings.
For $x_{2N} = P_0 x_0$, if there is a set of points $x^*_k$ ($k = 0, 1, \ldots, 2N$) computed by

$$g_k(x_{2(N-k)}^*, x_{2(N-k)+1}^*; p) = 0, \quad (k = 1, 2, \ldots, 2N)$$

(70)

After period-doubling, the period-1 flow becomes period-2 flow. The node points increase to $2N$ points during two periods ($2T$). The period-2 flow can be sketched in Fig. 3. The node points are determined through the discrete mapping with mathematical relation Eq. (69).

On the other hand,

$$T' = 2T = \frac{2(2\pi)}{\bar{\omega}} = \frac{2\pi}{\omega} = \frac{\Omega}{2}$$

(71)

During the period of $T'$, there is a periodic flow, which can be described by node points $x^*_k$ ($k = 1, 2, \ldots, N$). Since the period-1 flow is described by node points $x_k$ ($k = 1, 2, \ldots, N$) during the period $T$, due to $T' = 2T$, the period-2 flow can be described by $N' = 2N$ nodes. Thus the corresponding mapping $P_k$ is defined as

$$P_k : x_{2k-1}^{(2)} \rightarrow x_k^{(2)} \quad (k = 1, 2, \ldots, 2N)$$

(72)

![Fig. 3. Period-2 flow with 2N-nodes with short lines. The solid curve is for numerical results. The symbols are for node points on the periodic flow.](Image)

Discrete Implicit Mappings of Continuous Nonlinear Systems

and

$$g_k(x_{2(N-k)}^{(2)}, x_{2(N-k)+1}^{(2)}; p) = 0, \quad (k = 1, 2, \ldots, 2N)$$

(73)

$$x_0^{(2)} = x_{2N}^{(2)}.$$  

In general, for period $T' = mT$, there is a period-$m$ flow which can be described by $N' = mN$. The corresponding mapping $P_k$ is given by

$$P_k : x_{2(k-1)}^{(m)} \rightarrow x_k^{(m)} \quad (k = 1, 2, \ldots, mN)$$

(74)

and

$$g_k(x_{2(N-k)}^{(m)}, x_{2(N-k)+1}^{(m)}; p) = 0, \quad (k = 1, 2, \ldots, mN)$$

$$x_0^{(m)} = x_{mN}^{(m)}.$$  

(75)

From the above discussion, the period-$m$ flow in a nonlinear dynamical system can be described through $mN$ nodes for period $mT$. The corresponding theorem is presented as follows.

**Theorem 3.** Consider a nonlinear dynamical system in Eq. (1). If such a dynamical system has a period-$m$ flow $x^{(m)}(t)$ with finite norm $\|x^{(m)}\|$ and period $mT$ ($T = 2\pi/\bar{\omega}$), there is a set of discrete time $t_k$ ($k = 0, 1, \ldots, mN$) with $N \rightarrow \infty$ during $m$-periods ($mT$), and the corresponding solution $x^{(m)}(t_k)$ and vector fields $f(x^{(m)}(t_k), t_k, p)$ are exact. Suppose a discrete node $x_k^{(m)}$ is on the approximate solution of the periodic flow under $\|x^{(m)}(t_k) - x_k^{(m)}\| \leq \varepsilon_k$ with a small $\varepsilon_k \geq 0$ and

$$\|f(x^{(m)}(t_k), t_k, p) - f(x_k^{(m)}, t_k, p)\| \leq \delta_k$$

(76)

with a small $\delta_k \geq 0$. During a time interval $t \in [t_{k-1}, t_k]$, there is a mapping $P_k : x_{k-1}^{(m)} \rightarrow x_k^{(m)}$ ($k = 1, 2, \ldots, mN$) as

$$x_k^{(m)} = P_k x_{k-1}^{(m)} \quad \text{with} \quad g_k(x_{k-1}^{(m)}, x_k^{(m)}; p) = 0, \quad k = 1, 2, \ldots, mN.$$  

(77)

Consider a mapping structure as

$$P = P_{mN} \circ P_{mN-1} \circ \cdots \circ P_2 \circ P_1 : x_0^{(m)} \rightarrow x_{mN}^{(m)}$$

with $P_k : x_{k-1}^{(m)} \rightarrow x_k^{(m)}$ ($k = 1, 2, \ldots, mN$).
For \( x_{kN}^{(m)} = P x_{0}^{(m)} \), if there is a set of points \( x_{k}^{(m)} \)  
(k = 0, 1, \ldots, mN), then 
\[
\mathcal{g}_{k}(x_{k-1}^{(m)}, x_{k}^{(m)}, p) = 0, \quad (k = 1, 2, \ldots, mN) 
\]
\[
x_{0}^{(m)} = x_{mN}^{(m)}. 
\]
(79)

Then the points \( x_{k}^{(m)} \)  
(k = 0, 1, \ldots, mN) are approximations of points \( x^{(m)}(t_{k}) \) of the periodic solution. In the neighborhood of \( x_{k}^{(m)} \), with \( x_{k}^{(m)} = x_{k}^{(m)} + \Delta x_{k}^{(m)} \), the linearized equation is given by 
\[
\Delta x_{k}^{(m)} = D P_{k} \cdot \Delta x_{k-1}^{(m)} 
\]
with 
\[
\mathcal{g}_{k}(x_{k-1}^{(m)} + \Delta x_{k-1}^{(m)}, x_{k}^{(m)} + \Delta x_{k}^{(m)}, p) = 0 \quad (k = 1, 2, \ldots, mN). 
\]
(80)

The resultant Jacobian matrices of the periodic flow are 
\[
D P_{k(k-1)-1} = D P_{k} \cdot D P_{k-1} \cdots \cdot D P_{1}, 
\]
where 
\[
D P_{k} = \left[ \frac{\partial x_{k}^{(m)}}{\partial x_{k-1}^{(m)}} \right]_{(x_{k-1}^{(m)}, x_{k}^{(m)}, p)} = - \left[ \frac{\partial x_{k}^{(m)}}{\partial x_{k-1}^{(m)}} \right]_{(x_{k}^{(m)}, \Delta x_{k}^{(m)})}. 
\]
(82)

The eigenvalues of \( D P \) for such a periodic flow are determined by 
\[
|D P_{k(k-1)-1} - \lambda | = 0, \quad (k = 1, 2, \ldots, mN); 
\]
\[
|D P - \lambda | = 0. 
\]
(83)

Thus, the eigenvalues of \( D P_{k(k-1)-1} \) give the properties of \( \Delta x_{k} \) varying with \( \Delta x_{0} \). The stability and bifurcation of the periodic flow can be classified by the eigenvalues of \( D P(x_{k}^{(m)}) \) with 
\[
(\nu_{1}, \nu_{2}) : (\nu_{2}, \nu_{3}) : (\nu_{3}, \nu_{4}) : (\nu_{4}, \nu_{5}) : (\nu_{5}, \nu_{6}) : (\nu_{6}, \nu_{7}). 
\]
(84)

(i) If the magnitudes of all eigenvalues of \( D P^{(m)} \) are less than one (i.e., \( |\lambda| < 1, i = 1, 2, \ldots, m \)), the approximate period-\( m \) solution is stable.

(ii) If at least one eigenvalue of \( D P^{(m)} \) is greater than one (i.e., \( |\lambda| > 1, i \in \{1, 2, \ldots, m\} \)), the approximate period-\( m \) solution is unstable.

(iii) The boundaries between stable and unstable period-\( m \) flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. The discrete mapping for the period-\( m \) flow can be developed during \( t \in [t_{k}, t_{k+1}] \) as in Theorem 1. The proof is similar to Theorem 1.

The discrete mapping for a period-\( m \) flow with multiple steps can be developed by using many forward and backward nodes. The period-\( m \) flow in the nonlinear dynamical system can be determined through the following theorem.

**Theorem 4.** Consider a nonlinear dynamical system in Eq. (1). If such a dynamical system has a period-\( m \) flow \( x^{(m)}(t) \) with finite norm \( ||x^{(m)}|| \) and \( m \)-periods \( mT = 2\pi(2\pi) \), there is a set of discrete time \( t_{k} \) (\( k = 0, 1, \ldots, mN \)) with \( (N \rightarrow \infty) \) during \( m \)-period \( T \), and the corresponding solution \( x^{(m)}(t_{k}) \) and vector fields \( f(x^{(m)}(t_{k}), t_{k}, p) \) are exact. Suppose a discrete node \( x_{k}^{(m)} \) is on the approximate solution of the periodic flow under \( ||x^{(m)}(t_{k}) - x_{k}^{(m)}|| \leq \varepsilon_{k} \) with a small \( \varepsilon_{k} \geq 0 \) and 
\[
||f(x^{(m)}(t_{k}), t_{k}, p) - f(x_{k}^{(m)}, t_{k}, p)|| \leq \delta_{k} 
\]
with a small \( \delta_{k} \geq 0 \). During a time interval \( t \in [t_{k-1}, t_{k}] \), there is a mapping \( P_{k} : x_{k-1}^{(m)} \rightarrow x_{k}^{(m)} \) (\( k = 1, 2, \ldots, mN \)) as 
\[
x_{k}^{(m)} = P_{k}x_{k-1}^{(m)} \] 
with 
\[
\mathcal{g}_{k}(x_{k-1}^{(m)} + \Delta x_{k-1}^{(m)}, x_{k}^{(m)}, p) = 0, 
\]
\[
\Delta x_{i}^{(m)} = s_{i} \mod (k - j + mN, mN), \]
\[
j = -l_{2}, -l_{2} + 1, \ldots, 0, l_{1} - 1, l_{1}; 
\]
\[
l_{1}, l_{2} \in \{0, 1, 2, \ldots, mN\}, 1 \leq l_{2} + l_{1} \leq mN, \]
\[
l_{1} \geq 1, (k = 1, 2, \ldots, mN). 
\]
(85)
where \( g_k \) is an implicit vector function. Consider a mapping structure as
\[
P = P_{mN} \circ P_{mN-1} \circ \cdots \circ P_2 \circ P_1 : x^{(m)}_0 \rightarrow x^{(m)}_{mN};
\]
with \( P_k : x^{(m)}_{k-1} \rightarrow x^{(m)}_k \) \((k = 1, 2, \ldots, mN)\),

\[
(87)
\]
For \( x^{(m)}_n = P x^{(m)}_1 \), if there is a set of points \( x^{(m)*}_k \) \((k = 0, 1, \ldots, mN)\) computed by
\[
g_k(x^{(m)*}) = x^{(m)*}, \quad x^{(m)*}_{k+1} = x^{(m)*}_k + \Delta x^{(m)*}_k,
\]
then the points \( x^{(m)*}_k \) \((k = 0, 1, \ldots, mN)\) are approximations of points \( x^{(m)}(t_k) \) of the periodic solution. In the neighborhood of \( x^{(m)*}_k \), with \( x^{(m)}_k = x^{(m)*}_k + \Delta x^{(m)*}_k \), the linearized equation is given by
\[
\frac{\partial g_k}{\partial x^{(m)}_0} + \cdots + \frac{\partial g_k}{\partial x^{(m)}_{k-1}} + \frac{\partial g_k}{\partial x^{(m)}_k} \frac{\partial x^{(m)}_k}{\partial x^{(m)}_0} = 0.
\]

\[
\left[ \begin{array}{c}
\frac{\partial g_k}{\partial x^{(m)}_0} \\
\vdots \\
\frac{\partial g_k}{\partial x^{(m)}_{k-1}} \\
\frac{\partial g_k}{\partial x^{(m)}_k}
\end{array} \right] = - \left[ \begin{array}{cccc}
\frac{\partial g_1}{\partial x^{(m)}_0} & \cdots & \frac{\partial g_1}{\partial x^{(m)}_{mN}} & \frac{\partial g_1}{\partial x^{(m)}_{mN}} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial g_{mN-1}}{\partial x^{(m)}_0} & \cdots & \frac{\partial g_{mN-1}}{\partial x^{(m)}_{mN}} & \frac{\partial g_{mN-1}}{\partial x^{(m)}_{mN}} \\
\frac{\partial g_{mN}}{\partial x^{(m)}_0} & \cdots & \frac{\partial g_{mN}}{\partial x^{(m)}_{mN}} & \frac{\partial g_{mN}}{\partial x^{(m)}_{mN}}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\frac{\partial x^{(m)}_0}{\partial x^{(m)}_0} \\
\vdots \\
\frac{\partial x^{(m)}_{k-1}}{\partial x^{(m)}_0} \\
\frac{\partial x^{(m)}_k}{\partial x^{(m)}_0}
\end{array} \right].
\]

\[
(91)
\]
The properties of discrete points \( x_k \) \((k = 1, 2, \ldots, mN)\) can be estimated by the eigenvalues of \( DP_{k(k-1)} \) as
\[
|DP_{k(k-1)} - \lambda x_k| = 0 \quad (k = 1, 2, \ldots, mN).
\]

\[
(92)
\]
The eigenvalues of \( DP \) for such a periodic flow are determined by
\[
|DP - \lambda x_k| = 0.
\]

\[
(93)
\]
Thus, the stability and bifurcation of the period-
m flow can be classified by the eigenvalues of
\[
DP(x^{(m)*}) \text{ with }
\]
\[
|DP(x^{(m)*}) - \lambda x_k| = 0.
\]

\[
(94)
\]
(i) If the magnitudes of all eigenvalues of \( DP(x^{(m)*}) \) are less than one \((i.e. |\lambda| < 1, i = 1, 2, \ldots, n)\), the approximate period-

(ii) If at least the magnitude of one eigenvalue of \( DP(x^{(m)*}) \) is greater than one \((i.e. |\lambda| > 1,\)
nodes regular solution points, and the green, large circular
shown in Fig. 4. The small circular symbols are the
large symbols are the time-related points. The referenced point \( x_k \) and the correspond-
ing time-delay point \( x_k^* \) are labeled. The time-delay
point \( x_k^* \) can be estimated by the two vicinity points \( x_{k-1} \) and \( x_{k+1-\tau} \).

From the above discrete scheme, periodic flows in the
time-delay dynamical systems can be dis-
cussed herein. If a time-delay nonlinear system has a
periodic flow with a period of \( T = 2\pi/\Omega \), then such a
periodic flow can be expressed by discrete points
through discrete mappings of the time-delay con-
tinuous dynamical systems as aforementioned. The
method is stated through the following theorem.

**Theorem 5.** Consider a time-delay nonlinear
dynamical system as
\[
\dot{x} = f(x, x^*, t, p) \in \mathbb{R}^m
\]
where \( f(x, x^*, t, p) \) is a \( C^\infty \)-continuous nonlinear vector function \((r \geq 1)\) and \( x^* = x(t - \tau) \).

3.3 Time-delay nodes based on interpolation

If a time-delay nonlinear system has solution points \( x_k \approx x(t_k) \) and \( x_k^* \approx x(t_k - \tau) \) for \( k = 0, 1, 2, \ldots \), as shown in Fig. 4. The small circular symbols are the
regular solution points, and the green, large circular
symbols are the time-delay node points. The delay
nodes \( x_k^* \approx x(t_k - \tau) \) of \( x_k \approx x(t_k) \) will lie between \( x_{k-\tau}, \) and \( x_{k+1-\tau} \) (integer \( k > 0 \)), which can be expressed by interpolation of two points \( x_{k-1} \) and \( x_{k+1-\tau} \). From Eq. (95), we have
\[
x(t_k) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} f(x, x^*, t, p) dt.
\]

Consider an interpolation function between \( f(x_{k-1}, x_{k-1-\tau}, t_{k-1}, p) \) and \( f(x_k, x_k^*, t_k, p) \) to approximate \( f(x, x^*, t, p) \). Equation (96) becomes
\[
x_k = x_{k-1} + \tilde{g}_k(x_{k-1}, x_k; x_{k-1-\tau}, x_k^*, p).
\]
approximations of points

\[ x_j^* = h_j(x_{r_j-1}, x_r, \theta_{r_j}), \quad j = k, k - 1, \quad r_j = j - l_j \]

\[ \theta_{r_j} = \frac{1}{h_{r_j}} \left( \left. \prod_{i=1}^{l_j} h_{r_{j+i}} \right|_{r_{j+i}} \right), \]

(100)

where \( g_k \) is an implicit vector function and \( h_j \) is an interpolation vector function. Consider a mapping structure as

\[ P = P_N \circ P_{N-1} \circ \cdots \circ P_2 \circ P_1 : x_0 \rightarrow x_N; \]

with

\[ P_k : (x_{k-1}, x^*_k) \rightarrow (x_k, x^*_k) \quad (k = 1, 2, \ldots, N). \]

(101)

For \( (x_N, x^*_N) = P(x_0, x^*_0) \), if there is a set of points \( x^*_k (k = 0, 1, \ldots, N) \) computed by

\[ g_k(x_{k-1}, x_k, x^*_{k-1}, x^*_k, p) = 0, \quad x^*_k = h_k(x_{k-1}, x_k, \theta_{r_k}), \quad j = k, k-1 \]

\[ (k = 1, 2, \ldots, N) \]

\[ x_{r_j-1} = x_{\text{mod}(r_j-1+N,N)}, \quad x_{r_j} = x_{\text{mod}(r_j+N,N)}, \]

\[ x^*_0 = x^*_N \quad \text{and} \quad x^*_N = x^*_N, \]

(102)

then the points \( x^*_k \) and \( x^*_N \) \((k = 0, 1, \ldots, N)\) are approximations of points \( x(t_k) \) and \( x^*(t_k) \) of the periodic solution. In the neighborhoods of \( x^*_k \) and \( x^*_N \), with \( x_k = x^*_k + \Delta x_k \) and \( x^*_0 = x^*_N + \Delta x^*_N \), the linearized equation is given by

\[ \frac{\partial g_k}{\partial x_k} \Delta x_k + \frac{\partial g_k}{\partial x_{k-1}} \Delta x_{k-1} + \sum_{j=k+1}^{s} \frac{\partial g_k}{\partial x^*_j} \Delta x^*_j \]

\[ \times \left( \frac{\partial x^*_j}{\partial x_{r_j}} \Delta x_{r_j} + \frac{\partial x^*_j}{\partial x_{r_{j+i}}} \Delta x_{r_{j+i}} \right) \]

\[ = 0_{x_N} \quad \text{with} \quad r_j = j - l_j, \quad j = k - 1, k; \quad (k = 1, 2, \ldots, N). \]

(103)

The resultant Jacobian matrices of the periodic flow are

\[ DP_k = DP_{(k-1)-1} \begin{bmatrix} \frac{\partial y_k}{\partial y_0} \end{bmatrix}_{y_0^*} \]

\[ = A_k A_{k-1} \cdots A_1 \quad (k = 1, 2, \ldots, N), \quad \text{and} \]

\[ \Delta y_k = A_k \Delta y_{k-1}, \quad (104) \]

\[ A_k = \begin{bmatrix} \frac{\partial y_k}{\partial y_{k-1}} \end{bmatrix}_{y_{k-1}^*}, \quad (105) \]

\( \partial y_k/\partial y_{k-1} \)

with \( r_j = j - l_j, \quad j = k - 1, k; \)

\[ y_k = (x_k, x_{k-1}, \ldots, x_{r_k-1})^T, \]

\[ y_{k-1} = (x_{k-1}, x_{k-2}, \ldots, x_{r_{k-1}-1})^T, \]

\[ \Delta y_k = (\Delta x_k, \Delta x_{k-1}, \ldots, \Delta x_{r_k-1})^T, \]

\[ \Delta y_{k-1} = (\Delta x_{k-1}, \Delta x_{k-2}, \ldots, \Delta x_{r_{k-1}-1})^T; \]

\[ A_k = \begin{bmatrix} B_k & (a_k(r_k-1))_{n \times n} \\ I_k & B_k \end{bmatrix}_{n \times (n+1)}, \quad \left(106\right) \]

\[ B_k = \begin{bmatrix} (a_{(k-1)}(r_{k-1}))_{n \times n} \\ \vdots \\ 0 \end{bmatrix}_{n \times (n+1)}, \quad s = 1 + l_k \]

\[ I_k = \text{diag}(I_{n \times n}, I_{n \times n}, \ldots, I_{n \times n}), \]

\[ a_k = (0_{n \times n}, 0_{n \times n}, \ldots, 0_{n \times n}). \]

(107)

The properties of discrete points \( y_k (k = 1, 2, \ldots, N) \) can be estimated by the eigenvalues of \( DP_{(k-1)-1} \) as

\[ |DP_k - \Lambda_{(k-1)+1}^{n \times (n+1)}| = 0 \quad (k = 1, 2, \ldots, N). \]

(108)
Thus, the stability and bifurcation of the periodic flow can be classified by the eigenvalues of $DP(y)$ with
\begin{equation}
\begin{bmatrix}
[n_1^1, n_2^1] ; [n_1^2, n_2^2] ; [n_3, n_4] ; [n_5, n_6]
\end{bmatrix}, \quad (109)
\end{equation}
(i) If the magnitudes of all eigenvalues of $DP$ are less than one (i.e. $|\lambda| < 1$, $i = 1, 2, \ldots, n$), the approximate periodic solution is stable.
(ii) If at least the magnitude of one eigenvalue of $DP$ is greater than one (i.e. $|\lambda| > 1$, $i \in [1, 2, \ldots, n]$), the approximate periodic solution is unstable.
(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. If $f(x, x', t, p)$ is a $C^r$-continuous nonlinear function vector ($r \geq 1$), then the velocity $\mathbf{k}$ should be $C^r$-continuous ($r \geq 1$). If such a dynamical system has a periodic flow $x(t)$ and $x'(t)$ with finite norms $|x|$ and $|x'|$ with period $T = 2\pi/\Omega$, there is a set of discrete time $t_k$ ($k = 0, 1, \ldots, N$) with $(N \to \infty)$ during one period $T$. The corresponding solution $x(t_k)$ and $x'(t_k) = x(t_k - \tau)$ with vector fields $f(x(t_k), x'(t_k), t, p)$ are exact. Consider a time interval $t \in [t_{k-1}, t_k]$.

\begin{equation}
x(t) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} f(x, x', t, p) dt. \quad (110)
\end{equation}

For the time interval $[t_{k-1}, t_k]$ divided into nodes $t_i = t_{k-1} + \epsilon_i t_k$ with $\epsilon_i \in [0, 1]$ and $f(x(t_i), x'(t_i), t_i, p)$ ($i = 1, \ldots, s$) with $x'(t_i) = x(t_i - \tau)$, there is an approximate function $P(t, C)$ with unknown $C = (C_1, \ldots, C_s)^T$ and $C_i$ ($i = 1, \ldots, s$), and the following condition is satisfied, i.e.

\begin{equation}
f(x(t_i), x'(t_i), t_i, p) = P(t_i, C(t_i) - \tau, C), \quad i = 1, 2, \ldots, s. \quad (111)
\end{equation}

From the foregoing equation, the unknowns $C(t_k) = (C_1, \ldots, C_s)^T$ with $t_k = (t_1, \ldots, t_s)$ is $t_k(1, 1, \ldots, 1)^T + \epsilon_k (c_1, c_2, \ldots, c_s)^T$ are determined. For a small $\delta > 0$, if there is a relation

\begin{equation}
|P(t, t - \tau, C(t_k)) - f(x, x', t, p)| \leq \delta, \quad (112)
\end{equation}

for $t \in [t_{k-1}, t_k]$, Eq. (110) can be approximated by

\begin{equation}
x(t) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} [P(t, t - \tau, C(t_k)) + O(\delta)] dt;
\end{equation}

\begin{equation}
\mathfrak{R}(t) = \mathfrak{R}(t_{k-1}) + \int_{t_{k-1}}^{t_k} P(t, t - \tau, C(t_k)) dt
\end{equation}

and

\begin{equation}
\mathfrak{R}(t_k) \approx \mathfrak{R}(t_{k-1}) + \int_{t_{k-1}}^{t_k} P(t, t - \tau, C(t_k)) dt.
\end{equation}

Let $\mathfrak{R}(t_k) = x_{k-1}, \mathfrak{R}(t_k) = x_k$ and $\mathfrak{R}(t_k) = x'_k$. For any small $[\epsilon_{k-1}, \epsilon_k] > 0$, $\|x(t_k) - x_{k-1}\| \leq \epsilon_{k-1}$, $\|x(t_k) - x_k\| \leq \epsilon_k$, and $\|x(t) - x_k\| \leq \epsilon_k$, Eq. (114) gives

\begin{equation}
x_k = x_{k-1} + \epsilon_k \sum_{j=1}^{l} (x_{j-1}, x_j, x_{j+1}, x'_j, p),
\end{equation}

\begin{equation}
\epsilon_k \approx \left| \epsilon_{k-1} \sum_{j=1}^{l} (x_{j-1}, x_j, x_{j+1}, x'_j, p) \right|
\end{equation}

\begin{equation}
\frac{1}{h_{r_j}} \left[ \tau - \sum_{i=1}^{l_j} h_{r_{j+i}} \right]
\end{equation}

for $r_j = j - l_j$, $j = k - 1, k$.

\begin{equation}
g_k(x_{k-1}, x'_k, x_k, x'_k) = 0.
\end{equation}

From the discrete mapping, two points $x(t_{k-1})$ and $x(t_k)$ for the time interval $t \in [t_{k-1}, t_k]$ ($k = 0, 1, \ldots, N$) can be approximated by $x_{k-1}$ and $x_k$, respectively. If $f(x, x', t, p)$ is a $C^r$-continuous nonlinear vector function, we have $\|x(t)\| \leq L$ and $\|f(x)\| \leq L^r$ ($L$ and $L^r$ constant). Thus for $j = k - 1, k$

\begin{equation}
\|f(x(t_j), x'_j, t_j, p) - f(x_k, x'_k, t_j, p)\| \\
\leq L\|x(t_j) - x_k\| + L^r\|x'(t_j) - x'_k\|
\end{equation}

\begin{equation}
\leq L\epsilon_k + L^r\epsilon'_k.
\end{equation}

Once the mapping $P_k : (x_{k-1}, x'_k) \to (x_k, x'_k)$ ($k = 1, 2, \ldots, N$) with $g_k(x_{k-1}, x_k, x'_k, x'_k) = 0$
exists, the periodic flow can be formed by \( P : (x_0, x'_0) \rightarrow (x_N, x'_N) \) with \( P = P_N \circ P_{N-1} \circ \cdots \circ P_2 \circ P_1 \). For \( P_k : (x_{k-1}, x'_{k-1}) \rightarrow (x_k, x'_k) \), we have
\[
g_k(x_{k-1}, x'_{k-1}, x_k, x'_k, p) = 0,
\]
with the periodicity condition, we have
\[
x'_j = h_j(x_{j-1}, x_j, \theta_j),
\]
\[
r_j = j - l_j, j = k, k - 1 \ldots
\]
With the periodicity condition, we have
\[
x_j = x_{\text{mod}(r_N,N)}, \quad j = k, k - 1
\]
\[
x_0 = x_N \quad \text{and} \quad x'_0 = x'_N.
\]
Solving Eqs. (118) and (119) gives \( x'_k (k = 1, 2, \ldots, N) \) to get the period-1 flow. For the stability of such a periodic flow, consider \( x_k = x'_k + \Delta x_k \) and \( x''_k = x''_k + \Delta x''_k \) \( (k = 1, 2, \ldots, N) \) for \( x_k \in U(x'_k) \) and \( x''_k \in U(x''_k) \). Equation (118) becomes
\[
g_k(x_{k-1} + \Delta x_{k-1}, x'_k + \Delta x_k, x''_k - \Delta x''_{k-1} + \Delta x''_k + \Delta x''_{k-1}) = 0,
\]
\[
\Delta x_{k+1} = \Delta x_{k-1}, \quad \Delta x_{k-2} = \Delta x_{k-2}, \ldots,
\]
\[
\Delta x_{k-1} = \Delta x_{k-1}, \quad (k = 1, 2, \ldots, N)
\]
Thus, derivatives of \( g_k(x_{k-1}, x''_{k-1}, x_k, x''_k, p) = 0 \) with respect to \( x_0 \) gives
\[
\sum_{j=k-1}^k \left[ \frac{\partial g_k}{\partial x_j} \right] x'_j x''_j \cdots x'_2 x''_2 \Delta x_{k-j} + \left[ \frac{\partial g_k}{\partial x'_j} \right] x'_j x''_j \cdots x'_2 x''_2 \Delta x''_{k-j} = 0_{n \times 1}
\]
(121)
\[
\Delta x_{k+1} = \Delta x_{k-1}, \quad \Delta x_{k-2} = \Delta x_{k-2}, \ldots,
\]
\[
\Delta x_{k-1} = \Delta x_{k-1}, \quad (k = 1, 2, \ldots, N)
\]
with
\[
\Delta x'_j = \frac{\partial g_k'}{\partial x_j} \Delta x_{j+1} + \frac{\partial g_k''}{\partial x'_j} \Delta x_{j-1} \quad \text{with}
\]
\[
r_j = j - l_j, \quad j = k - 1, k
\]
Let
\[
a_{k,j} = \left[ \frac{\partial g_k}{\partial x_j} \right]^{-1} \frac{\partial g_k}{\partial x_k}
\]
Thus
\[
\Delta y_k = A_k \Delta y_{k-1} = \left[ \frac{\partial y_k}{\partial y_{k-1}} \right] (y'_k, y''_k),
\]
with
\[
B_k = \left[ \left[ a_{k-1}(r_{k-1}) \right]_{n \times n} \right]_{n+1},
\]
\[
A_k = \begin{bmatrix} B_k & 0_k \end{bmatrix} \begin{bmatrix} I_k & 0_k \end{bmatrix} = \begin{bmatrix} a_{k}(r_{k-1})_{n \times n} \end{bmatrix},
\]
\[
0_k = \begin{bmatrix} 0_k \end{bmatrix}, \quad \tau = s, \quad s = 1, k-1
\]
(125)
From the foregoing equations, we have \( \partial y_k \partial y_{k-1} \). Thus the linearized equation based on the initial point \( y_0 \) in Eq. (120) gives
\[
\Delta y_k = D P_{(k-1),1} \Delta y_0
\]
(126)
\[
\Delta y_N = D P_{N(N-1),1} \Delta y_0 = \left[ \frac{\partial y_k}{\partial y_0} \right] (y'_N, y''_N) \Delta y_0
\]
(127)
Setting $\Delta y_N = \lambda \Delta y_0$ and $\Delta y_{N - 1} = \lambda \Delta y_0$, the foregoing equation becomes

$$
\begin{align}
(D P_{k(\tau - 1)} - \lambda I_{|k(\tau - 1)|}) \Delta y_0 &= 0, \\
(D P - \lambda I_{|\tau + 1|}) \Delta y_0 &= 0.
\end{align}
$$

(127)

For any nontrivial solution ($\|\Delta y_0\| \neq 0$), we have

$$
\begin{align}
|D P_{k(\tau - 1)} - \lambda I_{|k(\tau - 1)|}| = 0, \\
|D P - \lambda I_{|\tau + 1|}| = 0.
\end{align}
$$

(128)

Thus, the eigenvalues of $D P_{k(\tau - 1)}$ give the changes of $\Delta y_N$ with $\Delta y_0$. In addition, the eigenvalues of $D P$ are computed for the periodic solution due to $y_N^* = y_0^*$. From the stability and bifurcation theory of dynamical systems at fixed points in the discrete nonlinear systems, the stability and bifurcation of the periodic solution of the time-delay nonlinear system can be classified as stated in the theorem. This theorem is proved.[]

For a time-delay system, a periodic solution is represented by $N$ discrete points $(x_k, k = 0, 1, 2, \ldots, N)$ and the corresponding time-delay points $(x_{k+1}, k = 0, 1, 2, \ldots, N)$, as shown in Fig. 5. The time-delay nodes are obtained by interpolation. The small, filled circular symbols are for discrete nodes, and the large, hollow circular symbols are for time-delay nodes. The periodicity requires $x_N = x_0$ and $x_{N-1} = x_0$. To reduce computation, the time-delay points $x_k^r (k = 0, 1, 2, \ldots, N)$ are interpolated by $x_{mod(k+N-1, N)}$ and $x_{mod(k+N-1, N)}$. For $k = 0$, $x_0^r$ is interpolated by $x_{N-1}$ and $x_{N-2}$ for periodic flow. For $k = N$, $x_N^r$ is interpolated by $x_{N-1}$ and $x_{N-2}$ for periodic flow. In fact, $x_k^r$ can be interpolated by multiple nodes around two points $x_{mod(k+N-1, N)}$ and $x_{mod(k+N-1, N)}$. For instance, $s_1 = s_2 + 1$ nodes, $s_{mod(k+N-1, N)}$, for interpolation of the time-delay $x_k^r$. At least, two points $x_{mod(k+N-1, N)}$ and $x_{mod(k+N-1, N)}$ should be used for interpolation with a better approximation. From the foregoing theorem, a set of nonlinear, time-delay, discrete mappings $P_h$ with $g_k(x_k, x_{k-1}, x_{k+1}, x_0^r, p) = 0$ is developed for a periodic flow. Such a mapping can be used for numerical simulations. For given $x_0^r$, and $x_{N-1}$, one can compute $x_k$ through the algebraic equation of $g_k(x_k, x_{k-1}, x_{k+1}, x_0^r, p) = 0$. In addition to a one-step time-delay mapping of $P_h$, one can develop a multistep (or r-steps) time-delay mapping of $P_h$ with

$$
g_k(x_{k-1}, x_{k+1}, x_k^r, x_0^r, p) = 0, \\
x_{k-1} = h_{k-1}(x_{k-2}, x_{k-1}, \theta_{k-1}),
$$

(129)

\begin{align}
\theta_{k-1} &= 1 \frac{1}{h_{k-1}} \left[ \tau - \sum_{l=1}^{s_{k-1}} h_{k-1} \right], \\
r_{k-1} &= k - j - l_{i_{k-1}}, \\
\theta_{k-1} &= k - j - l_{i_{k-1}}, \\
\theta_{k-1} &= j = 0, 1, 2, \ldots, r, \\
r_{k-1} &= \{1, 2, \ldots, k\} \\
N &= 1, 2, \ldots, N.
\end{align}

(i) If $r = 1$, the one-step time-delay mapping is recovered from the multistep time-delay mapping.

(ii) If $r = 2$, the two-step time-delay mapping is obtained from the multistep time-delay mapping as

$$
g_k(x_{k-1}, x_{k+1}, x_k^r, x_{k-1}, x_{k+1}, x_0^r, p) = 0
$$

(130)

\begin{align}
\theta_{k-1} &= 1 \frac{1}{h_{k-1}} \left[ \tau - \sum_{l=1}^{s_{k-1}} h_{k-1} \right], \\
\theta_{k-1} &= k - j - l_{i_{k-1}}, \\
\theta_{k-1} &= j = 0, 1, 2, \ldots, k, \\
N &= 1, 2, \ldots, N.
\end{align}
which can be expanded as
\[
g_1(x_0, x_1; x_0^*, x_1^*, p) = 0, \\
g_2(x_0, x_1; x_1^*, x_0^*, p) = 0, \\
\vdots \\
g_k(x_{k-2}, x_{k-1}, x_0, x_0^*, x_1^*, x_0^*, p) = 0, \\
(k = 1, 2, \ldots, N),
\]

(iii) If \( r = k \), the \( k \)-steps time-delay mapping is obtained
\[
g_1(x_0, x_1, \ldots, x_k; x_0^*, x_1^*, \ldots, x_k^*, p) = 0, \\
g_2(x_0, x_1; x_1^*, x_0^*, p) = 0, \\
\vdots \\
g_k(x_{k-2}, x_{k-1}, x_0; x_k^*, x_0^*, x_1^*, p) = 0.
\]

and the foregoing equations can be expanded as
\[
g_1(x_0, x_1; x_0^*, x_1^*, x_{k}^*; x_0^*, x_1^*, x_0^*, p) = 0, \\
g_2(x_0, x_1; x_1^*, x_0^*, x_{k}^*; x_0^*, x_1^*, x_0^*, p) = 0, \\
\vdots \\
g_k(x_{k-2}, x_{k-1}, x_0; x_k^*, x_0^*, x_{k}^*; x_0^*, x_1^*, x_0^*, p) = 0.
\]

From the multistep (or \( r \)-steps) mapping of \( P_k \) without \( k - j \geq 0 \), with the periodicity condition \( (x_0, x_1, \ldots, x_k) = (x_0, x_1, \ldots, x_k) \), the periodic flow can be obtained via
\[
g_1(x_0, x_1, \ldots, x_k; x_0^*, x_1^*, x_{k}^*; x_0^*, x_1^*, x_0^*, p) = 0; \\
x_{k}^* = h_{b_{kj}}(x_{r_{kj}}, x_{r_{kj}}, \theta_{r_{kj}}), \\
\theta_{r_{kj}} = \frac{1}{h_{b_{kj}}} \left[ r - \sum_{i=1}^{r_{kj}} h_{b_{kj}+i} \right],
\]

\[
\Delta x_{r_{kj}} = \frac{\partial g_{r_{kj}}}{\partial x_{r_{kj}}} \Delta x_{r_{kj}} = \frac{\partial g_{r_{kj}}}{\partial x_{r_{kj}}} \Delta x_{r_{kj}} = 0 \quad (k = 1, 2, \ldots, N; j \in \{1, 2, k\}).
\]

Let
\[
a_{r-1} = \begin{bmatrix} \frac{\partial g_{r-1}}{\partial x_{r-1}} \\ \vdots \\ \frac{\partial g_{r-1}}{\partial x_{r-1}} \end{bmatrix}, \\
a_{r} = \begin{bmatrix} \frac{\partial g_{r}}{\partial x_{r}} \\ \vdots \\ \frac{\partial g_{r}}{\partial x_{r}} \end{bmatrix}, \\
\Delta x_{r-1} = \begin{bmatrix} \Delta x_{r-1} \\ \vdots \\ \Delta x_{r-1} \end{bmatrix},
\]

and the univariate mapping of \( P_k \) is obtained
\[
y_k = a_k x_k, \\
x_{k+1} = a_{k+1} x_{k+1}, \\
\Delta y_k = \Delta x_{k+1}.
\]

Thus
\[
A_k = \begin{bmatrix} B_k & (a_k)_{k \times k} \\ I_k & 0_k \end{bmatrix}, \\
B_k = \begin{bmatrix} (a_k)_{k \times k} \\ \vdots \\ (a_k)_{k \times k} \end{bmatrix},
\]

(136)

(137)
Finally, we have
\[ \Delta y_k = A_{k} \Delta y_{k-1}. \]  
(138)

From the mapping structure, we have
\[ \Delta y_N = DP \cdot \Delta y_0 \quad \text{and} \quad DP = \left[ \frac{\partial y_N}{\partial y_0} \right] = A_N A_{N-1} \cdots A_1. \]  
(139)

Letting \( \Delta y_N = \lambda \Delta y_0 \), we have
\[ (DP - \lambda I_{n(s+1) \times n(s+1)}) \Delta y_0 = 0. \]  
(140)

The eigenvalues of \( DP \) are given by \(|DP - \lambda I_{n(s+1) \times n(s+1)}| = 0\). In addition, we have
\[ \Delta y_k = DP_{s(k-1)-1} \cdot \Delta y_0 \quad \text{and} \quad DP_{s(k-1)-1} = \left[ \frac{\partial y_k}{\partial y_0} \right] = A_k A_{k-1} \cdots A_1 \]  
(141)

\((k = 1, 2, \ldots, N)\).

Letting \( \Delta y_k = \lambda \Delta y_0 \), we have
\[ (DP_{s(k-1)-1} - \lambda I_{n(s+1) \times n(s+1)}) \Delta y_0 = 0. \]  
(142)

The eigenvalues of \( DP_{s(k-1)-1} \) are given by \(|DP_{s(k-1)-1} - \lambda I_{n(s+1) \times n(s+1)}| = 0\). Such eigenvalues tell effects of variation of \( y_0 \) on node points \( y_k \) in the corresponding neighborhood. The neighborhood of \( y_i^*, U_n(x_j^*) \), is presented in Fig. 6 through a large circle. Since the time-delay points are interpolated by the time-delay points, the variation of time-delay points can be determined by the neighborhoods of the time-delay points. In such a neighborhood, the eigenvalues can be used to measure the effects \( \Delta y_k \) of \( y_i^* \) varying with \( \Delta y_0 \) at \( y_0^* \). The eigenvalues of \( DP \) is given by \(|DP - \lambda I_{n(s+1) \times n(s+1)}| = 0\), which implies the stability and bifurcation of the period-1 flow.

(i) If \( r = 1 \), Eq. (135) becomes
\[ \sum_{j=0}^{1} \frac{\partial y_k}{\partial x_{kj}} \Delta x_{kj} = 0, \]  
(143)

Setting
\[ a_{k}(x_j) = \left[ \frac{\partial y_k}{\partial x_{kj}} \right], \]  
(144)

\[ x_k = (x_k, x_{k-1}, \ldots, x_1)^T, \]
\[ y_k = (y_k, y_{k-1}, \ldots, y_1)^T, \]
\[ \Delta y_k = (\Delta x_k, \Delta x_{k-1}, \ldots, \Delta x_1)^T, \]
\[ \Delta y_{k-1} = (\Delta x_{k-1}, \Delta x_{k-2}, \ldots, \Delta x_1)^T. \]

Thus
\[ A_k = \left[ \begin{array}{c} B_k \left( a_k(x_{k-1}) \right)_{n \times n} \\ I_k \end{array} \right], \]
\[ s = 1 + l_k \]
\[ B_k = \left[ \begin{array}{c} a_k(x_{k-1}) \right]_{n \times n} \left( \begin{array}{c} a_k(x_{k-1}) \right)_{n \times n}, \ldots, \left( \begin{array}{c} a_k(x_{k-1}) \right)_{n \times n} \end{array} \right) \]

Fig. 6. Neighborhoods of \( N \)-nodes for a period-1 flow of a time-delay system. The solid curve gives numerical results. The local shaded area is a small neighborhood of the solution at the 4th node. The red symbols are for node points of the periodic flow. The hollow symbols are for time-delay nodes of the periodic flow.
\[ I_k = \text{diag}(I_{k\times k}, I_{k\times k}, \ldots, I_{k\times k}) \]
\[ 0_k = (0_{k\times k}, 0_{k\times k}, \ldots, 0_{k\times k})^T. \]

Finally, we have
\[ \Delta y_k = A_k \Delta y_{k-1}. \]  

(146)

So we have
\[ DP = \begin{bmatrix} \frac{\partial y_N}{\partial y_0} \\ \vdots \\ \frac{\partial y_N}{\partial y_{N-1}} \end{bmatrix} = A_N A_{N-1} \cdots A_1. \]

(147)

(ii) For \( r = k \), Eq. (134) with periodicity condition \((x_0 = x_N)\) gives node points \( x_k^* \) \((k = 0, 1, 2, \ldots, N)\). The corresponding stability and bifurcation can be analyzed in the neighborhoods of \( x_k^* \) and \( x_k^{**} \) with \( x_k^* = x_k^* + \Delta x_k \) and \( x_k^{**} = x_k^{**} + \Delta x_k^* \). Equation (135) becomes
\[ \sum_{j=0}^{k} \frac{\partial g_k}{\partial x_{kj}} \Delta x_{kj} + \frac{\partial g_k}{\partial x_{kj}} \Delta x_{kj-1} = 0, \]
\[ j = 0, 1, \ldots, k; \quad k = 1, 2, \ldots, N. \]

(148)

Thus
\[ a_{kj} = - \frac{\partial g_k}{\partial x_{kj}} + \frac{\partial g_k}{\partial x_{kj}} \frac{\partial x_{kj}}{\partial x_{kj}}, \]
\[ \alpha_{kj} = - \frac{\partial g_k}{\partial x_{kj}} \frac{\partial x_{kj}}{\partial x_{kj}} \]
\[ a_{k(\tau_j-1)} = - \frac{\partial g_k}{\partial x_{kj}} \frac{\partial x_{kj}}{\partial x_{kj}} \]
\[ \text{with } \tau_j = k - j, \quad j = 0, 1, 2, \ldots, k; \]
\[ y_k = (x_k, x_{k-1}, \ldots, x_{k-\tau_j})^T, \]
\[ y_{k-1} = (x_{k-1}, x_{k-2}, \ldots, x_{k-1-\tau_j})^T, \]
\[ \Delta y_k = (\Delta x_k, \Delta x_{k-1}, \ldots, \Delta x_{k-\tau_j})^T, \]
\[ \Delta y_{k-1} = (\Delta x_{k-1}, \Delta x_{k-2}, \ldots, \Delta x_{k-1-\tau_j})^T. \]

(149)

Finally, we have
\[ \Delta y_k = A_k \Delta y_{k-1}, \]

(150)

Using \( \frac{\partial y_k}{\partial y_0} \), the eigenvalues are determined by
\[ |DP_k(k-1) - \lambda I_{(k+1)\times (k+1)}| = 0 \]

with
\[ DP_k(k-1) = \begin{bmatrix} \frac{\partial y_k}{\partial y_0} \\ \vdots \\ \frac{\partial y_k}{\partial y_{k-1}} \end{bmatrix} = A_k A_{k-1} \cdots A_1 \]

(151)

which is used to measure the properties of node points on the period-1 flow for the time-delay systems.

The multistep mappings are developed from the above-determined nodes of periodic motion. During time interval \( t \in [t_0, t_0 + T] \), the periodic flow can be determined by
\[ x(t) = x(t_0) + \int_{t_0}^{t} f(x, x^*, t, p) dt, \]
\[ t \in \{0, 1, 2, \ldots, k-1\}. \]

(152)

For such a periodic flow, all of \( N \)-nodes during the time interval of \( t \in [t_0, t_0 + T] \) are selected, and the corresponding points \( x(t_k) \) \((k = 1, 2, \ldots, N)\). Under \( \|x(t_k) - x_k\| \leq \varepsilon_k \) with \( \varepsilon_k = 0 \) in some \( t_k \),
\[ [f(x(t), x^*(t), t, p) - f(x_k, x_k^*, t, p)] \leq \delta_k. \]

(153)

If \( x_0, \ldots, x_N \) are given, \( f(x_k, x_k^*, t, p) \) \((k = 0, 1, \ldots, N)\) can be determined. An interpolation polynomial \( P(t, x_0, \ldots, x_N, x_k^*, t_0, \ldots, t_N, p) \) is used for an approximation of \( f(x, x^*, t, p) \). That is,
\[ f(x, x^*, t, p) \approx P(t, x_0, \ldots, x_N, x_k^*, t_0, \ldots, t_N, p) \]

(154)

and \( x(t_k) \approx x_k (k = 0, 1, \ldots, N) \) can be computed by
\[ x_k = x_{k-1} + \int_{t_{k-1}}^{t_k} P(t, x_0, \ldots, x_N, x_k^*, t_0, \ldots, t_N, p) dt. \]

(155)

Therefore, we have
\[ x_k = x_{k-1} + \sum_{t_{k-1}}^{t_k} P(t, x_0, \ldots, x_N, x_k^*, t_0, \ldots, t_N, p). \]

(156)

The mapping \( \phi_k \) \((k \in \{1, 2, \ldots, N\})\) is
\[ g_k(x_0, \ldots, x_k, x_k^*, x_k^{**}, \ldots, x_N, p) = 0, \]
\[ x_j^* = h_j(x_{j-1}, x_j), \quad r_j = j - l_j \]

(157)

\((j = 0, 1, 2, \ldots, N)\).
The periodic motions are determined by the mapping \( P_k \) \((k = 1, 2, \ldots, N)\) and periodicity conditions, i.e.,

\[
\begin{align*}
g_k(x_1^*, \ldots, x_{k-1}^*, x_k^*; \ldots, x_N^*, p) &= 0; \\
x_{k+j}^* &= h_{k+j}(x_{k+j-1}^*, x_{k+j-1}^*), \quad r_j = k - j - l_j \quad (j = 0, 1, 2, \ldots, N), \quad (k = 1, 2, \ldots, N)
\end{align*}
\]

From the foregoing equation, node points \( x_k^* \) and \( x_{k+1}^* \) \((k = 0, 1, 2, \ldots, N)\) can be determined. The corresponding dynamical characteristics in the neighborhood of \( x_k^* \) with \( x_k = x_k^* + \Delta x_k \) are discussed by variation of \( x_k \) in the neighborhood of \( x_k^* \) with \( x_0 = x_0^* + \Delta x_0 \). The differentiation of Eq. (158) gives

\[
\begin{align*}
\sum_{j=k-N}^{k} \frac{\partial h_k}{\partial x_j} \Delta x_j + \frac{\partial g_k}{\partial x_j} \Delta x_j^* \Delta x_k - 0 \quad (k = 1, 2, \ldots, N)
\end{align*}
\]

and

\[
\begin{align*}
ak_{k+j} &= \frac{\partial g_k}{\partial x_j} \Delta x_j + \frac{\partial g_k}{\partial x_j} \Delta x_j^* \Delta x_k - 0 \quad (k = 1, 2, \ldots, N);
nak_{k+j} &= \left[ \frac{\partial g_k}{\partial x_j} \Delta x_j + \frac{\partial g_k}{\partial x_j} \Delta x_j^* \Delta x_k \right] \quad \text{with} \quad r_j = k - j - l_j, \quad s_k = k - j, \quad j = 0, 1, 2, \ldots, N; \\
y_k &= (x_k, x_{k-1}, \ldots, x_{k-l_k})^T, \\
y_{k-1} &= (x_{k-1}, x_{k-2}, \ldots, x_{k-1-l_k})^T, \\
\Delta y_k &= (\Delta x_k, \Delta x_{k-1}, \ldots, \Delta x_{k-1-l_k})^T, \\
\Delta y_{k-1} &= (\Delta x_k, \Delta x_{k-1}, \ldots, \Delta x_{k-1-l_k})^T.
\end{align*}
\]

From the above discussion, the discrete mapping can be developed through many forward and backward nodes. The periodic flow in a nonlinear dynamical system can be determined through the following theorem.

**Theorem 6.** Consider a nonlinear dynamical system in Eq. (98). If such a dynamical system has a periodic flow \( x(t) \) with finite norm \( \|x\| \) and period \( T = 2\pi/\Omega \), there is a set of discrete time \( t_k \) \((k = 0, 1, \ldots, N)\) with \( N \rightarrow \infty \) during one period \( T \), and the corresponding solution \( x(t_k) \) and vector fields \( f(x(t_k), x'(t_k), t_k, p) \) are exact. Suppose discrete nodes \( x_k \) and \( x_{k+1} \) are on the approximate solution of the periodic flow under \( \|x(t_k) - x_k\| \leq \varepsilon_k \) and \( \|x'(t_k) - x_{k+1}'\| \leq \varepsilon_k' \) for small \( \varepsilon_k, \varepsilon_k' \geq 0 \) and

\[
\|f(x(t_k), x'(t_k), t_k, p) - f(x_k, x_{k+1}', t_k, p)\| \leq \delta_k
\]

with a small \( \delta_k \geq 0 \). During a time interval \( t \in [t_{k-1}, t_k] \), there is a mapping \( P_k : (x_{k-1}, x_k^*) \rightarrow (x_k, x_k^*) \) \((k = 1, 2, \ldots, N)\) as

\[
P_k : (x_{k-1}, x_k^*) \rightarrow (x_k, x_k^*)
\]

with

\[
g_k(x_{k-1}, \ldots, x_{k-l_k}, x_k^*) = 0,
\]

\[
x_{k+1} = h_k(x_{k-1}, x_k^*, \theta_{k-1}^k),
\]

\[
\theta_{k+1} = \frac{1}{h_k} \left[ \tau - \sum_{i=1}^{l_k} h_{k+i} \right],
\]

where \( g_k \) is an implicit vector function and \( h_k \) is an interpolation vector function. Consider a mapping structure as

\[
P = P_N \circ P_{N-1} \circ \cdots \circ P_1 \circ P_0 : (x_0, x_0^*) \rightarrow (x_N, x_N^*)
\]

with

\[
P_k : (x_{k-1}, x_k^*) \rightarrow (x_k, x_k^*) \quad (k = 1, 2, \ldots, N).
\]

For \( (x_N, x_N^*) = P(x_0, x_0^*) \), if there is a set of points \( x_k^* \) \((k = 0, 1, \ldots, N)\) computed by
are approximation of points $x$; the equation is given by

$$DP = \frac{\partial g}{\partial x_{k+r}} \Delta x_{k+r} + \frac{\partial g}{\partial x_{k+r}} \Delta x_{k+r-1} + \frac{\partial g}{\partial x_{k+r}} \Delta x_{k+r} = 0$$

with

$$\theta_{kj} = \frac{1}{n} \int_{t_{kj}}^{t_{kj+1}} \tau - \int_{t_{kj}}^{t_{kj+1}} \Delta x_{k+r} = 0$$

(164)

then the points $\mathbf{x}_k^*$ and $\mathbf{x}_k^{**}$ $(k = 0, 1, \ldots, N)$ are the approximation of points $\mathbf{x}(t_k)$ and $\mathbf{x}^*(t_k)$ of periodic solutions. In the neighborhood of $\mathbf{x}_k^*$ and $\mathbf{x}_k^{**}$, with $\mathbf{x}_k = \mathbf{x}_k^* + \Delta \mathbf{x}_k$ and $\mathbf{x}_k = \mathbf{x}_k^{**} + \Delta \mathbf{x}_k^*$, the linearized equation is given by

$$\sum_{j=1}^{m} \frac{\partial g}{\partial x_{k+r}} \Delta x_{kj} = 0$$

(165)

The resultant Jacobian matrices of the periodic flow are

$$DP_{k+1} = A_k A_{k-1} \cdots A_1$$

(166)

where

$$\Delta y_k = A_k \Delta y_{k-1}, \quad A_k = \left[ \frac{\partial y_k}{\partial y_{k-1}} \right]_{y_{k-1} = 0}$$

and

$$a_{k(k+1)} = \left[ \frac{\partial g}{\partial x_{k+r}} \right]^{-1} \frac{\partial g}{\partial x_{k+r}}$$

$$a_{k(k+1)} = \left[ \frac{\partial g}{\partial x_{k+r}} \right]^{-1} \frac{\partial g}{\partial x_{k+r}} \frac{\partial x_{k+r}}{\partial x_{k+r}}$$

$$a_{k(k+1)} = \left[ \frac{\partial g}{\partial x_{k+r}} \right]^{-1} \frac{\partial g}{\partial x_{k+r}} \frac{\partial x_{k+r}}{\partial x_{k+r}}$$

with

$$r_{kj} = k - j - I_{kj}, \quad s_{kj} = k - j;$$

$$j = -r_2, -r_2 + 1, \ldots, 0, 1, \ldots, r_1 - 1, r_1;$$

$$r_1, r_2 \in \{0, 1, 2, \ldots, N\};$$

$$1 \leq r_1 + r_2 \leq N, r_1 \geq 1;$$

$$y_k = (x_{k+r_1}, x_{k+r_2-1}, \ldots, x_{k+r_2})^T,$$

$$y_{k-1} = (x_{k+r_1-1}, x_{k+r_2-2}, \ldots, x_{k+r_2-1})^T,$$

$$\Delta y_k = (\Delta x_{k+r_1}, \Delta x_{k+r_2-1}, \ldots, \Delta x_{k+r_2})^T,$$

$$\Delta y_{k-1} = (\Delta x_{k+r_1-1}, \Delta x_{k+r_2-2}, \ldots, \Delta x_{k+r_2-1})^T$$

(167)

and

$$A_k = B_k \left( \begin{array}{c} a_{k(k+1)} \cdots a_{k+1} \\ 0 \end{array} \right)$$

$$B_k = (a_{k(k+1)} \cdots a_{k+1}) \cdots (a_{k(k+1)} \cdots a_{k+1})$$

$$I_k = \text{diag}(I_{m,n}, I_{m,n}, \ldots, I_{m,n})$$

$$0_k = (0_{m,n}, 0_{m,n}, \ldots, 0_{m,n})^T$$

(168)

The properties of discrete points $x_k$ $(k = 1, 2, \ldots, N)$ can be estimated by the eigenvalues of $DP_{k+1}$ as

$$\left| DP_{k+1}-1 \right| = 0$$

(169)

The eigenvalues of $DP$ for such periodic flow are determined by

$$\left| DP - \lambda_{k+1} I_{k+1} \right| = 0.$$
Thus, the stability and bifurcation of the periodic flow can be classified by the eigenvalues of \( DP(\lambda) \) with

\[
[p_1^m, n_1^m] : [p_2^m, n_2^m] : [n_3, n_4] : [n_5, n_6] : [n_7, n_8].
\]

(i) If the magnitudes of all eigenvalues of \( DP(\lambda) \) are less than one, the approximate periodic solution is stable.

(ii) If at least the magnitude of one eigenvalue of \( DP(\lambda) \) is greater than one, the approximate periodic solution is unstable.

(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. The proof is similar to Theorem 5. □

From the foregoing theorem, the stability and bifurcation analysis for the period-1 flow of the time-delay system can be completed from discrete mappings \( P_k \) with \( g_k(x_{k-1}, x_k, p) = 0 \) and \( x_k = h_k(x_{k-1}, x_k, \theta_j) \) \((j = k - 1, k = 1, 2, \ldots, N)\) under period \( T = 2\pi/\Omega \). If the period-doubling bifurcation occurs, the periodic flow will become a new periodic flow for the time-delay system under period \( T' = 2T \), and such a periodic flow is also called the period-2 flow of the time-delay system. Once again, owing to the period-doubling, 2\(N\) nodes of the period-2 flow of the time-delay system will be employed to describe the period-2 flow of the time-delay system and will be converted into the regular nodes rather than time-delay by interpolation. Thus, consider a mapping structure of the period-2 flow of the time-delay system with 2\(N\) mappings.

\[
P^{(0)} = P_{2N} \circ P_{2N-1} \circ \cdots \circ P_2 \circ P_1 : (x_0, x_k) \rightarrow (x_{2N}, x_{2N-k}),
\]

with \( P_k : (x_{k-1}, x_k) \rightarrow (x_k, x_{k+1}) \)

\((k = 1, 2, \ldots, 2N).\) \hspace{1cm} (173)

For \((x_{2N}, x_{2N-k}^{(0)}) = P^{(0)}(x_0, x_0^{(0)}),\) if there is a set of points \((x_k^{(0)}, x_k^{(1)})\) \((k = 0, 1, \ldots, 2N)\) computed by

\[
g_k(x_{k-1}^{(0)}, x_k^{(0)}, x_k^{(1)}, p) = 0,
\]

\[
x_k^{(0)} = h_k(x_{k-1}^{(0)}, x_k^{(0)}, \theta_s),
\]

\[
s_r = r - i; \quad r = k, k - 1
\]

\((k = 1, 2, \ldots, 2N)\) \hspace{1cm} (174)

\[
x_k^{(0)} = x_k^{(s)} \mod (e^{2\pi N} - 2N),
\]

\[
(x_k^{(2)} \cdot x_k^{(2)}) = (x_{2N}, x_{2N-k}^{(0)}).
\]

After period-doubling, the period-1 flow becomes period-2 flow. The node points increase to \(2N\) points during two periods \((2T)\). The period-2 flow can be sketched in Fig. 7. The node points are determined through the discrete mapping with mathematical relation in Eq. (174).

On the other hand,

\[
T' = 2T = \frac{2(2\pi)}{\Omega} = \frac{2\pi}{\omega} = \frac{\Omega}{2},
\]

\hspace{1cm} (175)

As discussed before, during the period of \(T'\), there is a periodic flow, which can be described by node points \(x_k (k = 0, 1, 2, \ldots, N).\) Since the period-1 flow is described by node points \(x_k (k = 0, 1, 2, \ldots, N)\) during the period \(T\), due to \(T' = 2T\), the period-2 flow can be described at least by \(N\) nodes. Thus the corresponding mapping \(P_k\) is defined as

\[
P_k : (x_{k-1}^{(2)} : x_k^{(2)}) \rightarrow (x_k^{(2)} : x_k^{(2)}), \hspace{1cm} (k = 1, 2, \ldots, 2N) \hspace{1cm} (176)
\]

Fig. 7. Period-2 flow with 2\(N\) nodes with short lines. Solid curve is for a numerical result. The filled symbols are node points on the periodic flow, and the hollow symbols are for time-delay nodes on the periodic flow.
and
\[
\begin{align*}
g_{s}(x^{(m)\parallel}\tau^{(m)}, x^{(m)}, \theta_k) &= 0, \\
x^{(m)\parallel} &= h_{s}(x^{(m)}, x^{(m)}, \theta_k), \\
s_r &= r - i, \quad r = k, \ldots, 1 \quad (k = 1, 2, \ldots, 2N) \quad (177)
\end{align*}
\]

In general, for period \( T' = mT \), there is a period-\( m \) flow described with \( N' \geq mN \). The corresponding mapping \( P_k \)
\[
P_k: (x^{(m)}_{k-1}, x^{(m)}_k) \rightarrow (x^{(m)}_{k-1}, x^{(m)}_k) \quad (k = 1, 2, mN) \quad (178)
\]

and
\[
\begin{align*}
g_{s}(x^{(m)\parallel}\tau^{(m)}, x^{(m)}, \theta_k) &= 0, \\
x^{(m)\parallel} &= h_{s}(x^{(m)}, x^{(m)}, \theta_k), \\
r &= k, k-1; \quad s_r = r - i \quad (k = 1, 2, \ldots, mN) \quad (179)
\end{align*}
\]

From the above discussion, the period-\( m \) flow in a nonlinear dynamical system can be described through \( mN \) nodes for period \( mT \). Theorem 7. Consider a time-delay nonlinear dynamical system in Eq. (98). If such a time-delay dynamical system has a period-\( m \) flow \( x^{(m)}(t) \) with finite norm \( \|x^{(m)}\| \) and period \( mT = 2\pi/\Omega \), there is a set of discrete time \( t_k \) with \( N = \infty \) during \( m \) periods \( (mT) \), and the corresponding solution \( x^{(m)}(t_k) \) and vector fields \( f(x^{(m)}(t_k), t_k, p) \) are exact. Suppose discrete nodes \( x^{(m)}_k \) and \( x^{(m)}_k \) are on the approximate solution of the periodic flow under \( \|x^{(m)}(t_k) - x^{(m)}_k\| \leq \epsilon_k \) and \( \|x^{(m)}(t_k) - x^{(m)}_k\| \leq \epsilon'_k \) with small \( \epsilon_k, \epsilon'_k \geq 0 \) and
\[
\begin{align*}
\text{[F(x^{(m)}(t_k), x^{(m)}(t_k), t_k, p) - F(x^{(m)}_k, x^{(m)}_k, t_k, p)]} &\leq \delta_k \\
\text{(180)}
\end{align*}
\]

with a small \( \delta_k \geq 0 \). During a time interval \( t \in [t_{k-1}, t_k] \), there is a mapping \( P_k: (x^{(m)}_{k-1}, x^{(m)}_k) \rightarrow (x^{(m)}_{k-1}, x^{(m)}_k) \) \( (k = 1, 2, \ldots, mN) \) as
\[
\begin{align*}
(x^{(m)}_{k}, x^{(m)}_{k}) &= P_k(x^{(m)}_{k-1}, x^{(m)}_k), \\
(x^{(m)}_{s}, x^{(m)}_{s}) &= h_{s}(x^{(m)}_{s-1}, x^{(m)}_{s}, \theta_k), \\
r_j &= j - i, \quad j = k - 1; \\
\theta_k &= \frac{1}{h_{s}} \left( \tau - \sum_{i=1}^{l_k} h_{s,i+1} \right) \quad (181)
\end{align*}
\]

where \( g_{s} \) is an implicit vector function and \( h_{s} \) is an interpolation vector function. Consider a mapping structure as
\[
P = P_{mN} \circ P_{mN-1} \circ \ldots \circ P_1: x^{(m)} \rightarrow x^{(m)}_{mN},
\]

with \( P_k: (x^{(m)}_{k-1}, x^{(m)}_k) \rightarrow (x^{(m)}_{k-1}, x^{(m)}_k) \)
\[
(k = 1, 2, \ldots, mN). \quad (182)
\]

For \( x^{(m)}_0 = P^{(m)}(x^{(m)}_0, x^{(m)}_0) \), if there is a set of points \( (x^{(m)}_k)_{k=0,1,\ldots} \) and computed by
\[
\begin{align*}
g_{s}(x^{(m)}_{k-1}, x^{(m)}_k) &= 0, \\
x^{(m)\parallel} &= h_{s}(x^{(m)}, x^{(m)}, \theta_k), \\
r &= k, \ldots, 1 \quad (k = 1, 2, \ldots, mN) \\
\text{such that } x^{(m)}_0 = x^{(m)}_0 \text{ and } x^{(m)}_s = x^{(m)}_s, \quad (s = 1, 2, \ldots, mN). \quad (183)
\end{align*}
\]

Then the points \( x^{(m)}_k \) and \( x^{(m)}_k \) \( (k = 0, 1, \ldots, mN) \) are the approximation of points \( x^{(m)}_k \) and \( x^{(m)}_k \) of periodic solutions. In the neighborhoods of \( x^{(m)}_k \) and \( x^{(m)}_k \), with \( x^{(m)}_0 = x^{(m)}_0 + \Delta x^{(m)}_0 \) and
\( x_k^{(m)} = x_k^{(m-1)} + \Delta x_k^{(m)}, \) the linearized equation is given by
\[
\sum_{j=0}^{k} \frac{\partial x_k^{(m)}}{\partial x_j^{(m)}} \Delta x_j^{(m)} + \frac{\partial x_k^{(m)}}{\partial x_j^{(m)}} \Delta x_j^{(m-1)} = 0
\]
with \( r_j = j - l_j, \quad j = k - 1, k; \)
\( (k = 1, 2, \ldots, mN). \) (184)

The resultant Jacobian matrices of the periodic flow are
\[
DP_{k(k-1)+1} = [\frac{\partial y_k^{(m)}}{\partial y_j^{(m)}}]_{j=1 \rightarrow mN-1} \omega_k
\]
\[= A_k A_{k-1} \cdots A_1 \quad (k = 1, 2, \ldots, mN), \quad \text{and} \]
\[
DP = DP_{mN(mN-1)+1}
\]
\[= [\frac{\partial y_k^{(m)}}{\partial y_j^{(m)}}]_{j=1 \rightarrow mN-1} \omega_k
\]
\[= A_m A_{m-1} \cdots A_1 \quad (k = 1, 2, \ldots, mN). \] (185)

where
\[
\Delta y_k^{(m)} = A_k^{(m)} \Delta y_{k-1}^{(m)},
\]
\[
A_k^{(m)} = [\frac{\partial y_k^{(m)}}{\partial y_j^{(m)}}]_{j=1 \rightarrow mN-1} \omega_k
\]
and
\[
a_k^{(m)} = [\frac{\partial y_k^{(m)}}{\partial x_j^{(m)}}]_{j=1 \rightarrow mN-1} \omega_k
\]
\[= \frac{\partial x_k^{(m)}}{\partial x_j^{(m)}} \Delta x_j^{(m)} + \frac{\partial x_k^{(m)}}{\partial x_j^{(m)}} \Delta x_j^{(m-1)} = 0
\]
with \( r_j = j - l_j, \quad j = k - 1, k; \)
\[
y_k^{(m)} = (x_k^{(m)}, x_{k-1}^{(m)}, \ldots, x_{k-1}^{(m)})^T,
\]
\[
y_{k-1}^{(m)} = (x_{k-1}^{(m)}, x_{k-2}^{(m)}, \ldots, x_{k-2}^{(m)})^T,
\]
\[
\Delta y_k^{(m)} = (\Delta x_k^{(m)}, \Delta x_{k-1}^{(m)}, \ldots, \Delta x_{k-1}^{(m)})^T,
\]
\[
\Delta y_{k-1}^{(m)} = (\Delta x_{k-1}^{(m)}, \Delta x_{k-2}^{(m)}, \ldots, \Delta x_{k-2}^{(m)})^T,
\]
\[
A_k^{(m)} = \begin{bmatrix} B_k^{(m)} & (a_k^{(m-1)} I_{mN})^{(1k(N-1))} \\ I_k^{(m)} & 0_k^{(m)} \end{bmatrix},
\]
\[
B_k^{(m)} = (a_k^{(m-1)} I_{mN}, 0_{mN}, \ldots, (a_k^{(m-1)} I_{mN})^{(1k(N-1))}),
\]
\[
A_k^{(m)} = (a_k^{(m-1)} I_{mN}, 0_{mN}, \ldots, (a_k^{(m-1)} I_{mN})^{(1k(N-1))})^T
\] (187)

The properties of discrete points \( x_k^{(m)} \) \( (k = 1, 2, \ldots, mN) \) can be estimated by the eigenvalues of \( DP_{k(k-1)+1} \) as
\[
|DP_{k(k-1)+1} - \lambda I_{mN(mN-1)}| = 0 \quad (k = 1, 2, \ldots, mN). \] (188)

The eigenvalues of \( DP \) for such periodic flow are determined by
\[
|DP - \lambda I_{mN(mN-1)}| = 0 \quad (189)
\]
and the stability and bifurcation of the periodic flow can be classified by the eigenvalues of \( DP(y_k) \) with
\[
[n_1^T, n_1^T] : [n_2^T, n_2^T] : [n_3^T, n_3^T] : [n_4^T, n_4^T] \]
\[n_5 : n_6 : [n_7^T, n_7^T]]. \] (190)

(i) If the magnitudes of all eigenvalues of \( DP \) are less than one (i.e. \( |\lambda| < 1 \), \( i = 1, 2, \ldots, n \)), the approximate periodic solution is stable.
(ii) If at least the magnitude of one eigenvalue of \( DP \) is greater than one (i.e. \( |\lambda| > 1 \), \( i \in \{1, 2, \ldots, n\} \)), the approximate periodic solution is unstable.
(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof: The discrete mapping for the period-\( m \) flow can be developed by using many forward and backward nodes. The period-\( m \) flow...
in a time-delay system can be determined through the following theorem.

**Theorem 8.** Consider a time-delay nonlinear dynamical system in Eq. (98). If such a dynamical system has a period-$m$ flow $x^{(m)}(t)$ with finite norm $\|x^{(m)}\|$ and $m$-periods $mT = 2\pi/\Omega$, there is a set of discrete time $t_k$ ($k = 0, 1, \ldots, mN$) with $N \to \infty$ during $m$-period $T$, and the corresponding solution $x^{(m)}(t)$ and $x^{(m)}(t_k)$ with vector fields $f(x^{(m)}(t_k), x^{(m)}(t_k), t_k, p)$ are exact. Suppose discrete nodes $x_k^{(m)}$ and $x_k^{(m)}$ are on the approximate solution of the periodic flow under $|x_k^{(m)}(t_k) - x_k^{(m)}(t_k)| \leq \varepsilon_k$ and $\|x_k^{(m)}(t_k) - x_k^{(m)}(t_k)\| \leq \varepsilon_k'$ with small $\varepsilon_k, \varepsilon_k' \geq 0$ and \[ \|f(x_k^{(m)}(t_k), x_k^{(m)}(t_k), t_k, p) - f(x_k^{(m)}(t_k), x_k^{(m)}(t_k), t_k, p)\| \leq \delta \] (191) with small $\delta \geq 0$. During a time interval $t \in [t_k-1, t_k]$, there is a mapping $P_k : (x_k^{(m)}, x_{k-1}^{(m)}) \rightarrow (x_k^{(m)}, x_k^{(m)})$ ($k = 1, 2, \ldots, mN$) as \[ (x_k^{(m)}, x_k^{(m)}) = P_k(x_{k-1}^{(m)}, x_{k-1}^{(m)}) \] with $B_k(x_k^{(m)}, \ldots, x_{k-1}^{(m)}, \ldots, x_{k-j}^{(m)}, \ldots, x_{k-j}^{(m)}), x_k^{(m)}, \ldots, x_{k-j}^{(m)}; x_{k-j}^{(m)} \in [0, 1, \ldots, mN], \] $1 \leq k + 1 \leq mN, r_j \geq 1; (k = 1, 2, \ldots, mN)$ (192) where $g_k$ is an implicit vector function and $h_k$ is an interpolation vector function. Consider a mapping structure as \[ P = P_{mN} \circ P_{mN-1} \circ \cdots \circ P_2 \circ P_1 : \] \[ (x_0^{(m)}, x_0^{(m)}) \rightarrow (x_{mN}^{(m)}, x_{mN}^{(m)}), \quad \text{with} \] \[ P_k : (x_k^{(m)}, x_{k-1}^{(m)}) \rightarrow (x_k^{(m)}, x_k^{(m)}) \] \[ (k = 1, 2, \ldots, mN). \] For $(x_0^{(m)}, x_0^{(m)}) = P(x_0^{(m)}, x_0^{(m)})$, if there is a set of points $x_k^{(m)} (k = 0, 1, \ldots, N)$ computed by $g_k(x_k^{(m)}, \ldots, x_{k-1}^{(m)}, x_k^{(m)}, \ldots, x_{k-1}^{(m)}, x_{k-1}^{(m)}, \ldots, x_{k-1}^{(m)}, x_{k-1}^{(m)}, x_{k-1}^{(m)}, \ldots, x_{k-1}^{(m)}; x_{k-1}^{(m)} = 0, \] $x_{k-1}^{(m)} = h_k(x_{k-1}^{(m)}, x_{k-1}^{(m)}, \theta), \] $\theta = \frac{1}{h_k} \left( \tau - \sum_{i=1}^{r_k} r_{k,i} \right)$, $r_{k,i} = k - j - s_{k,i}$, $s_{k,i} = k - j \quad \left( j = -r_2, -r_2 + 1, \ldots, 1, 0, 1, \ldots, r_2 - 1, r_1 \right), \quad r_1, r_2 \in [0, 1, \ldots, mN], \quad 1 \leq k + 1 \leq mN, r_j \geq 1; (k = 1, 2, \ldots, mN)$ (193) Then the points $x_k^{(m)}$ and $x_k^{(m)}$ ($k = 0, 1, \ldots, mN$) are the approximations of points in $x_k^{(m)}$ and $x_k^{(m)}$ of periodic solutions. In the neighborhood of $x_k^{(m)}$ and $x_k^{(m)}$, with $x_k^{(m)} = x_k^{(m)} + \Delta x_k^{(m)}$ and $x_k^{(m)} = x_k^{(m)} + \Delta x_k^{(m)}$, the linearized equation is given by \[ \sum_{j=1}^{r_k} \frac{\partial g_k}{\partial x_{k,j}} \Delta x_{k,j} + \frac{\partial g_k}{\partial x_{k,j}} \Delta x_{k,j-1} \] \[ + \frac{\partial g_k}{\partial x_{k,j}} \Delta x_{k,j} \Delta x_{k,j-1} = 0 \] (195) with $\frac{\partial g_k}{\partial x_{k,j}} = 0$ and $\frac{\partial g_k}{\partial x_{k,j}} = 0 \quad (a \neq s_{k,j}). \] $j = -r_2, -r_2 + 1, \ldots, r_2 - 1, r_1; \quad (k = 1, 2, \ldots, mN). \] The resultant Jacobian matrices of the periodic flow are \[ D \varphi_{(k-1)} = \frac{\partial \varphi_{(k-1)}}{\partial x_0} \] \[ \varphi_{(k)} = A_{k-1}^{(m)}A_{k-2}^{(m)} \cdots A_0^{(m)} \] \[ (k = 1, 2, \ldots, mN), \quad \text{and} \] \[ D \varphi = D \varphi_{mN(mN-1)} = \frac{\partial \varphi_{mN(mN-1)}}{\partial x_0} \] \[ \varphi_{mN(mN-1)} = A_{mN}^{(m)}A_{mN-1}^{(m)} \cdots A_0^{(m)} \] \[ (k = 1, 2, \ldots, mN). \]
A. C. J. Luo

where

$$\Delta y_k^{(m)} = \mathbf{A}_k^{(m)} \Delta y_{k-1}^{(m)},$$

$$A_k^{(m)} = \left[ \frac{\partial g_k}{\partial y_k^{(m)}} \right]_{y_k^{(m)} = y_k^{(m-1)}}$$

and

$$\Delta x_{k+1}^{(m)} = - \left[ \frac{\partial g_k}{\partial x_k^{(m)}} \right]_{x_k^{(m)} = x_k^{(m-1)}}$$

$$a_{k+1}^{(m)} = \left[ \frac{\partial g_k}{\partial x_k^{(m)}} \right]_{x_k^{(m)} = x_k^{(m-1)}}$$

$$x_{k+1}^{(m)} = \left[ \frac{\partial g_k}{\partial x_k^{(m)}} \right]_{x_k^{(m)} = x_k^{(m-1)}}$$

The properties of discrete points $x_k$ ($k = 1, 2, \ldots, mN$) can be estimated by the eigenvalues of $D\mathcal{P}_{k-1}^{(m)}$ as

$$|D\mathcal{P}_{k-1}^{(m)} - \mathcal{M}_{mN\times mN}| = 0$$

$$(k = 1, 2, \ldots, mN).$$

The eigenvalues of $D\mathcal{P}$ for such periodic flow are determined by

$$|D\mathcal{P} - \mathcal{M}_{mN\times mN}| = 0.$$ (201)

Thus, the stability and bifurcation of the periodic flow can be classified by the eigenvalues of $D\mathcal{P}(\mathbf{x}_1)$ with

$$([n_1^0, n_2^0], [n_2^0, n_3^0], [n_3^0, \kappa], [n_k])$$

(i) If the magnitudes of all eigenvalues of $D\mathcal{P}$ are less than one (i.e. $|\lambda| < 1$, $i = 1, 2, \ldots, n$), the approximate periodic solution is stable.

(ii) If at least the magnitude of one eigenvalue of $D\mathcal{P}$ is greater than one (i.e. $|\lambda| > 1$, $i = 1, 2, \ldots, n$), the approximate periodic solution is unstable.

(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. The proof is similar to Theorem 5.

3.2. Time-delay nodes based on integration

The time-delay nonlinear system has solution points $x_k \approx x(t_k)$ and $x_k \approx x(t_k - \tau)$ for $k = 0, 1, 2, \ldots$, as shown in Fig. 8. The small circular
symbols are the regular solution points, and the green, large circular symbols are the time-relayed points. Between two points \( x_{k-1} \) and \( x_{k} \), there is a time-delay related point \( x_{k}^\tau \approx x((k-1)x_{k-1} - \tau) \) where \( (k-1)x_{k-1} - \tau) \in [t_{k-1}, t_k] \) with an integer \( s_k \). From Eq. (96), we have
\[
\begin{align*}
\dot{x}(t_k) &= \dot{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} f(x, x^\tau, t, p) dt, \\
\dot{x}((k-1)x_{k-1} - \tau) &= \dot{x}(t_{k-1}) + \int_{t_{k-1}}^{(k-1)x_{k-1} - \tau} f(x, x^\tau, t, p) dt.
\end{align*}
\]  

Consider an interpolation function between \( f(x_{k-1}, x_{k-1}^\tau, t_{k-1}, p) \) and \( f(x_{k-1}, x_{k}^\tau, t_k, p) \) to approximate \( f(x, x^\tau, t, p) \). Equation (203) becomes
\[
\begin{align*}
x_k &= x_{k-1} + \int_{t_{k-1}}^{t_k} f(x, x^\tau, t, p) dt, \\
x_{k-1}^\tau &= x_{k-1} + \int_{t_{k-1}}^{x_{k-1}^\tau} f(x, x^\tau, t, p) dt.
\end{align*}
\]  

From the above discrete scheme for nondelay nodes and delay nodes, periodic flows in the time-delay dynamical systems can be discussed. If a time-delay nonlinear system has a periodic flow with a period of \( T = 2\pi/\Omega \), then such a periodic flow can be expressed by discrete points through discrete mappings of the time-delay continuous dynamical systems in Eq. (98). The method is stated through the following theorem.

**Theorem 9.** Consider a time-delay nonlinear dynamical system as
\[
\dot{x} = f(x, x^\tau, t, p) \in \mathbb{R}^n
\]  

where \( f(x, x^\tau, t, p) \) is a C1-continuous nonlinear vector function \( \Omega \geq 1 \) and \( x^\tau = x(t - \tau) \). If such a time-delay dynamical system has a periodic flow \( x(t) \) with finite norm \( \|x(t)\| \) and period \( T = 2\pi/\Omega \), there is a set of discrete time \( t_k (k = 0, 1, \ldots, N) \) with \( N \rightarrow \infty \) during one period \( T \), and the corresponding solution \( x(t_k) \) and \( x^\tau(t_k) = x(t_k - \tau) \) with vector fields \( f(x(t_k), x^\tau(t_k), t_k, p) \) are exact. Suppose discrete nodes \( x_k \) and \( x_k^\tau \) are on the approximate solution of the periodic flow under \( \|x(t_k) - x_k\| \leq \varepsilon_k \) and \( \|x^\tau(t_k) - x_k^\tau\| \leq \varepsilon_k^\tau \) for small \( \varepsilon_k, \varepsilon_k^\tau \geq 0 \) and
\[
\|f(x(t_k), x^\tau(t_k), t_k, p) - f(x_k, x_k^\tau(t_k), t_k, p)\| \leq \delta_k
\]  

then the points \( x_k^\tau \) and \( x_k^\tau \) \( (k = 0, 1, \ldots, N) \) are approximations of points \( x(t_k) \) and \( x^\tau(t_k) \) of the periodic solution. In the neighborhoods of \( x_k \) and \( x_k^\tau \), with \( x_k = x_k + \Delta x_k \) and \( x_k^\tau = x_k^\tau + \Delta x_k^\tau \), the linearized equation is given by
\[
\begin{align*}
\frac{\partial g_k}{\partial x_k} &\frac{\partial x_k}{\partial t_k} + \frac{\partial g_k}{\partial x_k^\tau} \frac{\partial x_k^\tau}{\partial t_k} + \frac{\partial g_k}{\partial p} \frac{\partial p}{\partial t_k} = 0, \\
\frac{\partial g_k}{\partial x_k} &\frac{\partial x_k}{\partial t_k} + \frac{\partial g_k}{\partial x_k^\tau} \frac{\partial x_k^\tau}{\partial t_k} + \frac{\partial g_k}{\partial p} \frac{\partial p}{\partial t_k} = 0.
\end{align*}
\]  

**Discrete Implict Mappings of Continuous Nonlinear Systems**
The boundaries between stable and unstable

\[ \lambda < 1, \quad i = 1, 2, \ldots, n, \]

(i) If the magnitudes of all eigenvalues of \( DP \) are less than one (i.e. \( |\lambda| < 1, \quad i = 1, 2, \ldots, n \)), the approximate periodic solution is stable.

(ii) If at least the magnitude of one eigenvalue of \( DP \) is greater than one (i.e. \( |\lambda| > 1, \quad i \in \{1, 2, \ldots, n\} \)), the approximate periodic solution is unstable.

(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. If \( f(x, x^*, t, p) \) is a \( C^r \)-continuous nonlinear function vector (\( r \geq 1 \)), then the velocity \( \mathbf{x} \) should...
be $C^r$-continuous ($r \geq 1$). If such a dynamical system has a periodic flow $x(t)$ and $x'(t)$ with finite norms $\|x\|$ and $\|x'\|$ with period $T = 2\pi/\Omega$, there is a set of discrete time $t_k$ ($k = 0, 1, \ldots, N$) with $(N \to \infty)$ during one period $T$. The corresponding solution $x(t_k)$ and $x'(t_k) = x(t_k - \tau)$ with vector fields $f(x(t_k), x'(t_k), t_k, p)$ are exact. For a time interval $t \in [t_{k-1}, t_k]$, we have

$$x(t) = x(t_{k-1}) + \int_{t_{k-1}}^{t} f(x, x', t, p) dt. \quad (217)$$

For the time interval $[t_{k-1}, t_k]$ divided into $s$-nodes $t_{k(i)} = t_{k-1} + \frac{i\Omega}{s}$ with $c_i \in [0, 1]$ and $f(x(t_{k(i)}))$, $x'(t_{k(i)}), t_{k(i)}(p)$ ($i = 1, \ldots, s$) with $x'(t_{k(i)}) = x(t_{k(i)} - \tau)$, there is an approximate function $P(t, C)$ with unknown $C = (C_1, \ldots, C_s)^T$ and $C_i$ ($i = 1, \ldots, s$), and the following condition is satisfied, i.e.

$$f(x(t_{k(i)}), x'(t_{k(i)}), t_{k(i)}(p)) = P(t_{k(i)}, t_{k(i)} - \tau, C), \quad i = 1, 2, \ldots, s$$

$$\frac{\partial P}{\partial C} \neq 0. \quad (218)$$

From the foregoing equation, the unknowns $C(t_k) = (C_1, \ldots, C_s)^T$ with $t_k = (t_{k(1)}, \ldots, t_{k(s)})^T = (t_1, t_1, \ldots, t_k)^T + h_k(c_1, \ldots, c_s)^T$ are determined. For a small $\delta > 0$, if there is a relation

$$|P(t, t - \tau, C(t_k)) - f(x, x', t, p)| \leq \delta \quad (219)$$

for $t \in [t_{k-1}, t_k]$, Eq. (217) can be approximated by

$$x(t) = x(t_{k-1}) + \int_{t_{k-1}}^{t} P(t, t - \tau, C(t_k)) dt + O(\delta); \quad (220)$$

and

$$\mathbf{x}(t_k) = \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} P(t, t - \tau, C(t_k)) dt, \quad (220)$$

and

$$\mathbf{x}(t_{k-1}) \approx \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} P(t, t - \tau, C(t_k)) dt.$$

Let $\mathbf{x}(t_k) = x_k, \mathbf{x}'(t_{k-1}) = x_{k-1}, \mathbf{x}'(t_{k-1}) = x_{k-1}'$ and $\mathbf{x}'(t_k) = x_k'$. For any small $(\epsilon_k, \epsilon_{k-1}^*) > 0$ and $(\epsilon_k, \epsilon_{k-1}^*) > 0$, under $\|x(t_{k-1}) - x_{k-1}\| \leq \epsilon_{k-1}$, $\|x'(t_{k-1}) - x_{k-1}'\| \leq \epsilon_{k-1}^*$, $\|x(t_k) - x_k\| \leq \epsilon_k$, and $\|x'(t_k) - x_k'\| \leq \epsilon_k^*$, Eq. (221) gives

$$x_k = x_{k-1} + \mathbf{g}_k(x_{k-1}, x_k; x_{k-1}', x_k'), \quad (222)$$

Thus, a discrete mapping relation is obtained by

$$g_k(x_{k-1}, x_k; x_{k-1}', x_k')$$

$$= 0. \quad (223)$$

From the discrete mapping, two points $x(t_{k-1})$ and $x(t_k)$ for the time interval $t \in [t_{k-1}, t_k]$ ($k = 1, 2, \ldots, N$) can be approximated by $x_k$ and $x_{k-1}$, respectively. If $f(x, x', t, p)$ is a $C^r$-nonlinear vector function, we have $\|f(x) \leq L$ and $\|f(x') \leq L'$ ($L$ and $L'$ constant). Thus

$$\|f(x(t_{k-1})), x'(t_{k-1}), t_{k-1}, p)\|$$

$$\leq L \|x(t_{k-1}) - x_{k-1}\| + L' \|x'(t_{k-1}) - x_{k-1}'\|$$

$$\leq L\epsilon_k + L'\epsilon_{k-1}^* = \delta_k; \quad (224)$$

and

$$\|f(x(t_k)), x'(t_k), t_k, p) - f(x_k, x_k', t_k, p)\|$$

$$\leq L \|x(t_k) - x_k\| + L' \|x'(t_k) - x_k'\|$$

$$\leq L\epsilon_k + L'\epsilon_k^* = \delta_k. \quad (225)$$
Thus, derivatives of $g$ with respect to the stability of such a periodic flow, consider $x$ exists with $x^* \pm 0$

Solving Eqs. (227) and (228) gives $(228)$.

Once the mapping $P_k : (x_{k-1}, x_k^*) \rightarrow (x_k, x_k^*)$ exists with

$$g_k(x_{k-1}, x_k^*, x_{k-1}^*, x_k^*, p) = 0,$$
$$h_k(x_{k-1}, x_k^*, x_{k-1}^*, x_k^*, p) = 0,$$

with $r_k = \text{mod}(k-1 + x_{k-1}, N)$

then the periodic flow can be formed by $P : (x_0, x_0^*) \rightarrow (x_N, x_N^*)$ with $P = P_N \circ \ldots \circ P_2 \circ P_1$, i.e.

$$g_k(x_{k-1}, x_k^*, x_{k-1}^*, x_k^*, p) = 0,$$
$$h_k(x_{k-1}, x_k^*, x_{k-1}^*, x_k^*, p) = 0,$$

where

$$x_0 = x_N, \quad x_0^* = x_N^*.$$  \hfill (228)

Solving Eqs. (227) and (228) gives $x_0^*$ and $x_k^*$ $(k = 0, 1, 2, \ldots, N)$ to get the period-1 flow. For the stability of such a periodic flow, consider $x_k = x_k^* + \Delta x_k$ and $x_k^* = x_k^* + \Delta x_k^*$ $(k = 1, 2, \ldots, N)$ for $x_k \in U(x_k^*)$ and $x_k^* \in U(x_k^*)$. Equation (227) becomes

$$g_k(x_{k-1}^* + \Delta x_{k-1}, x_k^*+ \Delta x_k, p) = 0,$$
$$h_k(x_{k-1}^* + \Delta x_{k-1}, x_k^*+ \Delta x_k, p) = 0,$$

Thus, derivatives of $g_k(x_{k-1}, x_k^*, x_{k-1}^*, x_k^*, p) = 0$

with respect to $x_0$ gives

$$y = A^{-1}b \quad \text{and} \quad y^* = A^{-1}b^*$$ \hfill (230)

where

$$A = (A_{kl})_{2N \times 2N}, \quad y = (y_1, y_2, \ldots, y_N)^T,$$
$$y^* = (y_1^*, y_2^*, \ldots, y_N^*)^T,$$
$$b = (b_1, b_2, \ldots, b_N)^T,$$
$$b^* = (b_1^*, b_2^*, \ldots, b_N^*)^T;$$

and

$$A_{kl} = \sum_{i=1}^{2N} A_{ki} \delta_i^l,$$ for $l_k = k-1, k, r_k; \quad l_k > 0$$

Setting $(\Delta x_k, \Delta x_k^*)^T = \lambda(\Delta x_0, \Delta x_0^*)^T$ and $(\Delta x_N, \Delta x_N^*)^T = \lambda(\Delta x_N, \Delta x_N^*)^T$, the foregoing equation

\begin{equation}
A_k = \begin{bmatrix}
a_{kj} & b_{kj}^T \\
b_{kj} & b_{kj}^T
\end{bmatrix}, \quad \begin{bmatrix}
\frac{\partial g_k}{\partial x_k} & \frac{\partial g_k}{\partial x_k}
\end{bmatrix} = \begin{bmatrix}
0_n & 0_n
\end{bmatrix}
\end{equation}

\begin{equation}
A_{kj} = \begin{bmatrix}
a_{kj} & b_{kj}^T \\
b_{kj} & b_{kj}^T
\end{bmatrix}, \quad \begin{bmatrix}
\frac{\partial g_k}{\partial x_k} & \frac{\partial g_k}{\partial x_k}
\end{bmatrix} = \begin{bmatrix}
0_n & 0_n
\end{bmatrix}
\end{equation}

\begin{equation}
\Delta x_k = DP_{k(k-1) - 1} \begin{bmatrix}
\Delta x_k \\
\Delta x_k^*
\end{bmatrix} \quad (k = 1, 2, \ldots, N)
\end{equation}

\begin{equation}
\Delta x_k = DP_{N(N-1) - 1} \begin{bmatrix}
\Delta x_k \\
\Delta x_k^*
\end{bmatrix}
\end{equation}

\begin{equation}
DP_{k(k-1) - 1} = \begin{bmatrix}
\frac{\partial g_k}{\partial x_k} & \frac{\partial g_k}{\partial x_k}
\end{bmatrix}
\end{equation}

\begin{equation}
DP = \begin{bmatrix}
\frac{\partial g_k}{\partial x_N} & \frac{\partial g_k}{\partial x_N}
\end{bmatrix}
\end{equation}
delay nodes. The periodicity requires
and the large, hollow circular symbols are for time-
small, filled circular symbols are for discrete nodes,

From the stability and bifurcation theory of dynam-

For any nontrivial solution \((x, \Delta x) \neq 0\), we have
\[
\frac{\partial P}{\partial x_{\Delta x}} = 0,
\]
\[
\frac{\partial P}{\partial x_{\Delta x}} = 0.
\]

Thus, the eigenvalues of \(DP_{k(k-1)}\) give the changes of \((\Delta x, \Delta x')\) with \((\Delta x_0, \Delta x_0')\). In ad-

tion, the eigenvalues of \(DP\) are computed for the
periodic solution due to \(x_0 = x_0'\) and \(x_0 = x_0'\). From
the stability and bifurcation theory of dynam-

For a time-delay system, a periodic solution
\[
\frac{\partial P}{\partial x_{\Delta x}} \neq 0
\]
For \(1 \leq k \leq N\), as shown in Fig. 9. The
time-delay nodes are obtained by integration. The
small, filled circular symbols are for discrete nodes,
and the large, hollow circular symbols are for time-
delay nodes. The periodicity requires \(x_N = x_0\) and
\(x_N = x_N'\).

Fig. 9 Period-1 flow with \(N\) nodes for a time-delay sys-

\[
\frac{\partial P}{\partial x_{\Delta x}} \neq 0
\]

From the foregoing theorem, a set of nonlinear,
time-delay, discrete mappings \(P_k\) with \(g_k(x_{k-1}, x_k,
x_{k-1}', x_k') = 0\) and \(h_k(x_{k-1}, x_k, x_{k-1}', x_k') = 0\) for \(k = 1, 2, \ldots, N\) are developed for a peri-
odic flow. In addition to a one-step time-delay map-
ing of \(P_k\), one can develop a multistep (or \(r\)-steps)
time-delay mapping of \(P_k\) with
\[
g_k(x_{k-1}, \ldots, x_{k-1}, x_k, x_{k-1}', \ldots, x_{k-1}', x_k') = 0,
\]
\[
h_k(x_{k-1}, x_k, x_{k-1}', x_k', \ldots, x_{k-1}', x_k') = 0,
\]
\[
\begin{cases}
g_k(x_{k-1}, \ldots, x_{k-1}, x_k, x_{k-1}', \ldots, x_{k-1}', x_k') = 0, \\
h_k(x_{k-1}, x_k, x_{k-1}', x_k', \ldots, x_{k-1}', x_k') = 0,
\end{cases}
\]
\[
(236)
\]
\[
(237)
\]
\[
(238)
\]
\[
(239)
\]
\[
(240)
\]
\[
(241)
\]
and the foregoing equations can be expanded as
\[
g_k(x_0, x_1; x_0', x_1') = 0.
\]
suppose node points \( x_k \), \( k = 0, 1, 2, \ldots, N \), of periodic flows are obtained, the corresponding stability and bifurcation can be analyzed in the neighborhood of \( x_k \) with \( x_0 = x_N \) and \( x'_0 = x'_N \). Let

\[
a_{k,j} = \frac{\partial g_k}{\partial x_j} \quad a'_{k,j} = \frac{\partial g_k}{\partial x'_j}
\]

\[
b_{k,j} = \frac{\partial h_k}{\partial x_j} \quad b'_{k,j} = \frac{\partial h_k}{\partial x'_j}
\]

\[
A = \frac{\partial^2 A}{\partial x_j \partial x'_j} \quad y = (y_1, y_2, \ldots, y_N)^T
\]

\[
y' = (y'_1, y'_2, \ldots, y'_N)^T
\]

\[
b = (b_1, b_2, \ldots, b_N)^T
\]

\[
b' = (b'_1, b'_2, \ldots, b'_N)^T
\]

\[
A_{k,j} = \sum_{l_k} A_{l_k} \delta_{k,l_k}
\]

\[
y = A^{-1}b
\]

\[
y' = A^{-1}b'
\]

From the mapping structure, we have

\[
\begin{bmatrix}
\Delta x_N \\
\Delta x'_N
\end{bmatrix} = DP \begin{bmatrix}
\Delta x_0 \\
\Delta x'_0
\end{bmatrix} = DP_{N(1)} \begin{bmatrix}
\Delta x_0 \\
\Delta x'_0
\end{bmatrix},
\]

\[
\text{with } DP = \begin{bmatrix}
\frac{\partial x_N}{\partial x_0} & \frac{\partial x'_N}{\partial x'_0} \\
\frac{\partial x'_N}{\partial x_0} & \frac{\partial x'_N}{\partial x'_0}
\end{bmatrix} \begin{bmatrix}
x_0 & x'_0
\end{bmatrix}
\]

\[(k = 1, 2, \ldots, N, j = k - r, \ldots, k - 1, k); \quad (k = 1, 2, \ldots, N).
\]
Letting \((\Delta x_N, \Delta x_k)^T = \lambda (\Delta x_0, \Delta x_k)^T\), we have
\[
(DP - \lambda I_{n+1}) \frac{\Delta x_0}{\Delta x_k} = 0.
\] (249)
The eigenvalues of \(DP\) are given by \(|DP - \lambda I_{n+1}| = 0\). In addition, we have
\[
\begin{bmatrix}
\Delta x_k \\
\frac{\Delta x_n}{\Delta x_k}
\end{bmatrix}
= DP_{k(k-1)-1} \begin{bmatrix}
\Delta x_0 \\
\frac{\Delta x_n}{\Delta x_k}
\end{bmatrix}
\quad (k = 1, 2, \ldots, N),
\]
with \(DP_{k(k-1)-1} = \left[\begin{array}{cc}
\frac{\partial x_n}{\partial x_0} & \frac{\partial x_n}{\partial x_0} \\
\frac{\partial x_n}{\partial x_k} & \frac{\partial x_n}{\partial x_k}
\end{array}\right]_{(k, k', \ldots, k'' \in N_N)}\).

Letting \((\Delta x_n, \Delta x_k)^T = \lambda (\Delta x_0, \Delta x_k)^T\), we have
\[
(DP_{k(k-1)-1} - \lambda I_{n+1}) \Delta x_0 = 0.
\] (250)
The eigenvalues of \(DP_{k(k-1)-1}\) are given by \(|DP_{k(k-1)-1} - \lambda I_{n+1}| = 0\). Such eigenvalues still indicate the effects of variation of \((x_0, x_k^n)\) on nodes points \((x_n, x_k^n)\) in the corresponding neighborhood. The neighborhoods of \(x_k^n\) and \(x_k^n\), \(U(x_k^n)\) and \(U(x_k^n)\), are presented in Fig. 10 through a large circle. In such neighborhoods, the eigenvalues can be used to measure the effects \(\Delta x_0\) and \(\Delta x_k^n\) at \(x_k^n\) and \(x_k^n\).

The eigenvalues of \(DP\) are given by \(|DP - \lambda I_{n+1}| = 0\), which imply the stability and bifurcation of the period-1 flow.

(i) If \(r = 1\), Eq. (244) becomes
\[
\begin{aligned}
&\sum_{j=0}^{1} \frac{\partial g_n}{\partial x_0} \frac{\partial x_k+j}{\partial x_0} + \frac{\partial f_n}{\partial x_0} \frac{\partial x_0^j}{\partial x_0} = 0_{n+1}, \\
&\sum_{j=0}^{1} \frac{\partial g_n}{\partial x_0} \frac{\partial x_k+j}{\partial x_0} + \frac{\partial f_n}{\partial x_0} \frac{\partial x_0^j}{\partial x_0} = 0_{n+1}, \\
&\sum_{j=0}^{1} \frac{\partial g_n}{\partial x_0} \frac{\partial x_k+j}{\partial x_0} + \frac{\partial f_n}{\partial x_0} \frac{\partial x_0^j}{\partial x_0} = 0_{n+1}, \\
&\sum_{j=0}^{1} \frac{\partial g_n}{\partial x_0} \frac{\partial x_k+j}{\partial x_0} + \frac{\partial f_n}{\partial x_0} \frac{\partial x_0^j}{\partial x_0} = 0_{n+1}, \\
&\sum_{j=0}^{1} \frac{\partial g_n}{\partial x_0} \frac{\partial x_k+j}{\partial x_0} + \frac{\partial f_n}{\partial x_0} \frac{\partial x_0^j}{\partial x_0} = 0_{n+1}, \\
&\sum_{j=0}^{1} \frac{\partial g_n}{\partial x_0} \frac{\partial x_k+j}{\partial x_0} + \frac{\partial f_n}{\partial x_0} \frac{\partial x_0^j}{\partial x_0} = 0_{n+1},
\end{aligned}
\]
with \(r_k = \text{mod}(k + 1 - s_k, N), (k = 1, 2, \ldots, N)\).

Let
\[
a_{kj} = \left[\begin{array}{c}
\frac{\partial g_n}{\partial x_0} \\
\frac{\partial f_n}{\partial x_0}
\end{array}\right]_{(k, k', \ldots, k'' \in N_N)},
\]
\[
b_{kj} = \left[\begin{array}{c}
\frac{\partial h_n}{\partial x_0} \\
\frac{\partial h_n}{\partial x_k} \\
\frac{\partial h_n}{\partial x_{k+1}}
\end{array}\right]_{(k, k', \ldots, k'' \in N_N)},
\]
\[
m_{kj} = \left[\begin{array}{c}
\frac{\partial h_n}{\partial x_0} \\
\frac{\partial h_n}{\partial x_k} \\
\frac{\partial h_n}{\partial x_{k+1}}
\end{array}\right]_{(k, k', \ldots, k'' \in N_N)} (j = k - 1, k).
\] (253)

Thus, we have
\[
A = (A_{kj})_{2N \times 2N},
\]
\[
y = (y_1, y_2, \ldots, y_N)^T,\quad y' = (y'_1, y'_2, \ldots, y'_N)^T,
\]
\[
b = (b_1, b_2, \ldots, b_N)^T,
\]
\[
b' = (b'_1, b'_2, \ldots, b'_N)^T;
\]
\[
A_{kj} = \sum_{l_k} A_{kkl} d_{kj}^l \quad \text{for} \quad l_k = k - 1, k, r_k; \quad l_k > 0;
\]
\[
y_j = \left[\begin{array}{c}
\frac{\partial x_k}{\partial x_0} \\
\frac{\partial x_k}{\partial x_k}
\end{array}\right]_{(k, k', \ldots, k'' \in N_N)}, \quad y'_j = \left[\begin{array}{c}
\frac{\partial x_k}{\partial x_0} \\
\frac{\partial x_k}{\partial x_k}
\end{array}\right]_{(k, k', \ldots, k'' \in N_N)}.
\] (254a)
Finally, Eq. (252) becomes
\[ A_{kj} = \left[ \begin{array}{cc} a_{ij} & a_{ij} \end{array} \right], \quad A_{kj} = \left[ \begin{array}{cc} 0_{n \times n} & 0_{n \times n} \end{array} \right]; \]
\[ b_{kj} = -\sum_{l_k} \left[ \frac{\partial g_j}{\partial x_0} \left| \frac{\partial x_0^l}{\partial x_0} \right| \right] T \delta_0^j, \]
\[ b_{kj} = -\sum_{l_k} \left[ \frac{\partial g_j}{\partial x_0} \left| \frac{\partial x_0^l}{\partial x_0} \right| \right] T \delta_0^j, \]
\[ (j = k - 1, k). \]  

Finally, Eq. (252) becomes
\[ y = A^{-1} b \quad \text{and} \quad y^* = A^{-1} b^*. \]  

So we have
\[ DP = \left[ \frac{\partial \mathcal{X}_y}{\partial \mathcal{X}_0} \right] \left[ \frac{\partial \mathcal{X}_y}{\partial \mathcal{X}_0} \right]^T, \]  

(256)
\[(ii) \text{ For } r = k, \text{ Eq. (244) with periodicity condition } (x_0 = x_N) \text{ gives node points } x_k^r \quad (k = 0, 1, 2, \ldots, N). \]

The corresponding stability and bifurcation can be analyzed in the neighborhoods of \( x_k^r \) and \( x_k^{r*} \) with \( x_k = x_0 + \Delta x_k \) and \( x_k = x_0 + \Delta x_k^r \) for the periodic motion. Equation (245) becomes
\[ \sum_{j} \frac{\partial \mathcal{X}_y}{\partial x_j} \left. \frac{\partial \mathcal{X}_y}{\partial x_j} \right|_{x_0} = 0_{n \times 1}, \]
\[ \sum_{j=0}^k \frac{\partial \mathcal{X}_y}{\partial x_j} \left. \frac{\partial \mathcal{X}_y}{\partial x_j} \right|_{x_0} = 0_{n \times 1}, \]
\[ \sum_{j=0}^k \frac{\partial \mathcal{X}_y}{\partial x_j} \left. \frac{\partial \mathcal{X}_y}{\partial x_j} \right|_{x_0} = 0_{n \times 1}, \]
\[ \sum_{j=0}^k \frac{\partial \mathcal{X}_y}{\partial x_j} \left. \frac{\partial \mathcal{X}_y}{\partial x_j} \right|_{x_0} = 0_{n \times 1}, \]
\[ (j = 0, 1, \ldots, k - 1, k). \]  

Thus, the eigenvalues are determined by
\[ |DP_{k(k-1)-1} - \lambda I_{x_0} | = 0 \quad \text{with} \]
\[ DP_{k(k-1)-1} = \left[ \begin{array}{cc} \frac{\partial \mathcal{X}_y}{\partial x_0} & \frac{\partial \mathcal{X}_y}{\partial x_0} \\ \frac{\partial \mathcal{X}_y}{\partial x_0} & \frac{\partial \mathcal{X}_y}{\partial x_0} \end{array} \right] \]  

(260)
for the properties of node points on the periodic flow of the time-delay system.

The multistep mappings are developed from the previous determined nodes of periodic motion.
During time interval $t \in [t_0, t_0 + T]$, the periodic flow can also be determined by

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x, x', t, p) dt, \quad l = 0, 1, 2, \ldots, k - 1. \quad (261)$$

For such a periodic flow, $N$-nodes during the time interval $t \in [t_0, t_0 + T]$ are selected, and the corresponding points $x(t_k)$ ($k = 0, 1, \ldots, N$). Under $|x(t_k) - x_0| \leq \varepsilon$ with $\varepsilon = \max \{\varepsilon_k \geq 0\}$ for $k \in (0, 1, \ldots, N)$,

$$\|f(x(t_k), x'(t_k), t_k, p) - f(x_0, x_0', t_k, p)\| \leq \delta_j. \quad (262)$$

Suppose that $x_0, \ldots, x_N$ are given, $f(x_k, x'_k, t_k, p)$ ($k = 0, 1, \ldots, N$) can be determined. An interpolation polynomial $P(t, x_0, \ldots, x_N; x'_0, t_0, \ldots, t_N; p)$ is determined, which can be used to approximate $f(x, x', t, p)$. That is,

$$f(x, x', t, p) \approx P(t, x_0, \ldots, x_N; x'_0, t_0, \ldots, t_N; p) \quad (263)$$

and $x(t_k) \approx x_k$ can be computed by

$$x_k = x_{k-1} + \int_{t_{k-1}}^{t_k} P(t, x_0, \ldots, x_N; x'_0, t_0, \ldots, t_N; p) dt,$$

$$x'_{k-1} = x_k - x_{k-1} + \int_{t_{k-1}}^{t_k} P(t, x_0, \ldots, x_N; x'_0, t_0, \ldots, t_N; p) dt. \quad (264)$$

Therefore, we have

$$x_k = x_{k-1} + \sum_{i=0}^{k} \sum_{j=0}^{N} (g_i(x_0, \ldots, x_N; x'_0, \ldots, x'_N; p), h_k(x_0, \ldots, x_N; x'_0, \ldots, x'_N; x'_k, t_k, p)) \quad (265)$$

The mapping $P_k$ ($k \in \{1, 2, \ldots, N\}$) is

$$g_k(x_0, \ldots, x_N; x'_0, \ldots, x'_N; p) = 0, \quad h_k(x_0, \ldots, x_N; x'_0, \ldots, x'_N, x'_k, t_k, p) = 0 \quad (266)$$

and $r_k = \text{mod}(k - 1 + s_{k-1}, N)$.

The periodic motions are determined by the mapping $P_k$ ($k = 1, 2, \ldots, N$) and periodicity conditions

$$g_k(x_0, \ldots, x_N; x'_0, \ldots, x'_N; p) = 0, \quad h_k(x_0, \ldots, x_N; x'_0, \ldots, x'_N, x'_k, t_k, p) = 0 \quad (267)$$

for $k = 1, 2, \ldots, N$

$$x_0 = x_N \quad \text{and} \quad x'_0 = x'_N.$$

From the foregoing equation, node points $x'_k$ and $x'_N$ ($k = 1, 2, \ldots, N$) can be determined. The corresponding stability and bifurcation is discussed in the neighborhood of $x'_k$ and $x'_N$ with $x_k = x'_k + \Delta x_k$ and $x'_N = x'_N + \Delta x'_N$. The derivative of Eq. (267) with respect to $x_0$ gives

$$\sum_{j=1}^{N} \frac{\partial g_k}{\partial x_j} \frac{\partial x'_0}{\partial x_0} + \frac{\partial g_k}{\partial x'_0} \frac{\partial x'_0}{\partial x'_0} = 0_{n \times 1};$$

$$\sum_{j=1}^{N} \frac{\partial h_k}{\partial x'_j} \frac{\partial x'_0}{\partial x_0} + \frac{\partial h_k}{\partial x'_0} \frac{\partial x'_0}{\partial x'_0} + \frac{\partial h_k}{\partial x'_{N}} \frac{\partial x'_{N}}{\partial x'_0} = 0_{n \times 1};$$

$$\sum_{j=1}^{N} \frac{\partial h_k}{\partial x'_j} \frac{\partial x'_0}{\partial x'_0} + \frac{\partial h_k}{\partial x'_{N}} \frac{\partial x'_{N}}{\partial x'_0} = 0_{n \times 1};$$

with $r_k = \text{mod}(k - 1 + s_{k-1}, N)$,

$$k = 1, 2, \ldots, N. \quad (268)$$

and

$$A = (A_{kl})_{2n \times 2n}; \quad y = (y_1, y_2, \ldots, y_N)^T, \quad y' = (y'_1, y'_2, \ldots, y'_N)^T; \quad b = (b_1, b_2, \ldots, b_N)^T; \quad b' = (b'_1, b'_2, \ldots, b'_N)^T.$$
where $\mathbf{b}_k$ and $\mathbf{b}_l^0$ are implicit vector functions for regular and time-delay nodes, respectively. Consider a mapping structure as

$$
P = P_N \circ P_{N-1} \circ \cdots \circ P_2 \circ P_1 :$$

$$(x_0, x_k^0) \rightarrow (x_N, x_k^0); \quad \text{with}$$

$$P_k : (x_{k-1}, x_{k-1}^0) \rightarrow (x_k, x_k^0) \quad (k = 1, 2, \ldots, N).$$

(272)

For $(x_N, x_N^0) = P(x_0, x_0^0)$, if there is a set of points $x_k^* (k = 0, 1, \ldots, N)$ computed by

$$
g_k(x_{k-1}^*, x_{k-1}^0, \ldots, x_{k-1}^*; \mathbf{p}) = 0$$

$$
b_k(x_{k-1}^*, x_{k-1}^0, \ldots, x_{k-1}^*; \mathbf{p}) = 0$$

(k = 1, 2, \ldots, N)

(273)

Then the points $x_k^*$ and $x_k^{**}$ (k = 0, 1, \ldots, N) is the approximation of points $x_k(t_k)$ and $x_k^{**}(t_k)$ of periodic solutions. In the neighborhood of $x_k^*$ and $x_k^{**}$, with $x_k = x_k^* + \Delta x_k$ and $x_k^{**} = x_k^{**} + \Delta x_k^{**}$, the linearized equations are given by

$$
\frac{\partial g_k}{\partial x_k} + \sum_{j=1}^{N} \frac{\partial g_k}{\partial x_j} \frac{\partial x_j}{\partial x_k} + \sum_{j=1}^{N} \frac{\partial g_k}{\partial x_0} \frac{\partial x_0}{\partial x_k} = 0,
$$

$$
\frac{\partial h_k}{\partial x_k} + \sum_{j=1}^{N} \frac{\partial h_k}{\partial x_j} \frac{\partial x_j}{\partial x_k} + \sum_{j=1}^{N} \frac{\partial h_k}{\partial x_0} \frac{\partial x_0}{\partial x_k} = 0
$$

(274)

with $\delta_k = \text{mod}(k-1 + s_k, -N)$,

$$
s_k = \text{mod}(k - j, N),$$

$$
\phi_k = \text{mod}(k - j, l, \ldots, l, l_1, l_2, \ldots, l_1, l_2, \ldots, l_1 + l_2 \leq N, l_1 \geq 1; (k = 1, 2, \ldots, N),$$

(271)
The resultant Jacobian matrices of the periodic flow are

\[ DP_{k(-1)-1} = \begin{bmatrix} \frac{\partial x_1}{\partial x_0} & \frac{\partial x_2}{\partial x_0} & \cdots & \frac{\partial x_N}{\partial x_0} \\ \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \cdots & \frac{\partial x_N}{\partial x_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial x_{N-1}} & \frac{\partial x_2}{\partial x_{N-1}} & \cdots & \frac{\partial x_N}{\partial x_{N-1}} \end{bmatrix} (x_0^0, x_1^0, \ldots, x_N^0) \]

\[ (k = 1, 2, \ldots, N), \]

\[ DP = \begin{bmatrix} \frac{\partial x_1}{\partial x_0} & \frac{\partial x_2}{\partial x_0} & \cdots & \frac{\partial x_N}{\partial x_0} \\ \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \cdots & \frac{\partial x_N}{\partial x_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_1}{\partial x_{N-1}} & \frac{\partial x_2}{\partial x_{N-1}} & \cdots & \frac{\partial x_N}{\partial x_{N-1}} \end{bmatrix} (x_0, x_1, \ldots, x_N) \]

(276)

where

\[ y = A^{-1}b \quad \text{and} \quad y^* = A^{-1}b^* \]

(277)

and

\[ A = (A_{kl})_{2N \times 2N}, \]

\[ A_{kl} = \sum_{l_0} A_{kl_0} \delta_{l_0}^{l_0} \]

for \( l_k = s_{kl_1}, \ldots, s_{kl_0}, \ldots, s_{kl_{(-1)}} \), \( k \neq 0 \);

\[ A_{kj} = \begin{bmatrix} a_{kj} & \bar{a}_{kj} \\ b_{kj} & \bar{b}_{kj} \end{bmatrix}, \quad A_{kj} = \begin{bmatrix} a_{n,n} & 0 \\ 0 & a_{n,n} \end{bmatrix}, \]

\[ a_{kj} = \left[ \frac{\partial a_{k2}}{\partial x_{j2}} \right], \quad b_{kj} = \left[ \frac{\partial a_{k1}}{\partial x_{j1}} \right], \]

\[ b_{kj}^* = \left[ \frac{\partial b_{k2}}{\partial x_{j2}} \right], \quad b_{kj}^* = \left[ \frac{\partial b_{k1}}{\partial x_{j1}} \right] \]

\[ (j = s_{kl_1}, \ldots, s_{kl_0}, \ldots, s_{kl_{(-1)}}); \]

(278)

and

\[ y = (y_1, y_2, \ldots, y_N)^T, \]

\[ y^* = (y_1^*, y_2^*, \ldots, y_N^*)^T, \]

\[ b = (b_1, b_2, \ldots, b_N)^T, \]

\[ b^* = (b_1^*, b_2^*, \ldots, b_N^*)^T, \]

\[ b_{k} = -\sum_{l_0} \left[ \frac{\partial g_{k}}{\partial x_{0}} \right] \delta_{0}^{l_0}, \]

\[ y_{k} = \left[ \frac{\partial b_{k}}{\partial x_{0}} \right]^T, \quad y_{k}^* = \left[ \frac{\partial b_{k}}{\partial x_{0}} \right]^T, \]

\[ (k = 1, 2, \ldots, N). \]

(279)

The eigenvalues of \( DP \) for such a periodic flow in the time-delay system are determined by

\[ |DP - \lambda_{n,n}| = 0. \]

(280)

The properties of discrete points \( x_k, x_k^* (k = 1, 2, \ldots, N) \) can be estimated by the eigenvalues of \( DP_{k(-1)-1} \) as

\[ |DP_{k(-1)-1} - \lambda_{n,n}| = 0 \quad (k = 1, 2, \ldots, N). \]

(281)

Thus, the stability and bifurcation of the periodic flow can be classified by the eigenvalues of \( DP(x_k^0, x_k^0) \) with

\[ ([n_1^*, n_1^*] : [n_2^*, n_2^*] : [\tau_1, \tau_2], \ldots, [\tau_1, \tau_2]), \ldots, [\tau_1, \tau_2] \]

\[ ([n_1^*, n_1^*] : [n_2^*, n_2^*] : [\tau_1, \tau_2]). \]

(282)

(i) If the magnitudes of all eigenvalues of \( DP(|\lambda_i| < 1, i = 1, 2, \ldots, n) \) are less than one, the approximate periodic solution is stable.

(ii) If at least the magnitude of one eigenvalue of \( DP(|\lambda_i| > 1, i = 1, 2, \ldots, n) \) is greater than one, the approximate periodic solution is unstable.

(iii) The boundaries between stable and unstable periodic flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. The proof is similar to Theorem 9. ■

As discussed in the previous section, from the stability and bifurcation analysis, the period-1 flow of the time-delay under period \( T = 2\pi/\Omega \) are stable and unstable that are based on the set of discrete mapping \( D_k : (x_{k-1}, x_{k-1}, x_k, x_k) \rightarrow (x_k, x_k) \) with \( g_k(x_k, x_k, x_k, x_k, p) = 0 \) and \( h_k(x_k, x_k, x_k, x_k, x_k, p) = 0 \); \( k = 1, 2, \ldots, N \).

As the period-doubling bifurcation of the period-1 flow occurs, the period-1 flow will become a new periodic flow under period \( T^* = 2T \). Thus, consider a mapping structure of the period-2 flow with \( 2N \).
mappings.

\[ P^{(2)} = P \circ P = P_{2N} \circ P_{2N-1} \circ \ldots \circ P_2 \circ P_1 : (x_0, x'_0) \rightarrow (x_{2N}, x'_{2N}); \]

with \( P_k : (x_{k-1}, x'_{k-1}) \rightarrow (x_k, x'_k) \) \( (k = 1, 2, \ldots, 2N) \). \hspace{1cm} (283)

For \( (x_{2N}, x'_{2N}) = P^{(2)}(x_0, x'_0) \), if there is a set of points \( (x_k^{(m)}, x'_k^{(m)}) \) \( (k = 0, 1, \ldots, 2N) \) computed by

\[
\begin{align*}
\quad g_k(x_{k-1}^{(m)}, x'_{k-1}^{(m)}, x_k^{(m)}, x'_k^{(m)}, p) &= 0, \\
g_k(x_{k-1}^{(m)}, x'_{k-1}^{(m)}, x_k^{(m)}, x'_k^{(m)}, p) &= 0 \\
\quad r_k &= \text{mod}(k - 1 + s_{k-1}, 2N) \\
x'_k &= x_k^{(m)}, \quad x''_k = x'_k^{(m)}. 
\end{align*}
\]

(284)

After period-doubling, the period-1 flow becomes a period-2 flow. The node points increase to \( 2N \) points during two periods \( (2T) \). The node points are determined through the discrete mapping with mathematical relation in Eq. (283). On the other hand

\[ T' = 2T = 2(2nT) = 2\pi \Rightarrow \omega = \Omega \] \hspace{1cm} (285)

Similarly, during the period of \( T' \), there is a periodic flow, which can be described by node points \( x_k \) \( (k = 1, 2, \ldots, 2N) \). Since the period-1 flow is described by node points \( x_k \) \( (k = 1, 2, \ldots, N) \) during period \( T \), due to \( T' = 2T \), the period-2 flow can be described at least by \( N' \geq 2N \) nodes. Thus the corresponding mapping \( P_k \) is defined as

\[ P_k : (x_k^{(2)}, x'_k^{(2)}) \rightarrow (x_k, x'_k) \] \hspace{1cm} (286)

and

\[
\begin{align*}
\quad g_k(x_k^{(2)}, x_k^{(2)}, x'_k^{(2)}, x''_k, p) &= 0, \\
g_k(x_k^{(2)}, x_k^{(2)}, x'_k^{(2)}, x''_k, p) &= 0 \\
\quad r_k &= \text{mod}(k - 1 + s_{k-1}, 2N) \\
x'_k^{(2)} &= x_k^{(2)} + x''_k = x'_k^{(m)}. 
\end{align*}
\]

(287)

In general, for period \( T' = mT \), there is a period-\( m \) flow which can be described at least by \( N' \geq mN \). The corresponding mapping \( P_k \)

\[ P_k : (x_k^{(m)}, x'_k^{(m)}) \rightarrow (x_k^{(m)}, x'_k^{(m)}), \quad (k = 1, 2, \ldots, mN) \] \hspace{1cm} (288)

and

\[
\begin{align*}
\quad g_k(x_k^{(m)}, x_k^{(m)}, x'_k^{(m)}, x''_k^{(m)}, x'_k^{(m)}, p) &= 0, \\
g_k(x_k^{(m)}, x_k^{(m)}, x'_k^{(m)}, x''_k^{(m)}, x'_k^{(m)}, p) &= 0 \\
\quad r_k &= \text{mod}(k - 1 + s_{k-1}, mN) \\
x'_k^{(m)} &= x_k^{(m)} + x''_k^{(m)}, \quad x''_k^{(m)} = x'_k^{(m)}. 
\end{align*}
\]

(289)

From the above discussion, the period-\( m \) flow in a time-delay dynamical system can be described through \( mN \) nodes for period \( mT \).

**Theorem 11.** Consider a nonlinear dynamical system in Eq. (285). If such a dynamical system has a period-\( m \) flow \( x^{(m)}(t) \) with finite norm \(||x^{(m)}||\) and period \( mT \) \( (T = 2\pi/\Omega) \), there is a set of discrete time \( t_k \) \( (k = 0, 1, \ldots, mN) \) with \((N \rightarrow \infty)\) during \( m \)-periods \((mT)\), and the corresponding solution \( x^{(m)}(t_k), x^{(m)}(t_k) \) and vector fields \( f(x^{(m)}(t_k), x^{(m)}(t_k), t_k, p) \) are exact. Suppose discrete nodes \( x_k^{(m)} \) and \( x'_k^{(m)} \) are on the approximate solution of the period-\( m \) flow under \( ||x^{(m)}(t_k) - x_k^{(m)}|| \leq \varepsilon_k \) and \( ||x^{(m)}(t_k) - x_k^{(m)}|| \leq \varepsilon_k' \) with small \( \varepsilon_k, \varepsilon_k' \geq 0 \) and

\[
\left| f(x_k^{(m)}, x'_k^{(m)}, t_k, p) \right| \leq \delta_k 
\]

(290)

with a small \( \delta_k \geq 0 \). During a time interval \( t \in [t_{k-1}, t_k] \), there is a mapping \( P_k : (x_k^{(m)}, x_k^{(m)}, x'_k^{(m)}) \rightarrow (x_k^{(m)}, x_k^{(m)}) \) \( (k = 1, 2, \ldots, mN) \) as

\[
\begin{align*}
\quad (x_k^{(m)}, x'_k^{(m)}) &= P_k(x_k^{(m)}, x_k^{(m)}), \\
g_k(x_k^{(m)}, x_k^{(m)}, x_k^{(m)}, x_k^{(m)}, x_k^{(m)}, p) &= 0, \\
\quad r_k &= \text{mod}(k + s_k, mN), \quad (k = 1, 2, \ldots, mN) \\
x'_k^{(m)} &= x_k^{(m)} + x''_k^{(m)} = x_k^{(m)}.
\end{align*}
\]

(291)

where \( g_k \) and \( h_k \) are implicit vector functions for regular and time-delay nodes, respectively. Consider
a mapping structure as
\[ P = P_N \circ P_{N-1} \circ \ldots \circ P_2 \circ P_1 : \]
\[ (x_0^{(m)}, x_0^{(m)}) \rightarrow (x_0^{(m)}, x_0^{(m)}) \], with
\[ P_k : (x_{k-1}^{(m)}, x_{k-1}^{(m)}) \rightarrow (x_k^{(m)}, x_k^{(m)}) \]
\[ (k = 1, 2, \ldots, mN). \]

For \((x_{mN}^{(m)}, x_{mN}^{(m)}) = P(x_0^{(m)}, x_0^{(m)})\), if there is a set of points \((x_k^{(m)}, x_k^{(m)})\) \((k = 0, 1, \ldots, mN)\) computed by
\[ g_k(x_{k-1}^{(m)}, x_{k-1}^{(m)}; x_k^{(m)}, x_k^{(m)}; p) = 0 \]
\[ h_k(x_{k-1}^{(m)}, x_{k-1}^{(m)}; x_k^{(m)}, x_k^{(m)}, x_k^{(m)}; p) = 0, \]
\[ (k = 1, 2, \ldots, mN) \]
\[ x_0^{(m)} = x_{mN}, \quad x_0^{(m)} = x_{mN}^{(m)} \]
then the points \(x_k^{(m)}\) and \(x_k^{(m)}\) \((k = 0, 1, \ldots, mN)\) are approximations of points \(x_k^{(m)}(t_k)\) and \(x_k^{(m)}(t_k)\) of the periodic solution. In the neighborhoods of \(x_k^{(m)}(p)\) and \(x_k^{(m)}(p)\), with \(x_k^{(m)} = x_k^{(m)} + \Delta x_k^{(m)}\) and \(x_k^{(m)} = x_k^{(m)} + \Delta x_k^{(m)}\), the linearized equation is given by
\[ \frac{\partial g_k}{\partial x_{k-1}^{(m)}} \frac{\partial x_{k-1}^{(m)}}{\partial x_0^{(m)}} + \frac{\partial g_k}{\partial x_k^{(m)}} \frac{\partial x_k^{(m)}}{\partial x_0^{(m)}} + \frac{\partial h_k}{\partial x_{k-1}^{(m)}} \frac{\partial x_{k-1}^{(m)}}{\partial x_0^{(m)}} + \frac{\partial h_k}{\partial x_k^{(m)}} \frac{\partial x_k^{(m)}}{\partial x_0^{(m)}} = 0, \]
\[ (k = 1, 2, \ldots, mN). \]

The resultant Jacobian matrices of the periodic flow are
\[ D P_k(k-1) = \begin{bmatrix} \frac{\partial g_k}{\partial x_{k-1}^{(m)}} & \frac{\partial g_k}{\partial x_k^{(m)}} \\ \frac{\partial h_k}{\partial x_{k-1}^{(m)}} & \frac{\partial h_k}{\partial x_k^{(m)}} \end{bmatrix} \]
\[ (k = 1, 2, \ldots, mN), \]
\[ \begin{bmatrix} \frac{\partial g_k}{\partial x_{k-1}^{(m)}} & \frac{\partial g_k}{\partial x_k^{(m)}} \\ \frac{\partial h_k}{\partial x_{k-1}^{(m)}} & \frac{\partial h_k}{\partial x_k^{(m)}} \end{bmatrix} \begin{bmatrix} x_{k-1}^{(m)} \\ x_k^{(m)} \end{bmatrix} = 0, \]
where
\[ y^{(m)} = (A^{(m)})^{-1} y^{(m)} \text{ and } y^{(m)} = (A^{(m)})^{-1} y^{(m)} \]
and

\[ A^{(m)} = (A_k^{(m)})_{2mN \times 2mN}, \]

\[ A_k^{(m)} = \sum_{i_k} A^{(m)} g_{i_k} \]

for \( i_k = k - 1, k, r_k; \ i_k > 0; \)

\[ A_k^{(m)} = \left[ \begin{array}{cc} A_k^{(m)} & \delta_i^{(m)} \\ \delta_i^{(m)} & A_k^{(m)} \end{array} \right]; \]

\[ A_k^{(m)} = \begin{bmatrix} \delta_i^{(m)} & b_k^{(m)} \\ b_k^{(m)} & \delta_i^{(m)} \end{bmatrix}, \]

\[ b_k^{(m)} = \begin{bmatrix} \frac{\partial g_k}{\partial \delta_i^{(m)}} \\ \frac{\partial g_k}{\partial \delta_j^{(m)}} \end{bmatrix} ; \]

\[ (j = k - 1, k) \quad \text{and} \quad (k = 1, 2, \ldots, mN) \]

(297)

and

\[ y^{(m)} = (y_1^{(m)}, y_2^{(m)}, \ldots, y_{mN}^{(m)})^T, \]

\[ y^{(m)} = (y_1^{(m)}, y_2^{(m)}, \ldots, y_{mN}^{(m)})^T, \]

\[ h^{(m)} = (h_1^{(m)}, h_2^{(m)}, \ldots, h_{mN}^{(m)})^T; \]

\[ b^{(m)} = (b_1^{(m)}, b_2^{(m)}, \ldots, b_{mN}^{(m)})^T; \]

\[ b_k^{(m)} = -\sum_{l_k} \frac{\partial g_k}{\partial \delta_i^{(m)}} \frac{\partial g_k}{\partial \delta_{i_k}^{(m)}} \delta_{i_k}^{(m)}; \]

\[ b_k^{(m)} = -\sum_{l_k} \frac{\partial g_k}{\partial \delta_i^{(m)}} \frac{\partial g_k}{\partial \delta_{i_k}^{(m)}} \delta_{i_k}^{(m)}; \]

(298)

\[ y^{(m)} = \left[ \begin{array}{c} \frac{\partial g_k}{\partial \delta_i^{(m)}} \\ \frac{\partial g_k}{\partial \delta_j^{(m)}} \end{array} \right] \quad \text{and} \quad (k = 1, 2, \ldots, mN). \]

The properties of discrete points \((x_k^{(m)}, x^r_k^{(m)}) (k = 1, 2, \ldots, mN)\) can be estimated by the eigenvalues of \(DP_{k(k-1)-1}\) as

\[ |DP_{k(k-1)-1} - \Lambda_{i_{k,x}}| = 0 \quad (k = 1, 2, \ldots, mN), \]

(299)

The eigenvalues of DP for such a periodic flow in the time-delay system are determined by

\[ |DP - \Lambda_{i_{x,x}}| = 0. \]

(300)

Thus, the stability and bifurcation of the periodic flow can be classified by the eigenvalues of \(DP^{(m)}(x_0^{(m)}, x_0^{(r)})\) with

\[ (n_1^*, n_2^*) : [n_1^*, n_2^*] : [n_1, n_2] : \]

\[ [n_1, n_2] : [m_1, l, k_2] \].

(301)

(i) If the magnitudes of all eigenvalues of \(DP^{(m)}\) are less than one (i.e. \(\|\lambda\| < 1, i = 1, 2, \ldots, mN\)), the approximate period-m solution is stable.

(ii) If at least the magnitude of one eigenvalue of \(DP^{(m)}\) is greater than one (i.e. \(\|\lambda\| > 1, i \in \{1, 2, \ldots, n\}\)), the approximate period-m solution is unstable.

(iii) The boundaries between stable and unstable period-m flows with higher order singularity give bifurcation and stability conditions with higher order singularity.

Proof. The discrete mapping for the period-m flow for the time-delay nonlinear system can be developed during \(t \in [t_k, t_{k+1}, t_2, t_3, \ldots, t_{mN}]\) as in Theorem 9. The proof is similar to Theorem 9. \(\square\)

The discrete mapping for a period-m flow with multiple steps can be developed by using many forward and backward nodes. The period-m flow in time-delay nonlinear dynamical system can be determined through the following theorem.

Theorem 12. Consider a time-delay nonlinear dynamical system in Eq. (295). If such a system has a period-m flow \(x^{(m)}(t)\) with finite norm \(\|x^{(m)}(t)\|\) and period \(mN\) \((T = 2mN/T)\), there is a set of discrete time \(t_k (k = 0, 1, \ldots, mN)\) with \((N \rightarrow \infty)\) during \(mN\) periods \((mN)\), and the corresponding solution \(x^{(m)}(t_k) = x^{(m)}(t_k - \tau)\) for the vector fields \(f(x^{(m)}(t_k), x^{(m)}(t_k), t_k)\) are exact. Suppose discrete nodes \(x_k^{(m)}\) and \(x_k^{(r)}\) are on the approximate solution of the period-m flow under \(\|x^{(m)}(t_k - \tau) - x_k^{(m)}\| \leq \epsilon_k\) and \(\|x^{(m)}(t_k) - x_k^{(m)}\| \leq \epsilon_k\).
with small \( \varepsilon_k, \varepsilon_k^2 \geq 0 \) and

\[
|f(x^{(m)}(t_k), x^{(m)}(t_k), t_k, p) - f(x_k^{(m)}, x_k^{(m)}(t_k), t_k, p)| \leq \delta_k
\]  

(302)

with a small \( \delta_k \geq 0 \). During a time interval \( t \in [t_{k-1}, t_k] \), there is a mapping \( P_k : (x^{(m)}_{k-1}, x^{(m)}_{k-1}) \rightarrow (x^{(m)}_k, x^{(m)}_k) \) as

\[
(x^{(m)}_k, x^{(m)}_k) = P_k(x^{(m)}_{k-1}, x^{(m)}_{k-1})
\]

(303)

where \( g_k \) and \( h_k \) are implicit vector functions for regular and time-delay nodes, respectively. Consider a mapping structure as

\[
P = P_{mN} \circ P_{mN-1} \circ \cdots \circ P_2 \circ P_1:
\]

\[
(x^{(m)}_0, x^{(m)}_0) \rightarrow (x^{(m)}_{mN}, x^{(m)}_{mN})
\]

(304)

For \( (x^{(m)}_k, x^{(m)}_k) = P_k(x^{(m)}_{k-1}, x^{(m)}_{k-1}) \), if there is a set of points \( (x_k^{(m)}, x_k^{(m)}) \) \( (k = 0, 1, \ldots, mN) \) computed by

\[
g_k(x^{(m)}_{k-1}, \ldots, x^{(m)}_{k-1}) = 0,
\]

\[
h_k(x^{(m)}_{k-1}, \ldots, x^{(m)}_{k-1}) = 0,
\]

(305)

Then the points \( x^{(m)}_k \) and \( x^{(m)}_k \) \( (k = 0, 1, \ldots, mN) \) are approximations of points \( x^{(m)}(t_k) \) and \( x^{(m)}(t_k) \) of the periodic solution. In the neighborhoods of \( x^{(m)}_k \) and \( x^{(m)}_k \), with \( x_k^{(m)} = x^{(m)}_k + \Delta x^{(m)}_k \) and \( x_k^{(m)} = x^{(m)}_k + \Delta x^{(m)}_k \), the linearized equation is given by

\[
\frac{\partial g_k}{\partial x^{(m)}_0} \sum_{j=1}^{N} \frac{\partial g_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_0}{\partial x^{(m)}_{0j}} + \sum_{j=1}^{N} \frac{\partial g_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_0}{\partial x^{(m)}_{0j}} = 0,
\]

\[
\frac{\partial h_k}{\partial x^{(m)}_0} \sum_{j=1}^{N} \frac{\partial h_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_0}{\partial x^{(m)}_{0j}} + \sum_{j=1}^{N} \frac{\partial h_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_0}{\partial x^{(m)}_{0j}} = 0;
\]

\[
\frac{\partial g_k}{\partial x^{(m)}_{0j}} \sum_{j=1}^{N} \frac{\partial g_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_{0j}}{\partial x^{(m)}_{0j}} + \sum_{j=1}^{N} \frac{\partial g_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_{0j}}{\partial x^{(m)}_{0j}} = 0;
\]

\[
\frac{\partial h_k}{\partial x^{(m)}_{0j}} \sum_{j=1}^{N} \frac{\partial h_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_{0j}}{\partial x^{(m)}_{0j}} + \sum_{j=1}^{N} \frac{\partial h_k}{\partial x^{(m)}_{0j}} \frac{\partial x^{(m)}_{0j}}{\partial x^{(m)}_{0j}} = 0;
\]

\[
\frac{\partial g_k}{\partial x^{(m)}_{0j}} = 0 \quad \text{and} \quad \frac{\partial g_k}{\partial x^{(m)}_{0j}} = 0 \quad (\alpha \neq s_k),
\]

\[
\frac{\partial h_k}{\partial x^{(m)}_{0j}} = 0 \quad \text{and} \quad \frac{\partial h_k}{\partial x^{(m)}_{0j}} = 0 \quad (\alpha \neq s_k),
\]

\[
j = -l_2, -l_2 + 1, \ldots, l_1 - 1, l_1.
\]

(307)
A. C. J. Luo

The resultant Jacobian matrices of the periodic flow are

$$DP^{(m)}_{k(k-1)-1} = \begin{bmatrix}
\frac{\partial x^{(m)}_1}{\partial x^{(m)}_0} & \frac{\partial x^{(m)}_1}{\partial x^{(m)}_0} & \cdots & \frac{\partial x^{(m)}_1}{\partial x^{(m)}_n} \\
\frac{\partial x^{(m)}_2}{\partial x^{(m)}_0} & \frac{\partial x^{(m)}_2}{\partial x^{(m)}_0} & \cdots & \frac{\partial x^{(m)}_2}{\partial x^{(m)}_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x^{(m)}_n}{\partial x^{(m)}_0} & \frac{\partial x^{(m)}_n}{\partial x^{(m)}_0} & \cdots & \frac{\partial x^{(m)}_n}{\partial x^{(m)}_n}
\end{bmatrix}
$$

$(k = 1, 2, \ldots, mN)$,

$$DP^{(m)} = \begin{bmatrix}
\frac{\partial y^{(m)}_1}{\partial x^{(m)}_0} & \frac{\partial y^{(m)}_1}{\partial x^{(m)}_0} & \cdots & \frac{\partial y^{(m)}_1}{\partial x^{(m)}_n} \\
\frac{\partial y^{(m)}_2}{\partial x^{(m)}_0} & \frac{\partial y^{(m)}_2}{\partial x^{(m)}_0} & \cdots & \frac{\partial y^{(m)}_2}{\partial x^{(m)}_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial y^{(m)}_n}{\partial x^{(m)}_0} & \frac{\partial y^{(m)}_n}{\partial x^{(m)}_0} & \cdots & \frac{\partial y^{(m)}_n}{\partial x^{(m)}_n}
\end{bmatrix}
$$

$(j = s_{k_1}, \ldots, s_{k_0}, \ldots, s_{k-(k-1)}, 0)$.

where

$$y^{(m)}(m) = (A^{(m)})^{-1}b^{(m)}$$

and

$$y^{(m)} = (A^{(m)})^{-1}b^{(m)}$$

and

$$A^{(m)} = (A_{kl}^{(m)})_{2mN \times 2mN},$$

$$A_{kl}^{(m)} = \sum_{l_k} A_{kl}^{(m)} d_k^{l},$$

for $l_k = s_{k_1}, \ldots, s_{k_0}, \ldots, s_{k-(k-1)}, r_k; l_k \neq 0$.

$$A_{kj}^{(m)} = \begin{bmatrix}
A_{kj}^{(m)} & A_{kj}^{(m)} \\
A_{kj}^{(m)} & A_{kj}^{(m)}
\end{bmatrix},$$

$$A_{kj}^{(m)} = \begin{bmatrix}
\frac{\partial g_k^{(m)}}{\partial x^{(m)}_j} \\
\frac{\partial h_k^{(m)}}{\partial x^{(m)}_j}
\end{bmatrix},$$

$$b_{kj}^{(m)} = \begin{bmatrix}
\frac{\partial h_k^{(m)}}{\partial x^{(m)}_j} \\
\frac{\partial h_k^{(m)}}{\partial x^{(m)}_j}
\end{bmatrix},$$

$$b_{kj}^{(m)} = \begin{bmatrix}
\frac{\partial h_k^{(m)}}{\partial x^{(m)}_j} \\
\frac{\partial h_k^{(m)}}{\partial x^{(m)}_j}
\end{bmatrix},$$

$(j = s_{k_1}, \ldots, s_{k_0}, \ldots, s_{k-(k-1)});$

and

$$y^{(m)} = (y^{(m)}_1, y^{(m)}_2, \ldots, y^{(m)}_n)^T,$$

$$y^{(m)} = (y^{(m)}_1, y^{(m)}_2, \ldots, y^{(m)}_n)^T,$$

$$b^{(m)} = (b_1^{(m)}, b_2^{(m)}, \ldots, b_{mN}^{(m)})^T;$$

$$b^{(m)} = (b_1^{(m)}, b_2^{(m)}, \ldots, b_{mN}^{(m)})^T,$$

$$b_k^{(m)} = \sum_l \frac{\partial g_k^{(m)}}{\partial x^{(m)}_j} d_k^{l},$$

$$b_k^{(m)} = \sum_l \frac{\partial g_k^{(m)}}{\partial x^{(m)}_j} d_k^{l},$$

$$y_k^{(m)} = (y_k^{(m)})^T,$$

$$y_k^{(m)} = (y_k^{(m)})^T,$$

$(k = 1, 2, \ldots, mN).$

The properties of discrete points $x_k^{(m)}$ and $y_k^{(m)}$

$(k = 1, 2, \ldots, mN)$ can be estimated by the eigenvalues of $DP^{(m)}_{k(k-1)-1}$ as

$$|DP^{(m)}_{k(k-1)-1} - \lambda_{2mN} I| = 0 \quad (k = 1, 2, \ldots, mN).$$

The eigenvalues of $DP^{(m)}_{k(k-1)-1}$ for such a periodic flow in the time-delay system are determined by

$$|DP^{(m)}_{k(k-1)-1} - \lambda_{2mN} I| = 0,$$

Thus, the stability and bifurcation of the periodic flow can be classified by the eigenvalues of
Discrete Implicit Mappings of Continuous Nonlinear Systems

The boundaries between stable and unstable for the time-delay nonlinear system can be developed during time with/without time delay, and such a dynamical system has a period-

Definition 1. Consider a nonlinear dynamical system with/without time delay, and such a dynamical system has a period-

Proof. The discrete mapping for the period-

4. Discrete Fourier Series

Consider a nonlinear dynamical system with/without time delay. If such a dynamical system has a period-

From the Fourier series theory of periodic function, we have the following definition.

Definition 1. Consider a nonlinear dynamical system with/without time delay, and such a dynamical system has a flow \( x(t) \) on the time interval \( t \in (0, T) \). Assume there are node points \( t_j \) \( (j = 0, 1, 2, \ldots, N) \) with \( t_0 = 0 \) and \( t_N = T \). If \( x(t_j) \) is finite \( (j = 0, 1, 2, \ldots, N) \) and \( x(t) \) is continuous for \( t \in (t_{j-1}, t_j) \) \( (j = 1, 2, \ldots, N) \), such a flow \( x(t) \) is said to be piecewise continuous on the time interval \( t \in (0, T) \).

Definition 2. Consider a nonlinear dynamical system with/without time delay, and such a dynamical system has a period-

If \( x(t) \) is a piecewise continuous flow on \( t \in (0, mT) \), then the Fourier series \( \mathbf{S}^{(m)}(t) \) \( (m \in \mathbb{N}) \) is as follows.

\[
\mathbf{S}^{(m)}(t) = a_0^{(m)} + \sum_{j=1}^{\infty} b_j^{(m)} \cos \left( \frac{j \Omega}{m} t \right) + c_j^{(m)} \sin \left( \frac{j \Omega}{m} t \right) \quad (316)
\]

If \( \mathbf{S}^{(m)}(t) = x^{(m)}(t) \), the coefficients \( a_j^{(m)}, b_j^{(m)}, c_j^{(m)} \) in Eq. (315) are by the Euler’s formulas

\[
a_0^{(m)} = \frac{1}{mT} \int_0^{mT} x^{(m)}(t) dt,
\]

\[
b_j^{(m)} = \frac{2}{mT} \int_0^{mT} x^{(m)}(t) \cos \left( \frac{j \Omega}{m} t \right) dt \quad (j = 1, 2, \ldots),
\]

\[
c_j^{(m)} = \frac{2}{mT} \int_0^{mT} x^{(m)}(t) \sin \left( \frac{j \Omega}{m} t \right) dt \quad (j = 1, 2, \ldots)
\]

and

\[
\mathbf{a}^{(m)} = (a_0^{(m)}, a_{12}^{(m)}, \ldots, a_n^{(m)})^T \in \mathbb{R}^n;
\]

\[
\mathbf{b}^{(m)} = (b_1^{(m)}, b_2^{(m)}, \ldots, b_n^{(m)})^T \in \mathbb{R}^n,
\]

\[
\mathbf{c}^{(m)} = (c_1^{(m)}, c_2^{(m)}, \ldots, c_n^{(m)})^T \in \mathbb{R}^n.
\]
where \( \mathbf{x}^{(m)}(t) \) and \( \mathbf{x}^{(m)}(t^+) \) are the left-hand and right-hand limits, respectively. Thus, the Fourier series of \( \mathbf{x}^{(m)}(t) \) can be expressed as in Eq. (319).

Proof. The proof can be found from Kreyszig [1988]. Since the basis of the Fourier series are continuous with infinite derivatives, if the period-\( m \) flow of a dynamical system can be expressed by the Fourier series, then the period-\( m \) flow should be at least continuous. Suppose a period-\( m \) flow in the dynamical system is continuous, which can be written by the Fourier series as

\[
\mathbf{x}^{(m)}(t) = a_0^{(m)} + \sum_{j=1}^{\infty} b_{j/m} \cos \left( \frac{j \Omega t}{m} \right) + c_{j/m} \sin \left( \frac{j \Omega t}{m} \right).
\]

(i) The foregoing equation is averaged in the time interval \( t \in (0, mT) \), thus we have

\[
\frac{1}{mT} \int_0^{mT} \mathbf{x}^{(m)}(t) dt = a_0^{(m)} + \sum_{j=1}^{\infty} b_{j/m} \frac{1}{m} \Omega \int_0^{mT} \cos \left( \frac{j \Omega t}{m} \right) dt + c_{j/m} \frac{1}{m} \Omega \int_0^{mT} \sin \left( \frac{j \Omega t}{m} \right) dt.
\]

Thus, we have

\[
a_0^{(m)} = \frac{1}{mT} \int_0^{mT} \mathbf{x}^{(m)}(t) dt.
\]

(ii) Multiplication of \( \cos(\Omega t/m) \) to the Fourier expression gives

\[
\mathbf{x}^{(m)}(t) \cos \left( \frac{\Omega t}{m} \right) = a_0^{(m)} \cos \left( \frac{\Omega t}{m} \right) + \sum_{j=1}^{\infty} b_{j/m} \cos \left( \frac{j \Omega t}{m} \right) + \sum_{j=1}^{\infty} c_{j/m} \sin \left( \frac{j \Omega t}{m} \right).
\]

\[
\mathbf{x}^{(m)}(t) \sin \left( \frac{\Omega t}{m} \right) = a_0^{(m)} \sin \left( \frac{\Omega t}{m} \right) + \sum_{j=1}^{\infty} b_{j/m} \sin \left( \frac{j \Omega t}{m} \right) + \sum_{j=1}^{\infty} c_{j/m} \sin \left( \frac{j \Omega t}{m} \right).
\]

\[
\mathbf{x}^{(m)}(t) = a_0^{(m)} + \sum_{j=1}^{\infty} b_{j/m} \cos \left( \frac{j \Omega t}{m} \right) + \sum_{j=1}^{\infty} c_{j/m} \sin \left( \frac{j \Omega t}{m} \right).
\]

If \( j \neq 1 \), all integrals on the right-hand side are zero. For \( j = 1 \), only the integral for the term of \( \cos(j - 1 \Omega t/m) \) is not zero, and other integrals are zero. They are based on the orthogonality of the basis of sine and cosine in the Fourier series expansion. So, we have

\[
b_{1/m} = \frac{2}{mT} \int_0^{mT} \mathbf{x}^{(m)}(t) \cos \left( \frac{\Omega t}{m} \right) dt.
\]

(iii) Multiplication of \( \sin(\Omega t/m) \) to the Fourier expression gives

\[
\mathbf{x}^{(m)}(t) \sin \left( \frac{\Omega t}{m} \right) = a_0^{(m)} \sin \left( \frac{\Omega t}{m} \right) + \sum_{j=1}^{\infty} b_{j/m} \sin \left( \frac{j \Omega t}{m} \right) + \sum_{j=1}^{\infty} c_{j/m} \sin \left( \frac{j \Omega t}{m} \right).
\]

\[
\mathbf{x}^{(m)}(t) \sin \left( \frac{\Omega t}{m} \right) = a_0^{(m)} \sin \left( \frac{\Omega t}{m} \right) + \sum_{j=1}^{\infty} b_{j/m} \sin \left( \frac{j \Omega t}{m} \right) + \sum_{j=1}^{\infty} c_{j/m} \sin \left( \frac{j \Omega t}{m} \right).
\]
The integration of the foregoing equation gives

\[
\int_a^c \sin \left( \frac{j}{m} \Omega t \right) dt = \frac{mT}{2} c_{j/m},
\]

which is also based on the orthogonality of the basis of sine and cosine in the Fourier series expansion. Thus, we have

\[
c_{j/m} = \frac{2}{mT} \int_0^{mT} x^{(m)}(t) \sin \left( \frac{j}{m} \Omega t \right) dt.
\]

In Eq. (320), the piecewise flow is enforced to be continuous, which can be expanded by the Fourier series. This theorem is proved. 

**Remarks**

(i) The piecewise continuous periodic flows in nonlinear dynamical systems cannot be expressed by the Fourier series expansion. Such piecewise continuous periodic flow should be determined through

the discontinuous dynamical system theory (e.g. Filippov, 1988; Luo, 2009, 2012b). (ii) If a periodic flow possesses the kth derivatives that are continuous, then the Fourier series expansion of the periodic flow is convergent with \(1/j^k\). The detailed discussion of the Fourier series theory for periodic functions can be referred to [Churchill, 1941].

**Definition 3.** Consider a nonlinear dynamical system with/without time delay, and such a dynamical system has a period-\(m\) flow \(x^{(m)}(t)\) with finite norm \(||x^{(m)}||\) and period \(mT\) \((T = 2\pi/\Omega)\). If \(x^{(m)}(t)\) is a continuous flow on \(t \in (0, mT)\), there is the finite Fourier series \(T_M^{(m)}(t) \in \mathbb{R}^n\) for the period-\(m\) flow \(x^{(m)}(t) \in \mathbb{R}^n\) as

\[
T_M^{(m)}(t) = a_0^{(m)} + \sum_{j=1}^{M} b_{j/m} \cos \left( \frac{j}{m} \Omega t \right) + c_{j/m} \sin \left( \frac{j}{m} \Omega t \right)
\]

which is called a trigonometric polynomial of order \(M\).

From discrete mapping structures, the node points of periodic flows are computed. Consider the node points of period-\(m\) flows as \(x^{(m)}_k = (x^{(m)}_{1k}, x^{(m)}_{2k}, \ldots, x^{(m)}_{mk})^T\) for \(k = 0, 1, 2, \ldots, mN\) in a nonlinear dynamical system. The approximate expression for period-\(m\) flow is determined by the Fourier series as

\[
x^{(m)}(t) \approx a_0^{(m)} + \sum_{j=1}^{M} b_{j/m} \cos \left( \frac{j}{m} \Omega t \right) + c_{j/m} \sin \left( \frac{j}{m} \Omega t \right)
\]

There are \((2M + 1)\) unknown vector coefficients of \(a_0^{(m)}, b_{j/m}, c_{j/m}\). To determine such unknowns, at least we have the given nodes \(x^{(m)}_k\) \((k = 0, 1, 2, \ldots, mN)\) with \(mN + 1 \geq 2M + 1\). In other words, we have \(M \leq mN/2\). The node points \(x^{(m)}_k\) on the period-\(m\) flow can be expressed by the finite Fourier series as for \(t_k \in [0, mT]\)

\[
x^{(m)}(t_k) \equiv x^{(m)}_k = a_0^{(m)} + \sum_{j=1}^{mN/2} b_{j/m} \cos \left( \frac{j}{m} \Omega t_k \right) + c_{j/m} \sin \left( \frac{j}{m} \Omega t_k \right)
\]
A. C. J. Luo

where

$$T_{mN/2}(t) = a_0^{(m)} + \sum_{j=0}^{mN/2} b_j^{(m)} \cos \left( \frac{k}{m N} \right)$$

Taking the derivative of function $F$ gives the following three cases.

(i) For constant term $a_0^{(m)}$, we have

$$\sum_{k=0}^{mN} [x^{(m)}(t_k) - T_{mN/2}^{(m)}(t_k)] = 0,$$

$$\sum_{k=0}^{mN} x^{(m)}(t_k) - \sum_{k=0}^{mN} a_0^{(m)}$$

$$- \sum_{j=1}^{mN/2} b_j^{(m)} \cos \left( \frac{k}{m N} \right) t_k$$

$$+ \sum_{j=1}^{mN/2} c_j^{(m)} \sin \left( \frac{k}{m N} \right) t_k = 0.$$

Then a trigonometric polynomial $T_{mN/2}^{(m)}(t)$ with minimization of

$$\sum_{k=0}^{mN} (x^{(m)}(t_k) - T_{mN/2}^{(m)}(t_k))^2$$

exists and $x^{(m)}(t)$ can be approximated by $T_{mN/2}^{(m)}(t)$ (i.e.

$$x^{(m)}(t) \approx T_{mN/2}^{(m)}(t).$$

That is,

$$x^{(m)}(t) \approx a_0^{(m)} + \sum_{j=1}^{mN/2} b_j^{(m)} \cos \left( \frac{k}{m N} \right) t_k$$

$$+ c_j^{(m)} \sin \left( \frac{k}{m N} \right) t_k.$$

Proof. Let

$$F = \sum_{k=0}^{mN} (x^{(m)}(t_k) - T_{mN/2}^{(m)}(t_k))^2,$$

where

$$T_{mN/2}^{(m)}(t) = a_0^{(m)} + \sum_{j=1}^{mN/2} b_j^{(m)} \cos \left( \frac{k}{m N} \right)$$

+ $c_j^{(m)} \sin \left( \frac{k}{m N} \right).$

Thus

$$a_0^{(m)} = \frac{1}{mN} \sum_{k=0}^{mN} x^{(m)}(t_k).$$

(ii) For constant term $b_j^{(m)}$, we have

$$\sum_{k=0}^{mN} (x^{(m)}(t_k) - T_{mN/2}^{(m)}(t_k)) \cos \left( \frac{k}{m N} \right) t_k = 0.$$
and

\[ \sum_{k=0}^{mN} x^{(m)}(t_k) \cos \left( \frac{j}{mN} \Omega t_k \right) - a_0^{(m)} \sum_{k=0}^{mN} \cos \left( \frac{j}{mN} \Omega t_k \right) \]

- \sum_{j_1=1}^{mN/2} b_{j_1/m} \sum_{k=0}^{mN} \cos \left( \frac{j_1}{mN} \Omega t_k \right) \cos \left( \frac{j}{mN} \Omega t_k \right) \]

- \sum_{j_1=1}^{mN/2} c_{j_1/m} \sum_{k=0}^{mN} \sin \left( \frac{j_1}{mN} \Omega t_k \right) \cos \left( \frac{j}{mN} \Omega t_k \right) = 0.\]

Further

\[ \sum_{k=0}^{mN} x^{(m)}(t_k) \cos \left( \frac{j}{mN} \Omega t_k \right) - a_0^{(m)} \sum_{k=0}^{mN} \cos \left( \frac{j}{mN} \Omega t_k \right) \]

- \sum_{j_1=1}^{mN/2} b_{j_1/m} \sum_{k=0}^{mN} \cos \left( \frac{j_1}{mN} \Omega t_k \right) + \cos \left( \frac{j_1 + j}{mN} \Omega t_k \right) \]

- \sum_{j_1=1}^{mN/2} c_{j_1/m} \sum_{k=0}^{mN} \sin \left( \frac{j_1}{mN} \Omega t_k \right) \cos \left( \frac{j_1 + j}{mN} \Omega t_k \right) + \sin \left( \frac{j_1 + j}{mN} \Omega t_k \right) = 0.\]

If \( j_1 \neq j \), we have

\[ \sum_{k=0}^{mN} \cos \left( \frac{j_1 - j}{m} \Omega t_k \right) \approx 0 \quad \text{and} \quad \sum_{k=0}^{mN} \cos \left( \frac{j_1 + j}{m} \Omega t_k \right) \approx 0. \]

\[ \sum_{k=0}^{mN} \sin \left( \frac{j_1 - j}{m} \Omega t_k \right) \approx 0 \quad \text{and} \quad \sum_{k=0}^{mN} \sin \left( \frac{j_1 + j}{m} \Omega t_k \right) \approx 0. \]

If \( j_1 = j \), we have

\[ \sum_{k=0}^{mN} \cos \left( \frac{j_1 - j}{m} \Omega t_k \right) = mN \quad \text{and} \quad \sum_{k=0}^{mN} \sin \left( \frac{j_1 - j}{m} \Omega t_k \right) = 0. \]
where $\Omega = \pi / h$ and period $mT = (2\pi / \Omega)$, consider the node points of period-$m$ flows in a nonlinear dynamical system as $\mathbf{x}_k^{(m)} = (x_k^{(m)}, x_{k+1}^{(m)}, \ldots, x_{k+N-1}^{(m)})^T$ for $k = 0, 1, 2, \ldots, mN$. The integration in the coefficients of the Fourier series is by the interpolation of the discrete nodes. Let $h = \Delta t = T/N$ where $T = 2\pi / \Omega$ and $\mathbf{x}_k^{(m)}(t_0) = \mathbf{x}_k^{(m)}(t_0)$. For simplicity, let $t_0 = 0$. Application of the trapezoidal rules to Euler formulas for the coefficients of the Fourier series produces the discrete Euler formulas.

(i) The constant term $a_0^{(m)}$ is discussed as follows.

\[
a_0^{(m)} = \frac{1}{mT} \int_0^{mT} x_k^{(m)}(t) \, dt
\]

\[
= \frac{1}{mT} \left[ \frac{1}{2} x_k^{(m)}(t_0) + x_k^{(m)}(t_1) + \cdots + x_k^{(m)}(t_{mN-1}) + \frac{1}{2} x_k^{(m)}(t_{mN}) \right] h
\]

\[
- \frac{1}{12mT} \sum_{k=1}^{mN} \left| \frac{d^2 x_k^{(m)}(t)}{dt^2} \right|_{t=t_k} t_{k-1}^{t_k} \tag{327}
\]

where $t_k \in [t_{k-1}, t_k]$ for $k = 1, 2, \ldots, mN$. Letting $\max(||d^2 x_k^{(m)}(t)/dt^2||_{t_{k-1}^{t_k}}) = L$, we have

\[
\left\| a_k^{(m)} - \frac{1}{mN} \sum_{j=0}^{mN} x_k^{(m)}(t_j) \right\| \leq \frac{h^2}{12} L. \tag{328}
\]

Thus,

\[
a_0^{(m)} \approx \frac{1}{mN} \sum_{k=0}^{mN} x_k^{(m)}(t_k) \approx \frac{1}{mN} \sum_{k=0}^{mN} x_k^{(m)}. \tag{329}
\]

(ii) The cosine terms coefficients $b_{j/m}$ ($j = 1, 2, \ldots, mN/2$) are discussed as follows.

\[
b_{j/m} = \frac{2}{mT} \int_0^{mT} x_k^{(m)}(t) \cos \left( \frac{j}{m} \Omega t \right) \, dt
\]

\[
= \frac{2}{mT} \left[ \frac{1}{2} x_k^{(m)}(t_0) \cos \left( \frac{j}{m} \Omega t_0 \right) \right]
\]

\[
+ x_k^{(m)}(t_1) \cos \left( \frac{j}{m} \Omega t_1 \right) + \cdots
\]

\[
+ x_k^{(m)}(t_{mN}) \cos \left( \frac{j}{m} \Omega t_{mN} \right) h
\]

\[
- \frac{h^3}{12mT} \sum_{k=1}^{mN} \left| \frac{d^2 x_k^{(m)}(t)}{dt^2} \right|_{t=t_k} \tag{330}
\]

From the foregoing equations, we have

\[
\left\| b_{j/m} - \frac{2}{mN} \sum_{k=0}^{mN} x_k^{(m)}(t_k) \cos \left( \frac{j}{m} \Omega t_k \right) \right\| \leq \frac{h^2}{6} L_1. \tag{331}
\]

where $\max(||d^2 x_k^{(m)}(t) \cos(j\Omega t)/dt^2||_{t_{k-1}^{t_k}}) = L_1$. Thus, the cosine coefficients in discrete Fourier series is

\[
b_{j/m} \approx \frac{2}{mN} \sum_{k=0}^{mN} x_k^{(m)}(t_k) \cos \left( \frac{j}{m} \Omega t_k \right)
\]

\[
\approx \frac{2}{mN} \sum_{k=0}^{mN} x_k^{(m)} \cos \left( \frac{j}{m} \Omega t_k \right). \tag{332}
\]

(iii) The sine terms coefficients $c_{j/m}$ ($j = 1, 2, \ldots, mN/2$) can be discussed similarly. That is,

\[
c_{j/m} = \frac{2}{mT} \int_0^{mT} x_k^{(m)}(t) \sin \left( \frac{j}{m} \Omega t \right) \, dt
\]

\[
= \frac{2}{mT} \left[ \frac{1}{2} x_k^{(m)}(t_0) \sin \left( \frac{j}{m} \Omega t_0 \right) \right]
\]

\[
+ x_k^{(m)}(t_1) \sin \left( \frac{j}{m} \Omega t_1 \right) + \cdots
\]

\[
+ x_k^{(m)}(t_{mN}) \sin \left( \frac{j}{m} \Omega t_{mN} \right) h
\]

\[
- \frac{h^3}{12mT} \sum_{k=1}^{mN} \left| \frac{d^2 x_k^{(m)}(t)}{dt^2} \right|_{t=t_k} \tag{333}
\]

\[
\left\| c_{j/m} - \frac{2}{mN} \sum_{k=0}^{mN} x_k^{(m)}(t_k) \sin \left( \frac{j}{m} \Omega t_k \right) \right\| \leq \frac{h^2}{6} L_2. \tag{334}
\]

\[
\left\| c_{j/m} - \frac{2}{mN} \sum_{k=0}^{mN} x_k^{(m)}(t_k) \sin \left( \frac{j}{m} \Omega t_k \right) \right\| \leq \frac{h^2}{6} L_2. \tag{334}
\]
The foregoing equation can be expressed as

\begin{align*}
+ x^{(m)}(t_1) \sin \left( \frac{1}{m} \Omega t_1 \right) + \ldots
+ x^{(m)}(t_{mN-1}) \sin \left( \frac{1}{m} \Omega t_{mN-1} \right)
+ \frac{1}{2} b^{(m)}(t_mN) \sin \left( \frac{1}{m} \Omega t_mN \right) \right) h
- \frac{b^2}{6m^2} \sum_{k=1}^{mN} \frac{1}{m^2} \left[ x^{(m)}(t) \sin \left( \frac{1}{m} \Omega t \right) \right]_{t=t_k}^t \, dt.
\end{align*}

From the foregoing equations, we have

\begin{align}
|b_{j/m} - \frac{2}{mN} \sum_{k=0}^{mN} x^{(m)}(t_k) \cos \left( \frac{j}{m} \Omega t_k \right)| & \leq \frac{h^2}{6} L_2, \quad (334)
\end{align}

where \( \max_k ||d^2[x^{(m)}(t) \sin(\Omega t)/m]/dt^2||_{t=t_k} = L_2 \). Thus, the cosine coefficients in discrete Fourier series is

\begin{align}
b_{j/m} & \approx \frac{2}{mN} \sum_{k=0}^{mN} x^{(m)}(t_k) \cos \left( \frac{j}{m} \Omega t_k \right),
\end{align}

In fact, other interpolation can be used to obtain the Euler formulas, which is not presented.

The harmonic amplitudes and harmonic phases for period-\( m \) motion are

\begin{align}
A_j/m & = \sqrt{b_{j/m}^2 + c_{j/m}^2},
\phi_j/m & = \arctan \frac{c_{j/m}}{b_{j/m}} \quad (s = 1, 2, \ldots, n).
\end{align}

Thus the approximate expression for period-\( m \) motion in Eq. (324) is determined by

\begin{align}
x^{(m)}(t) \approx a_0^{(m)} + \sum_{j=1}^{mN/2} b_{j/m} \cos \left( \frac{j}{m} \Omega t \right) + c_{j/m} \sin \left( \frac{j}{m} \Omega t \right).
\end{align}

The foregoing equation can be expressed as

\begin{align}
x^{(m)}_s(t) = a_0^{(m)} + \sum_{j=1}^{mN/2} A_{j/m} \cos \left( \frac{j}{m} \Omega t - \phi_{j/m} \right)
\end{align}

\begin{align}
(s = 1, 2, \ldots, n).
\end{align}

\section{5. An Application}

Consider the Duffing oscillator as:

\begin{align}
\ddot{x} + \delta \dot{x} - \alpha x + \beta x^3 = Q_0 \cos \Omega t. \quad (339)
\end{align}

The state equation of the above equation in state space is

\begin{align}
\dot{x} = y 
\dot{y} = Q_0 \cos \Omega t - \delta \dot{x} + \alpha x - \beta x^3.
\end{align}

The differential equation in Eq. (339) can be discretized by a midpoint scheme for the time interval \( t \in [t_k, t_{k+1}] \) to form a map \( P_k \) (\( k = 0, 1, 2, \ldots \)) as

\begin{align}
P_k : (x_{k-1}, y_{k-1}) \rightarrow (x_k, y_k)
\Rightarrow (x_k, y_k) = P_k(x_{k-1}, y_{k-1})
\end{align}

with the implicit relation as

\begin{align}
x_k = x_{k-1} + \frac{1}{2} h(y_{k-1} - y_k),
y_k = y_{k-1} + h \left[ Q_0 \cos \Omega \left( t_{k-1} + \frac{1}{2} \right) \right.
\end{align}

\begin{align}
- \frac{1}{2} \delta(y_{k-1} - y_k) + \frac{1}{2} \alpha(x_{k-1} + x_k)
\end{align}

\begin{align}
- \frac{1}{8} \beta(x_{k-1} + x_k)^3 \right].
\end{align}

Once the period-doubling bifurcation of the period-1 motions occurs, the period-2 motions will appear. If the period-doubling bifurcation of the period-2 motion occurs, the period-4 motions will appear, and so on. In addition, other periodic motions will exist. In general, to predict the period-\( m \) motions in such a Duffing oscillator analytically, consider a mapping structure as follows

\begin{align}
P = P_{mN} \circ P_{mN-1} \circ \cdots \circ P_3 \circ P_2 : x^{(m)}_{mN} \rightarrow x^{(m)}_{mN}.
\end{align}
The corresponding periodicity condition is
\[ m \text{obtained, the stability of periodic motion can be determined by} \]
\[ \frac{\partial x_k^{(m)}}{\partial x_{k-1}^{(m)}} = \frac{\partial y_k^{(m)}}{\partial y_{k-1}^{(m)}} \]
\[ \Rightarrow \begin{cases} x_k^{(m)} & = P_k(x_{k-1}^{(m)}, y_{k-1}^{(m)}) \\ y_k^{(m)} & = P_k(x_{k-1}^{(m)}, y_{k-1}^{(m)}) \end{cases} \quad (k = 1, 2, \ldots, mN). \]

From Eq. (343), the corresponding algebraic equations are
\[ x_k^{(m)} = x_{k-1}^{(m)} + \frac{1}{2} h(y_{k-1}^{(m)} + y_k^{(m)}), \]
\[ y_k^{(m)} = y_{k-1}^{(m)} + h \left[ \Omega y_{k-1}^{(m)} \frac{t_{k-1} + \frac{1}{2} k}{2} \right] \]
\[ + \frac{1}{2} h(x_{k-1}^{(m)} + x_k^{(m)}) - \frac{1}{8} h(x_{k-1}^{(m)} + x_k^{(m)}) \]
\[ P_k \quad (k = 1, 2, \ldots, mN). \]

(345)

The corresponding periodicity condition is
\[ (x_k^{(m)}, y_k^{(m)}) = (x_0^{(m)}, y_0^{(m)}). \]

(346)

From Eqs. (345) and (346), values of nodes at the discretized Duffing oscillator can be determined by \(2(mN + 1)\) equations. Once the node points \(x_k^{(m)}, y_k^{(m)}\) in period-\(m\) motion are obtained, the stability of period-\(m\) motion can be discussed by the corresponding Jacobian matrix.

For a small perturbation in vicinity of \(x_k^{(m)}, y_k^{(m)}\), \(x_k^{(m)} = x_k^{(m)} + \Delta x_k^{(m)}, (k = 0, 1, 2, \ldots, mN)\), we have
\[ \Delta x_{mN} = D P \Delta x_k^{(m)} \]
\[ = \left( D P_m \right) \left( \cdots \left( D P_1 \right) \Delta x_k^{(m)} \right) \quad \text{mN-scaling} \]

(347)

with
\[ \Delta x_k^{(m)} = D P_k \Delta x_{k-1}^{(m)} \]
\[ \Rightarrow \frac{\partial x_k^{(m)}}{\partial x_{k-1}^{(m)}} \Delta x_{k-1}^{(m)} = \Delta x_k^{(m)} \quad \text{for} \quad (k = 1, 2, \ldots, mN). \]

(348)

where
\[ D P_k = \begin{bmatrix} \frac{\partial x_k^{(m)}}{\partial x_{k-1}^{(m)}} & \frac{\partial x_k^{(m)}}{\partial y_{k-1}^{(m)}} \\ \frac{\partial y_k^{(m)}}{\partial x_{k-1}^{(m)}} & \frac{\partial y_k^{(m)}}{\partial y_{k-1}^{(m)}} \end{bmatrix} \left( x_k^{(m)}, y_k^{(m)} \right) \]

(349)

To measure stability and bifurcation of period-\(m\) motion, the eigenvalues are computed by
\[ \left| D P - \lambda \mathbf{I} \right| = 0 \]

(350)

where
\[ D P = \prod_{k=1}^{mN} \left( \frac{\partial x_k^{(m)}}{\partial x_{k-1}^{(m)}} \right) \quad \text{mN-scaling} \]

(351)

Similarly, the stability and bifurcation conditions are determined by the eigenvalue analysis.

5.1. Analytical predictions of bifurcation trees

From the foregoing section, the node points of period-\(m\) motions for the Duffing oscillator can be computed, and the set of node points of periodic motions with \(N\) points per period \(T = 2\pi/\Omega\) is defined as
\[ \left\{ (x_k, y_k) \ \middle| t_k = t_0 + \frac{kT}{N}, t_0 = 0; \right\} \quad \text{with} \quad T = \frac{2\pi}{\Omega} \quad k = 0, 1, 2, \ldots. \]

(352)

The periodicity of period-\(m\) motion is \( (x_k, y_k) = (x_{k+mN}, y_{k+mN}) \). From all analytical predictions of the node points of periodic motion, the FFT can provide the harmonic amplitudes and phases, which will be presented in this section. To avoid presenting all node points of periodic motions, the node points relative to the initial condition point for each period

1550044-54
are collected in the Poincaré mapping section for period-
motions \((m = 1, 2, \ldots, 4)\), as defined by

\[
\sum_{i=0}^{m-1} \left\{ \left( x_{\text{mod}(i,N)}; y_{\text{mod}(i,N)} \right) \left| t_i = t_0 + \frac{kT}{N} \right. \right\}
\]

where \(t_0 = 0; \quad T = \frac{2\pi}{\Omega}; \quad k = 0, 1, 2, \ldots \). \hspace{1cm} (353)

which will be used to present periodic motions.

In this section, analytical predictions of the bifurcation trees of period-1 motions to chaos in the Duffing oscillator will be presented, and the corresponding stability and bifurcation analysis will be completed through the eigenvalue analysis of discrete mapping structures of periodic motions. Consider system parameters

\[
\delta = 1.0; \quad \alpha = 5.5; \quad \beta = 20.0; \quad Q_0 = 10.0.
\hspace{1cm} (354)
\]

As in [Luo & Guo, 2014], the bifurcation trees of the period-1 to period-4 motions are presented in Figs. 11(a) and 11(b). \hspace{1cm} (355)

Periodic Node Displacement, \(x_{\text{mod}(i,N)}\)

Periodic Node Velocity, \(y_{\text{mod}(i,N)}\)

Fig. 11. Analytical prediction of bifurcation trees of period-1 motions to chaos: (a) Periodic node displacement \(x_{\text{mod}(i,N)}\). (b) Periodic node velocity \(y_{\text{mod}(i,N)}\). (\(\alpha = 5.5, \beta = 20.0, \delta = 1.0, Q_0 = 10.0\).)
and they are the saddle-node bifurcation for the period-4 motions. The period-4 motions are in the range of Ω ∈ (1.52, 1.90) for the second branch and Ω ∈ (4.97, 6.58) for the third branch. For the third branch, the stable period-4 motions are in Ω ∈ (4.97, 5.03) and Ω ∈ (6.49, 6.58), and the unstable period-4 motions are in Ω ∈ (5.03, 6.49). The period-doubling bifurcations of period-4 motion in the third branch are at Ω ≈ (5.03, 6.49), which is the saddle-node bifurcation for period-8 motion. Thus, the period-8 motions exist for Ω ∈ (5.03, 6.49). Continuously, we can obtain period-16 motions to chaos. Because the stable motions for period-8 or higher-order periodic motions exist for the short range of excitation frequency, the bifurcation tree of period-1 motion to chaos will not be computed anymore further.

5.2. Bifurcation trees of period-1 motion to chaos

For simplicity, only the excitation frequency-amplitude curves for displacement $x^{(m)}(t)$ are presented. Similarly, the frequency-amplitudes for velocity $y^{(m)}(t)$ can also be determined. Thus the displacement can be expressed as

$$x^{(m)}(t) \approx a_0^{(m)} + \sum_{j=1}^{mN/2} b_{j/m} \cos \left( \frac{2 \pi j m t}{m} \right) \cos \left( \frac{2 \pi \Omega t}{m} \right)$$

$$+ c_{j/m} \sin \left( \frac{2 \pi j m t}{m} \right) \sin \left( \frac{2 \pi \Omega t}{m} \right)$$

(355)

and

$$x^{(m)}(t) \approx a_0^{(m)} + \sum_{j=1}^{mN/2} A_{j/m} \cos \left( \frac{2 \pi j m t}{m} \right) - \varphi_{j/m}$$

(356)

where

$$A_{j/m} = \sqrt{b_{j/m}^2 + c_{j/m}^2}, \quad \varphi_{j/m} = \arctan \frac{c_{j/m}}{b_{j/m}}$$

(357)

To discuss nonlinear behaviors of period-m motions for the Duffing oscillator, the frequency-amplitude for displacement will be presented as follows. The acronyms SN and PD are the saddle-node and period-doubling bifurcations for period-m motions, respectively. In all plots, the unstable and stable solutions of period-m motions are represented by the dashed and solid curves, respectively.

As in [Luo & Guo, 2014], the bifurcation trees of period-1 motion to chaos will be presented in Fig. 12 through the period-1 to period-4 motions. The given parameters are listed in Eq. (355). The constant term $a_0^{(m)}$ ($m = 1, 2, 4$) is presented in Fig. 12(a) for the solution center on the right side of the y-axis. The bifurcation tree is clearly observed. For the solution center on the left side of the y-axis, we have $a_0^{(m)} = -a_0^{(m)} R$. For the symmetric period-m motion, we have $a_0 = 0$, labeled by “S”. However, for asymmetric period-m motion, we have $a_0^{(m)} \neq 0$, labeled by “A”. For the symmetric period-1 motion to an asymmetric period-1 motion, the saddle-node bifurcation will occur. The saddle-node bifurcations are at Ω ≈ 1.016, 1.23, 1.50, 2.63, 4.528. For such saddle-node bifurcations, the asymmetric periodic motions appear, and the symmetric motions are from the stable to unstable solution or from the unstable to stable solution. The saddle-node bifurcations for symmetric motion jumping points are at Ω ≈ 1.46, 1.513, 3.86, 5.98. The symmetric period-1 motion are only from the stable to unstable solution or from the unstable to stable solution. When the asymmetric period-1 motion experiences a period-doubling bifurcation, the period-2 motions will appear and the asymmetric period-1 motion is from the stable to unstable solution. The frequencies of Ω ≈ 1.517, 1.97, 4.528, 7.27 are not only for the period-doubling bifurcations of the asymmetric period-1 motions but also for the saddle-node bifurcations of the period-2 motion. When the period-2 motion possesses a period-doubling bifurcation, the period-4 motion appears and the period-2 motion is from the stable to unstable solution. The frequencies of Ω ≈ 1.52, 1.90, 4.97, 6.58 are for the period-doubling bifurcations of period-2 motions and for the saddle-node bifurcation for the period-4 motions. The frequencies of Ω ≈ 5.03, 6.49 are for the period-doubling bifurcations of period-4 motions and for the saddle-node bifurcation for the period-8 motions. All period-2 and period-4 motions are on the branches of asymmetric period-1 motions, and the centers of the periodic motions are on the right side of the y-axis. In Fig. 12(b), the harmonic amplitude $A_{j/m}$ is presented. For period-1 and period-2 motions, $A_{j/2} = 0$. The saddle-node bifurcations are at Ω ≈ 4.97, 6.58 for period-1 motion, and the period-doubling bifurcations are at Ω ≈ 5.03, 6.49. The bifurcation points are clearly observed, and the quantity of the harmonic
Fig. 12. Frequency-amplitude characteristics for bifurcation trees of period-1 to period-4 motions: (a) $a_0^{(m)}$ ($m = 1, 2, 4$). (b)-(f) $A_k/n$ ($m = 4, k = 1, 2, 4, 84, 244$). ($\alpha = 5.5, \beta = 20.0, \delta = 1.0, Q_0 = 10$); $\text{mod}(k, N) = 0$. 

1550044-57
amplitude for period-4 motion is $A_{1/4} \approx 7 \times 10^{-2}$. In Fig. 12(c), the harmonic amplitude $A_{1/2}$ for period-2 and period-4 motions are presented. For the second branch, only the period-2 motion are presented because the stability range of period-4 motion is very small and more discrete nodes are needed to obtain such a period-4 motion. For the third branch, the bifurcation trees for period-2 to period-4 motions are clearly illustrated. The period-doubling bifurcations are at $\Omega \approx 5.03, 6.49$ for the third branch. The saddle-node bifurcations of the period-2 motion are at $\Omega \approx 1.517, 1.97, 4.528, 7.27$ for the second and third branches. The quantity level of the harmonic amplitude $A_{1/2}$ is $A_{1/2} \sim 1.5 \times 10^{-1}$. In Fig. 12(d), the primary harmonic amplitudes $A_1$ versus excitation frequency $\Omega$ are presented for the period-1 to period-4 motion. The bifurcation points are presented as discrete nodes. The initial conditions and the corresponding periodic motions are attached to the symmetric period-1 motions and the relative period-2 and period-4 motions are presented. For $\Omega > 1$, we can use about 80 harmonic terms to approximate period-1, period-2 and period-4 motions; for $\Omega < 1$, we can use 250 harmonic terms to approximate period-1, period-2, and period-4 motions. For $\Omega \approx 0$, the infinite harmonic terms should be adopted to approximate the periodic motions.

5.3. Numerical simulations

In this section, numerical illustrations are given from the semi-analytical solutions and numerical integration schemes. The initial conditions in numerical simulations are obtained from analytical prediction of periodic solutions. In all plots for illustration, circular symbols gives analytical predictions, and solid curves give numerical simulation results. Acronym "IC" represents initial conditions. The initial points and the corresponding periodic points are depicted by the large circular symbols.

As in [Luo & Guo, 2014], consider excitation frequency $\Omega = 1.05$ to demonstrate period-1 motion in Fig. 13. Other parameters are presented in Eq. (354). The analytical prediction results match very well with numerical simulation results. The harmonic amplitude spectrum is presented in Fig. 13(b). The constant term is $a_0 \approx 0.1235$. The main harmonic amplitudes are $A_1 \approx 0.9413$, $A_2 \approx 0.0949$, $A_3 \approx 0.1620$, $A_4 \approx 0.0924$, $A_5 \approx 0.2699$, $A_6 \approx 0.978$, $A_7 \approx 0.1035$, $A_8 \approx 0.0277$, $A_9 \approx 0.0195$, $A_{10} \approx 0.0166$, $A_{11} \approx 0.0153$, $A_{12} \approx 0.0134$. The other harmonic amplitudes are $A_j \in (10^{-9}, 10^{-7})$ ($j = 13, 14, \ldots, 50$) and $A_{50} \approx 3.6100e^{-9}$. The harmonic amplitudes decrease very slowly with harmonic order. For this
is a trum is presented in Fig. 13(b). The constant term simulation results. The harmonic amplitude spec-
motions is presented in Fig. 13(a). The analytical harmonic amplitude. Initial condition (\(x_0, y_0\)) \(\approx (1.231856, 0.699067)\). The trajectory of period-2 motions is presented in Fig. 13(a). The analytical prediction results match very well with numerical simulation results. The harmonic amplitude spectrum is presented in Fig. 13(b). The constant term is \(a_0^{(2)} \approx 0.3830\). The main harmonic amplitudes are \(A_{1/2} \approx 0.0669, A_1 \approx 0.8889, A_{11/2} \approx 0.0498, A_3 \approx 0.3134, A_{13/2} \approx 0.2163, A_4 \approx 0.1829, A_{17/2} \approx 0.0247, A_5 \approx 0.0302, A_{19/2} \approx 0.0622, A_7 \approx 0.02779, A_{11/2} \approx 0.0307, A_8 \approx 0.0186, A_{13/2} \approx 0.0131, A_9 \approx 5.1982e-3, A_{17/2} \approx 0.0118\). The other harmonic amplitudes are \(A_{j/2} \in (10^{-9}, 10^{-2}) (j = 16, 17, \ldots, 60)\) and \(A_{70} \approx 7.2782e-10\). For this period-2 motion, we cannot use a few harmonic terms to approximate the periodic solutions. From the harmonic amplitudes, at least 15 harmonic terms plus constant term should be included to obtain the rough estimate of periodic motion. Other period-1 to period-4 motions for such a Duffing oscillator can be found from [Luo & Gao, 2014].

6. Conclusions

In this paper, the methodology for periodic flows in continuous nonlinear dynamical systems was discussed through discretization of differential equations of the nonlinear dynamical systems. Based on mapping structures, periodic flows in nonlinear dynamical systems were predicted and the corresponding stability and bifurcations of the periodic flows were determined through the eigenvalue analysis. The periodic flows predicted by the single-step implicit maps were presented first, and the periodic flows predicted by the multiple-step implicit maps were discussed as well. The periodic flows in time-delay nonlinear dynamical systems discussed by the single-step and multiple-step implicit maps were presented. The time-delay nodes in discretization of time-delay nonlinear systems were treated by both the interpolation and direct integration. Based on the discrete nodes of periodic flows in nonlinear dynamical systems with/without time-delay nonlinear systems, the discrete Fourier series responses of periodic flows were presented, and the harmonic amplitudes in the discrete Fourier series can be determined through the discrete nodes of periodic flows. To demonstrate the methodology, the Duffing oscillator was presented as a sampled problem. The analytical prediction of periodic motion in the bifurcation trees of period-1 motion to chaos was presented, and the bifurcation trees were presented through the harmonic amplitudes varying with excitation frequency. Numerical simulation results compared to the analytically predicted periodic motions was completed. The numerical results and predicted results match very well. This method is based on the implicit maps that are obtained from discretization of differential equations of nonlinear dynamical
systems. This method is a semi-analytical method. The analytical prediction is based on the mapping structures of discrete implicit maps. The method presented in this paper can be applied to nonlinear dynamical systems, which cannot be solved by full analytical methods presented in [Luo, 2014a, 2014b].

References
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