The origin of bursts and heavy tails in human dynamics

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Abstract

The dynamics of many social, technological and economic phenomena are driven by individual human actions, turning the quantitative understanding of human behavior into a central question of modern science. Current models of human dynamics, used from risk assessment to communications, assume that human actions are randomly distributed in time and thus well approximated by Poisson processes [1, 2, 3]. In contrast, there is increasing evidence that the timing of many human activities, ranging from communication to entertainment and work patterns, follow non-Poisson statistics, characterized by bursts of rapidly occurring events separated by long periods of inactivity [4, 5, 6, 7, 8]. Here we show that the bursty nature of human behavior is a consequence of a decision based queuing process [9, 10]: when individuals execute tasks based on some perceived priority, the timing of the tasks will be heavy tailed, most tasks being rapidly executed, while a few experience very long waiting times. In contrast, priority blind execution is well approximated by uniform interevent statistics. These findings have important implications from resource management to service allocation in both communications and retail.

Humans participate on a daily basis in a large number of distinct activities, ranging from electronic communication, such as sending emails or making phone calls, to browsing the web, initiating financial transactions, or engaging in entertainment and sports. Given the number of factors that determine the timing of each action, ranging from work and sleep patterns to resource availability, it appears impossible to seek regularities in human dynamics, apart from the obvious daily and seasonal periodicities. Therefore, in contrast with the accurate predictive tools common in physical sciences, forecasting human and social patterns remains a difficult and often elusive goal.

Current models of human activity are based on Poisson processes, and assume that in a dt time interval an individual (agent) engages in a specific action with probability qdt, where q is the overall frequency of the monitored activity. This model predicts that the time interval between two consecutive actions by the same individual, called the waiting or inter-event time, follows an exponential distribution (Fig. 1, a-c) [1]. Poisson processes are widely used to quantify the consequences of human actions, such as modelling traffic flow patterns or accident frequencies [1], and are commercially used in call center staffing [2], inventory control [3], or to estimate the number of congestion caused blocked calls in mobile communications [4]. Yet, an increasing number of recent measurements indicate that the timing of many human actions systematically deviate from the Poisson prediction, the waiting or inter-event times being better approximated by a heavy tailed or Pareto distribution (Fig. 1, d-f). The differences between Poisson and heavy tailed behavior is striking: a Poisson distribution decreases exponentially, forcing the consecutive events to follow each other at relatively regular time intervals and forbidding very long waiting times. In contrast, the slowly decaying heavy tailed processes allow for very long periods of inactivity that separate bursts of intensive activity (Fig. 1).

To provide direct evidence for non-Poisson activity patterns in individual human behavior, we study the communication between several thousand email users based on a dataset capturing the sender, recipient, time and size of each email [11, 12]. As Figure 2a shows, the distribution of the time differences between consecutive emails sent by a selected user is best approximated with $P(\tau) \sim \tau^{-\alpha}$, where $\alpha \simeq 1$, indicating that an individual's email pattern has a bursty non-Poisson character: during a single session a user sends out several emails in a quick succession, followed by long periods of no email activity. This behavior is not limited to email communications. Measurements capturing the distribution of the time differences between consecutive instant messages sent by individuals during online chats [5] show a similar pattern. Professional tasks, such as the timing of job submissions on a supercomputer [6], directory listings and file transfers (FTP requests) initiated by individual users [7], or the timing of printing jobs submitted by users [13] were also reported to display non-Poisson features. Similar patterns emerge in the time interval distribution between individual trades in currency futures [8]. Finally, heavy tailed distributions characterize entertainment related events, such as the time intervals between consecutive online games played by the same user [14].

The fact that a wide range of human activity patterns follow non-Poisson statistics suggests that the observed bursty character reflects some fundamental and potentially generic feature of human dynamics. Yet, the mechanism responsible for these striking non-random features remain unknown. Here we show that the bursty nature of human dynamics is a consequence of a queuing process driven by human decision making: whenever an individual is presented with multiple tasks and chooses among them based on some perceived priority parameter, the waiting time of the various tasks will be Pareto distributed. In contrast, first-come-first-serve and random task execution, common in most service oriented or computer driven environments, lead to a uniform Poisson like dynamics.

Most human initiated events require an individual to weigh and prioritize different activities. For example, at the end of each activity an individual needs to decide what to do next: send an email, do some shopping, or place a phone call, allocating time and resources for the chosen activity. Consider an agent operating with a priority list of L tasks. After a task is executed, it is removed from the list, offering the opportunity to add another task. The agent assigns to each task a priority parameter x, which allows it to compare the urgency of the different tasks on the list. The question is, how long will a given task have to wait before it is executed. The answer depends on the method the agent uses to choose the task to be executed next. In this respect three selection protocols [10] are particularly relevant for human dynamics:

(i) The simplest is the first-in-first-out protocol, executing the tasks in the order they were added to the list. This is common in service oriented processes, like the first-come-first-serve execution of orders in a restaurant or getting help from directory assistance and consumer support. The time period an item stays on the list before execution is determined by the cumulative time required to perform all tasks added to the list before it. If the time necessary to perform the individual tasks are chosen from a bounded distribution (*i.e.* the second moment of the distribution is finite), then the waiting time distribution will develop an exponential tail, indicating that most tasks experience approximately the same waiting time.

(ii) The second possibility is to execute the tasks in a random order, irrespective of their priority or time spent on the list. This is common, for example, in educational settings, when students are called on randomly, and in some packet routing protocols. The waiting time distribution of the individual tasks (i.e. the time between two calls on the same student) in this case is also exponential.

(iii) In most human initiated activities task selection is not random, but the individual executes the highest priority item on its list. The resulting execution dynamics is quite different from (i) and (ii): high priority tasks will be executed soon after their addition to the list, while low priority items will have to wait until all higher priority tasks are cleared, forcing them to stay on the list for considerable time intervals. In the following we show that this selection mechanism, practiced by humans on a daily basis, is the likely source of the fat tails observed in human initiated processes.

We assume that an individual has a priority list with L tasks, each task being assigned a priority parameter $x_i,\ i = 1, ..., L$, chosen from a $\rho(x)$ distribution. At each time step the agent selects the highest priority task from the list and executes it, removing it from the list. At that moment a new task is added to the list, its priority x_i being again chosen from $\rho(x)$. This simple model ignores the possibility that the agent occasionally selects a low priority item for execution before all higher priority items are done, common, for example, for tasks with deadlines. This can be incorporated by assuming that the agent executes the highest priority item with probability p, and with probability $1 - p$ executes a randomly selected task, independent of its priority. Thus the $p \to 1$ limit of the model describes the deterministic (iii) protocol, when always the highest priority task is chosen for execution, while $p \to 0$ corresponds to the random choice protocol discussed in (ii).

To establish that this priority list model can account for the observed fat tailed interevent time distribution, we first studied its dynamics numerically with priorities chosen from a uniform distribution $x_i \in [0, 1]$. Computer simulations show that in the $p \to 1$ limit the probability that a task spends τ time on the list has a power law tail with exponent $\alpha = 1$ (Fig 3a), in agreement with the exponent obtained in email communications (Fig 2a). In the $p \to 0$ limit $P(\tau)$ follows an exponential distribution (Fig. 3b), as expected for the case (ii). As the typical length of the priority list differs from individual to individual, it is particularly important for the tail of $P(\tau)$ to be independent of L. Numerical simulations indicate that this is indeed the case: changes in L do not affect the scaling of $P(\tau)$. The fact that the scaling holds for $L = 2$ indicates that it is not necessary to have a long priority list: as long as individuals balance at least two tasks, a bursty heavy tailed interevent dynamics will emerge.

To determine the tail of $P(\tau)$ analytically we consider a stochastic version of the model in which the probability to choose a task with priority x for execution in a unit time is $\Pi(x) \sim x^\gamma$, where γ is a parameter that allows us to interpolate between the random choice limit (ii) ($\gamma = 0$, $p = 0$) and the deterministic case, when always the highest priority item is chosen for execution (iii) ($\gamma = \infty$, $p = 1$). Note that this parameterization captures the scaling of the model only in the $p \to 0$ and $p \to 1$ limits, but not for intermediate p values, thus it is chosen only for mathematical convenience. The probability that a task with priority x waits a time interval t before execution is $f(x,t) = (1 - \Pi(x))^{t-1}\Pi(x)$. The average waiting time of a task with priority x is obtained by averaging over t weighted with $f(x,t)$, providing

$$\tau(x) = \sum_{t=1}^{\infty} t f(x,t) = \frac{1}{\Pi(x)} \sim \frac{1}{x^\gamma}, \tag{1}$$

i.e. the higher an item's priority, the shorter is the average time it waits before execution. To calculate $P(\tau)$ we use the fact that the priorities are chosen from the $\rho(x)$ distribution, i.e. $\rho(x)dx = P(\tau)d\tau$, which gives

$$P(\tau) \sim \frac{\rho(\tau^{-1/\gamma})}{\tau^{1+1/\gamma}}. \tag{2}$$

In the $\gamma \to \infty$ limit, which converges to the strictly priority based deterministic choice ($p = 1$) in the model, Eq. (2) predicts $P(\tau) \sim \tau^{-1}$, in agreement with the numerical results (Fig 3a), as well as the empirical data on the email interarrival times (Fig 2a). In the $\gamma = 0$ ($p = 0$) limit $\tau(x)$ is independent of x, thus $P(\tau)$ converges to an exponential distribution, as shown in Fig. 3b (see Supplementary Information).

The apparent dependence of $P(\tau)$ on the $\rho(x)$ distribution from which the agent chooses the priorities may appear to represent a potential problem, as assigning priorities is a subjective process, each individual being characterized by its own $\rho(x)$ distribution. According to Eq. (2), however, in the $\gamma \to \infty$ limit $P(\tau)$ is independent of $\rho(x)$. Indeed, in the deterministic limit the uniform $\rho(x)$ can be transformed into an arbitrary $\rho'(x)$ with a parameter change, without altering the order in which the tasks are executed [10]. This insensitivity of the tail to $\rho(x)$ explains why, despite the diversity of human actions, encompassing both professional and personal priorities, most decision driven processes develop a heavy tail.

To obtain empirical evidence for the validity of the proposed queuing mechanism we consider the email activity pattern of an individual [11, 12]. Once in front of the computer, an individual will reply immediately to a high priority message, while placing the less urgent or the more difficult ones on its priority list to compete with other non-email activities. We propose, therefore, that the observed interevent time distribution is in fact rooted in the uneven waiting times experienced by different tasks. To test this hypothesis we need to determine directly the waiting time for each task. In the email dataset we have the time, sender and recipient of each email transmitted over several months by each user, thus we can determine the time it takes for a user to reply to a received message [11]. As Fig. 2b shows, we find that the waiting time distribution $P(\tau_w)$ for the user whose $P(\tau)$ is shown in Fig. 2a is best approximated by $P(\tau_w) \sim \tau_w^{-\alpha_w}$ with exponent $\alpha_w = 1$, supporting our hypothesis that the heavy tailed waiting time distribution drives the observed bursty email activity patterns.

As in the $p \to 1$ limit of the model the priority list is dominated by low priority tasks, new tasks will often be executed immediately. This results in a peak at $P(\tau = 1)$(see Fig. 3 in the Supplementary Information), which, while in some cases may represent a model artifact, in the email context is not unrealistic: most emails are either deleted right away (which is one kind of task execution), or are immediately replied to. Only the more difficult or time consuming tasks will queue on the priority list. The email dataset does not allow us to resolve this peak, however: a message to which the user replies right away will appear to have some waiting time, given the delay between the arrival of the message and the time the user has a chance to read it.

While we illustrated the queuing process on emails, in general the model is better suited to capture the competition between different kinds of activities an individual is engaged in, i.e. the switching between various work, entertainment and communication events. Indeed, most datasets displaying heavy tailed interevent times in a specific activity reflect the outcome of the competition between tasks of different nature. For example, the starting of an online gaming session implies that all higher priority work and entertainment related activities have been already executed.

Detailed models of human activity require us to consider the impact of a number of additional mechanisms on the queuing process. First, in the priority list model we assumed that the time necessary to execute a task (service time) is the same for all tasks. The size distribution of emails is heavy tailed [16, 17], however, thus the waiting time distribution could be driven entirely by the time it takes to read an email, i.e. the message size. Yet, as Fig. 2c shows, we fail to find a correlation between the size of the email received by a user and the time the user takes to reply to it. While a detailed analysis should also consider the role of attachments, Fig. 2c suggests that the priority of a response is more important than the message size. Furthermore, the priorities assigned to tasks are often driven by optimization processes, as agents aim to maximize profits or minimize the overall time spent on some activity.

A natural extension of the model is to assume that tasks arrive at a rate λ and are executed at a rate μ, allowing the length of the priority list L to change in time. In this case the model maps into Cobham's priority queue model [9], which has a power law distributed waiting time with $\alpha = 3/2$ only when $\lambda = \mu$ (see Supplementary Information). Thus to account for the power law waiting times the model requires an additional mechanism that guarantees $\lambda = \mu$ (which, as Fig. 3d indicates, is not satisfied for most email users). In contrast, in the proposed priority list model we assumed that for humans the length of the priority list remains relatively unchanged (i.e. L is constant). To understand the origin of this assumption we must realize that for $\lambda = \mu$ the length of the priority list fluctuates widely and can occasionally grow very long. While keeping track of a long priority list is not a problem for a computer, it is well established that the immediate memory of humans has finite capacity [15]. In other words, the number of priorities we can easily remember, and therefore the length of the priority list, is bounded, motivating our choice of a finite L.

While other generalizations are possible and often required, our main finding is that the observed fat tailed activity distributions can be explained by a simple hypothesis: humans execute their tasks based on some perceived priority, setting up queues that generate very uneven waiting time distributions for different tasks. In this respect the proposed priority list model represents only a minimal framework that allows us to demonstrate the potential origin of the heavy tailed activity patterns, and offers room for further extensions to capture more complex human behavior. As the exponent of the tail could depend on the details of the prioritizing process, future work may allow the empirical data to discriminate between different queuing hypotheses. A mapping into punctuated equilibrium models (see Supplementary Information [18, 19]), with the mathematical framework of queuing theory could help the systematic classification of the various temporal patterns generated by human behavior.

There is overwhelming evidence that Internet traffic is characterized by heavy-tailed statistics [20], rooted in the Pareto size distribution of the transmitted files [16, 17]. As we have shown above (Fig 2c), we find that a user's email activity does not correlate with the email size. Similarly, the timing of online games [14] or sending an instant message [5] cannot be driven by file sizes either. This suggests that Internet traffic is in fact driven by two separate processes: The heavy tailed size distribution of the files sent by the users and the human decision driven timing of various Internet mediated activities individuals engage in. In some environments this second mechanism, whose origin is addressed in this paper, can be just as important as the much investigated first one. Given the differences in routing performance under Poisson and Pareto arrival time distributions [20, 21, 22], a queuing based model of human-driven arrival times could contribute to a better understanding of Internet traffic as well.

Uncovering the mechanisms governing the timing of various human activities has significant scientific and commercial potential. First, models of human behavior are indispensable for large-scale models of social organization, ranging from detailed urban models [23, 24], to modeling the spread of epidemics and viruses, the development of panic [25] or capturing financial market behavior [26]. Understanding the origin of the non-Poisson nature of human dynamics could fundamentally alter the dynamical conclusions these models offer. Second, models of human behavior are crucial for better resource allocation and pricing plans for phone companies, to improve inventory and service allocation in both online and brick-and-mortal retail, and potentially to understand the bursts of ideas and memes emerging in communication and publication patterns [27]. Finally, heavy tails have been observed in the foraging patterns of birds as well [28], raising the intriguing possibility that animals also utilize some evolution-encoded priority based queuing mechanisms to decide between competing tasks, like caring for offsprings, gathering food, or fighting off predators.

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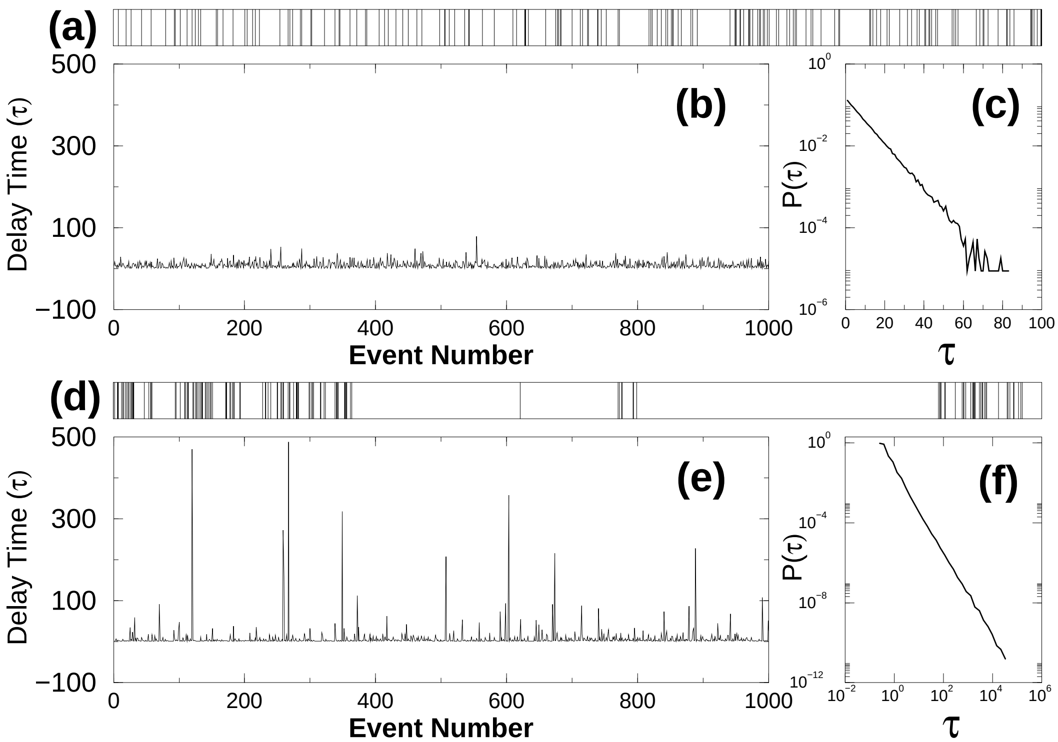

FIG. 1: The difference between the activity patterns predicted by a Poisson process (top) and the heavy tailed distributions observed in human dynamics (bottom). **(a)** Succession of events predicted by a Poisson process, which assumes that in any moment an event takes place with probability q. The horizontal axis denotes time, each vertical line corresponding to an individual event. Note that the interevent times are comparable to each other, long delays being virtually absent. **(b)** The absence of long delays is visible on the plot showing the delay times τ for 1,000 consecutive events, the size of each vertical line corresponding to the gaps seen in (a). **(c)** The probability of finding exactly n events within a fixed time interval is $\mathcal{P}(n;q) = e^{-qt}(qt)^n/n!$, which predicts that for a Poisson process the inter-event time distribution follows $P(\tau) = qe^{-q\tau}$, shown on a log-linear plot in (c) for the events displayed in (a, b). **(d)** The succession of events for a heavy tailed distribution. **(e)** The waiting time τ of 1,000 consecutive events, where the mean event time was chosen to coincide with the mean event time of the Poisson process shown in (a-c). Note the large spikes in the plot, corresponding to very long delay times. (b) and (e) have the same vertical scale, allowing to compare the regularity of a Poisson process with the bursty nature of the heavy tailed process. **(f)** Delay time distribution $P(\tau) \simeq \tau^{-2}$ for the heavy tailed process shown in (d,e), appearing as a straight line with slope -2 on a log-log plot. The signal shown in (d-f) was generated using $\gamma = 1$ in the stochastic priority list model discussed in the Supplementary Information.

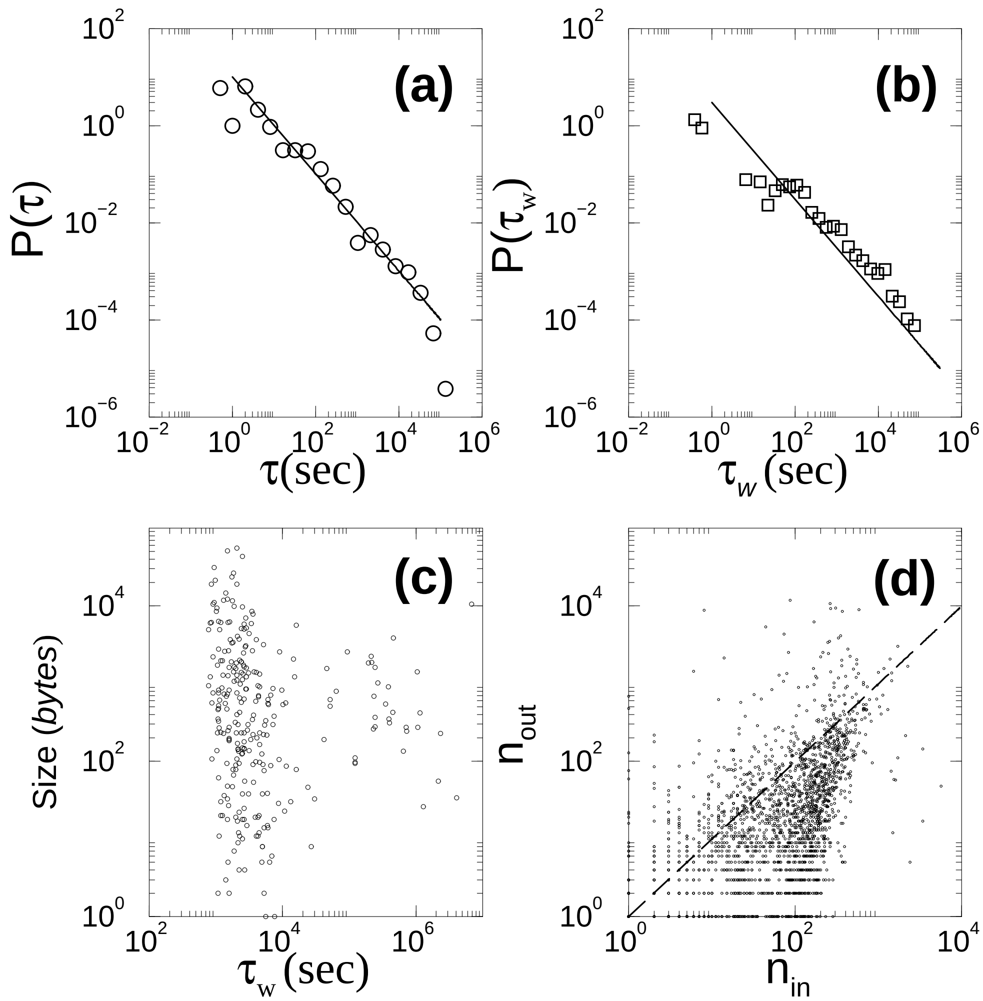

FIG. 2: Heavy tailed activity patterns in email communications. **(a)** The distribution of the time intervals between consecutive emails sent by a single user over a three month time interval, indicating that $P(\tau) \sim \tau^{-1}$ (the solid line in the log-log plot has slope -1). While the exponent differs slightly from user to user, it is typically centered around $\alpha = 1$. **(b)** The distribution of the time taken by the user to reply to a received message. To determine τ_w we recorded the time the user received an email from a specific user, and the time it sent a response to it, the difference between the two providing τ_w. For consistency the figure shows the data for the user whose interevent time distribution is shown in (a). The solid line in the log-log plot has slope -1. **(c)** A scatter plot showing the waiting time (τ_w) and the size for each email responded to by the user discussed in (a,b), indicating that the file size and the response time do not correlate. **(d)** Scatter plot showing the number of emails received and sent by 3,188 users during a three month interval. Each point corresponds to a different user, indicating that there are significant differences between the number of received and responded emails. The dashed line corresponds to $n_{in} = n_{out}$, capturing the case when the classical queuing models also predict a power law waiting time distribution (see Supplementary Information), albeit with exponent $\alpha = 3/2$.

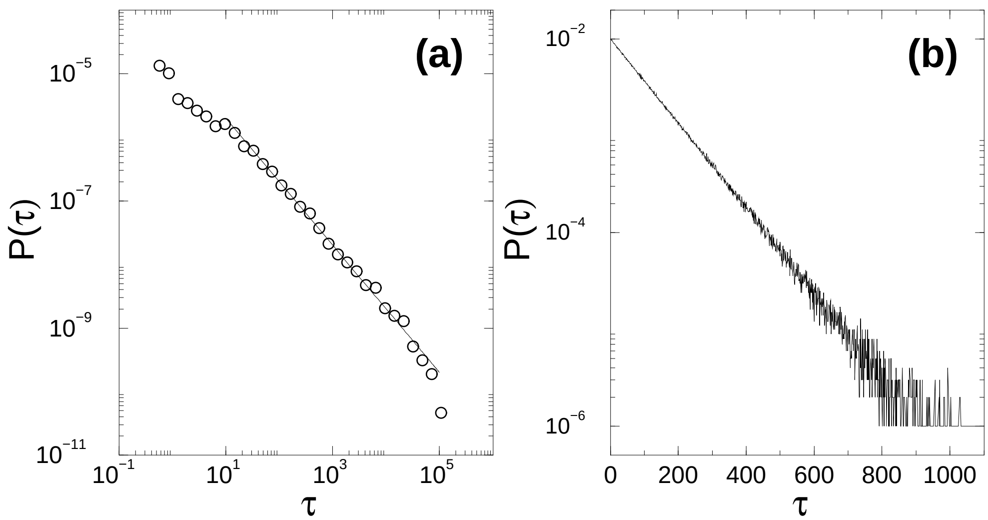

FIG. 3: The waiting time distribution predicted by the investigated queuing model. The priorities were chosen from a uniform distribution $x_i \in [0, 1]$, and we monitored a priority list of length $L = 100$ over $T = 10^6$ time steps. **(a)** Log-log plot of the tail of probability $P(\tau)$ that a task spends τ time on the list obtained for $p = 0.99999$, corresponding to the deterministic limit of the model. The continuous line on the log-log plot correspond to the scaling predicted by (2), having slope -1, in agreement with the numerical results and the analytical predictions. The data was log-binned, to reduce the uneven statistical fluctuations common in heavy tailed distributions, a procedure that does not alter the slope of the tail. For the full curve, including the $\tau = 1$ peak, see Fig. 3 in the Supplementary Information. **(b)** Linear-log plot of the $P(\tau)$ distribution for $p = 0.00001$, corresponding to the random choice limit of the model. The fact that the curve follows a straight line on a linear-log plot indicates that $P(\tau)$ decays exponentially.