Covering Triples by Quadruples: An Asymptotic Solution

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Let $C(3,4,n)$ be the minimum number of four-element subsets (called blocks) of an $n$-element set, $X$, such that each three-element subset of $X$ is contained in at least one block. Let $L(3,4,n)=\lceil n/4 \rceil \lceil n-1/3 \rceil \lceil n-2/2 \rceil$. Schoenheim has shown that $C(3,4,n) \geq L(3,4,n)$. The construction of Steiner quadruple systems of all orders $n \equiv 2$ or $4 \pmod{6}$ by Hanani (Canad. J. Math. 12 (1960), 145-157) can be used to show that $C(3,4,n)=L(3,4,n)$ for all $n \equiv 2, 3, 4 \text{ or } 5 \pmod{6}$ and all $n \equiv 1 \pmod{12}$. The case $n \equiv 7 \pmod{12}$ is made more difficult by the fact that $C(3,4,7)=L(3,4,7)+1$ and until recently no other value for $C(3,4,n)$ with $n \equiv 7 \pmod{12}$ was known. In 1980 Mills showed by construction that $C(3,4,499)=L(3,4,499)$. We use this construction and some recursive techniques to show that $C(3,4,n)=L(3,4,n)$ for all $n \geq 52423$. We also show that if $C(3,4,n)=L(3,4,n)$ for $n=31, 43, 55$ and if a certain configuration on 54 points exists then $C(3,4,n)=L(3,4,n)$ for all $n \neq 7$ with the possible exceptions of $n=19$ and $n=67$. If we assume only $C(3,4,n)=L(3,4,n)$ for $n=31$ and 43 we can deduce that $C(3,4,n)=L(3,4,n)$ for all $n \neq 7$ with the possible exceptions of $n \in \{19, 55, 67, 173, 487\}$.

1. INTRODUCTION AND MAIN RESULTS

Let \( C(t, k, v) \) be the minimum number of \( k \)-element subsets (called blocks) of an \( n \)-element set \( X \), such that every \( t \)-element subset is contained in at least one block.

The problem of evaluating \( C(t, k, v) \) is a generalization of existence problem for Steiner systems, since \( C(t, k, v) = (\binom{v}{t})/(\binom{k}{t}) \) if and only if a Steiner system \( S(t, k, v) \) exists. Several authors have studied the problem and the reader is referred to [2] for a survey and to [12] for an up to date bibliography of the question and more recent results.

Let \( L(t, k, u) = \frac{rv}{krv - 1/k - 1} \cdots \frac{v - t + 1/k - t + 1}{v - t + 1} \). Schoenheim [5] noted that \( C(t, k, v) \geq L(t, k, v) \) for all \( v \geq k \geq t \geq 1 \).

We are concerned in this paper with the evaluation of \( C(3, 4, v) \).

Mills [11] has shown that \( C(3, 4, v) = L(3, 4, v) \) for all \( v \equiv 7 \pmod{12} \) and several authors have noted that \( C(3, 4, 7) = L(3, 4, 7) + 1 \). Recently Mills [3] showed that \( C(3, 4, 499) = L(3, 4, 499) \).

The purpose of this paper is to show that Mills' configuration on 499 points can be used to give \( C(3, 4, v) = L(3, 4, v) \) for all \( v \geq 52423 \). We also show that the construction of any smaller configuration on \( 12n + 7 \) points covering the triples in \( L(3, 4, 12n + 7) \) blocks would considerably improve the lower bound on \( v \).

In subsequent sections we give constructions of the combinatorial designs used as building blocks in the determination of \( C(3, 4, v) \).

Let \( m, g, k, t \) be non-negative integers. We define an \( H(m, g, k, t) \) design to be an ordered triple \((X, G, B)\), where \( X \) is a set of cardinality \( mg \) whose elements are called points and \( G = \{G_1, G_2, \ldots, G_m\} \) is a partition of \( X \) into \( m \) sets of cardinality \( g \); the members of \( G \) are called groups. A transverse of \( G \) is a subset of \( X \) which meets each \( G_i \) in at most one point. The set \( B \) contains \( k \)-element transverses of \( G \) called blocks, with the property that each \( t \)-element transverse of \( G \) is contained in precisely one block.

These designs were introduced by Hanani in [5], where they were denoted by the symbol \( P_g^{m}[k, l, mg] \); the notation given here is due to Mills [11].

In [5] Hanani constructed \( H \)-designs with the parameters \( H(m, g, 4, 3) \), \( m = 4, 6, g = 2, 3 \), for \((m, g) \neq (6, 2)\). Mills [11] independently constructed each of these and also constructed an \( H(5, 6, 4, 3) \). It is clear from the definition that an \( H(3, g, 4, 3) \) cannot exist for any \( g > 0 \) and that all \( H(m, g, 4, 3) \) with \( m < 3 \) exist (with \( B = \emptyset \)).

In Section 2 we prove the following result.

**Theorem 1.** An \( H(m, 6, 4, 3) \) design exists for all positive integers \( m \neq 3 \), with the possible exceptions of \( m = 9, 27 \) or 81.

Our main tool in the study of \( C(3, 4, n) \) is the concept of an \( H \)-design
frame or simply frame. Frames have been used explicitly in the construction of other combinatorial configurations (e.g., [14, 17]) and they are also implicit in the works of Hanani [4, 5] and Wilson [18].

For non-negative integers \( m, g, k \) and \( t \) we define an \( H(m, g, k, t) \) frame to be an ordered four-tuple \((X, G, B, F)\), where \( X \) is a set of \( mg \) points, \( G = \{G_1, G_2, \ldots, G_m\} \) is an equipartition of \( X \) into \( m \) groups and \( F \) is a family \( \{F_i\} \) of subsets of \( G \) called holes which is closed under intersections. Hence each hole \( F_i \in F \) is of the form \( F_i = \{G_{i1}, G_{i2}, \ldots, G_{ik}\} \), and if \( F_i \) and \( F_j \) are holes than \( F_i \cap F_j \) is also a hole. The number of groups in a hole is its size.

The set of blocks \( B \) is a set of \( k \)-element transverses of \( G \) with the property that every \( t \)-element transverse of \( G \) which is not a \( t \)-element transverse of some frame \( F_i \in F \) is contained in precisely one block, and no block contains a \( t \)-element transverse of any hole.

Intuitively, a frame is a combinatorial design with a family of hypothetical subdesigns or flats deleted.

In Section 2 we also prove the following existence theorems for \( H(m, 6, 4, 3) \) frames.

**Theorem 2.** If there exists an \( H(9, 6, 4, 3) \) design, then for all odd \( m \geq 5 \) there exists an \( H(m, 6, 4, 3) \) frame containing exactly one hole of size 5, 7 or 9 and in which the other groups are paired forming holes of size 2.

We remark that the proof of this theorem guarantees that when \( m = 13 \) or 15 the odd hole is of size 5, and that holes of size 9 are only needed when \( m = 9, 27 \) and 81.

**Theorem 3.** For all odd \( m \geq 8737 \) there exists an \( H(m, 6, 4, 3) \) frame containing exactly one hole of size 83 and in which the other groups are paired, forming holes of size 2.

In Section 3 we investigate the existence of \( H(m, 6, 3, 2) \) frames and prove the following:

**Theorem 4.** For all odd \( m \geq 2f + 3 \) and \( f = 5, 7 \) or 9 there exists an \( H(m, 6, 3, 2) \) frame containing one hole size \( f \) and in which the remaining groups are paired in the formation of holes of size 2.

**Theorem 5.** For all odd \( m \geq 209 \) there exists an \( H(m, 6, 3, 2) \) frame containing one hole of size 83 and in which the remaining groups are paired, forming holes of size 2.

These results enable us to prove our main theorems:
THEOREM 6. If an H(9, 6, 4, 3) design exists, and if \( C(3, 4, v) = L(3, 4, v) \) for \( v = 31, 43 \) and 55, then \( C(3, 4, v) = L(3, 4, v) \) for all \( v \equiv 7 \pmod{12}, v \neq 7 \), with the possible exceptions of \( v = 19 \) and \( v = 67 \).

Proof. Let \( f \) be an odd number. We begin by noting that

\[ L(3, 4, 6(n + f) + 1) \text{ is equal to the sum of} \]

(i) the number of blocks in an \( H(2n + f, 6, 4, 3) \) frame with exactly one hole of size \( f \); (and \( n \) disjoint holes of size 2);

(ii) the number of blocks in an \( H(2n + f, 6, 3, 2) \) frame with one hole of size \( f \) and \( n \) disjoint holes of size 2;

(iii) \( n \) times \( C(3, 4, 13) \);

(iv) \( L(3, 4, 6f + 1) \); and

(v) \( 5.3^2[2n(2n + f - 2) + 2nf]/2 \).

This fact may be verified by elementary counting techniques and by algebraic manipulation.

For \( v > 79, v \equiv 7 \pmod{12} \) write \( v = 6m + 1 \) with \( m \) odd, \( m \geq 13 \). By Theorem 2, the remark following it and Theorem 4 there exist frames \( H(m, 6, 4, 3) \) and \( H(m, 6, 3, 2) \) containing one hole of size \( f = 5, 7, \) or \( 9 \) and \( n \) disjoint holes of size 2 with \( 2n + f = m \), and each group in precisely one hole. Let

\[ F_i = \{G_{2i-1}, G_{2i}\}, \quad i = 1, 2, \ldots, n \]

be the holes of size two, and let

\[ I = \bigcup_{i=1}^{f} G_{2n+i} \]

be the set of points in the groups of the large hole. Let \( B_1 \) be the block set of the \( H(m, 6, 4, 3) \) frame and let \( B_2 \) be the block set of the \( H(m, 6, 3, 2) \) frame on the same set of points. Let \( \infty \) be a new point and let \( B_2 \) be the set of blocks of size 4 formed by adding \( \infty \) to each block in \( B_2 \). Let \( B_3 \) be the union of \( n \) sets of quadruples which cover the 3-subsets of \( G_{2i-1} \cup G_{2i} \cup \{\infty\} \) in precisely \( L(3, 4, 13) \) blocks. Let \( B_4 \) be a covering of the 3-subsets of \( I \cup \{\infty\} \) by quadruples in \( L(3, 4, |I| + 1) \) blocks. These covering configurations exist by Mills' result [11] for \( B_3 \), and by hypothesis for \( B_4 \).

Finally, let \( F_1 F_2 F_3 \cdots F_f^5 \) be any one-factorization of the complete graph with vertex set \( G_i \), and let

\[ B_5 = \{[a, b, c, d] : [a, b] \in F_j^l, [c, d] \in F_p^l, \quad j = 1, 2, \ldots, 5 \]

\[ 1 \leq i < p \leq 2n + f \text{ with } \{G_i, G_p\} \not\text{together in any hole}\} \]

The reader may verify that every 3-subset of \( \{\infty\} \cup \bigcup_{i=1}^{2n+f} G_i \) is contained
in at least one block of $\bigcup_{i=1}^{s} B_i$, and by the opening remark $|\bigcup_{i=1}^{s} B_i| = L(3, 4, 6m + 1)$, therefore $C(3, 4, 6m + 1) = L(3, 4, 6m + 1)$. 

A similar proof, using Theorem 3, Theorem 5, the fact that $C(3, 4, 499) = L(3, 4, 499)$, (Mills [13]), and noting that $499 = 6(83) + 1$, yields the following result.

**Theorem 7.** For all $v \equiv 7 \pmod{12}$ with $v \geq 52423 = 6(8737) + 1$, $C(3, 4, v) = L(3, 4, v)$.

We also show that the existence of an $H(9, 6, 4, 3)$ is not essential for the following conditional result.

**Theorem 8.** If $C(3, 4, v) = L(3, 4, v)$ for $v = 31$ and $43$ then $C(3, 4, v) = L(3, 4, v)$ for all $v \neq 7$ with the possible exceptions of $v = 10, 55, 67, 163$ and $487$.

### 2. Construction of $H(m, 6, 4, 3)$ Frames

We begin by proving some elementary properties of $H$ frames.

**Lemma 2.1 (Replacement of subdesigns).** Let $(X, G, B, F)$ be an $H(m, g, k, t)$ frame, let $F$ be a maximal hole in $F$ of size $s$ and let $E$ be the set of all holes in $F$ which are properly contained in $F$. If there exists an $H(s, g, k, t)$ frame $(U, F, C, E)$ then $(X, G, B \cup C, F - \{F\})$ is an $H(m, g, k, t)$ frame.

The group-inflation theorem of Hanani [5, Proposition 5] and Mills [11, Lemma 1] for $H$-designs also holds for $H$ frames, as follows. The symbol $\mathbb{Z}_n$ denotes the cyclic group of integers modulo $n$ under addition.

**Lemma 2.2.** If $(X, G, B, F)$ is an $H(m, g, k, k - 1)$ frame, then there exists an $H(m, ng, k, k - 1)$ frame $(X', F', B', F')$ for any integer $n > 0$.

**Proof.** Let

$X' = X \times \mathbb{Z}_n$,

$G'_i = G_i \times \mathbb{Z}_n$ for each group $G_i \in G$,

$F' = \{G'_1, G'_2, \ldots, G'_s\}$ for each hole $F = \{G_{i_1}, G_{i_2}, \ldots, G_{i_s}\} \in F$,

and for each block $B = \{x_1, x_2, \ldots, x_k\} \in B$ construct $n^{k-1}$ new blocks

$B' = \{(x_1, a_1)(x_2, a_2)\cdots(x_k, a_k)\}$,

one new block for each solution to $\sum_{i=1}^{k} a_i \equiv 0 \pmod{n}$. 


The construction given above may be generalized for $t \neq k - 1$ by the use of orthogonal arrays, however, we only use the construction when $t = 3$ and $k = 4$.

We shall make repeated use of the elementary three-wise balanced design construction for $H$-designs.

Let $v$ be a positive integer and let $K$ be a set of non-negative integers.

A three-wise balanced design of order $v$ and with block sizes from $K$, is a pair $(X, B)$, where $X$ is a $v$-element set and $B$ is a subset of $X$, called blocks, with the properties that (i) every block has its cardinality in $K$ and (ii) every 3-element subset of $X$ is contained in precisely one block.

**Lemma 2.3.** If $(X, B)$ is a three-wise balanced design of order $v$ with block sizes from $K$ and for every $k \in K$ there exists an $H(k, g, l, 3)$ design then there exists an $H(v, g, l, 3)$ design.

**Proof.** From $(X, B)$ we may construct an $H(v, 1, l, 3)$-frame with point set $X$, trivial partition $G$, empty block set and hole set $B \cup \binom{X}{2} \cup \binom{X}{3} \cup \emptyset$ (using $\binom{X}{r}$ to denote the set of all $r$-subsets of $X$). We may now apply Lemma 2.2 and then use Lemma 2.1 on each non-trivial hole in turn to obtain the required design.

We are now able to prove:

**Theorem 2.4.** For any even positive integer $m$ an $H(m, 6, 4, 3)$ design exists.

**Proof.** Hanani [5, Lemma 1] has shown that for every even positive integer $m$ there exists a three-wise balanced design of order $m$ with block sizes from $[4, 6]$. An $H(4, 1, 4, 3)$ design exists trivially, so by Lemma 2.2 there exists an $H(4, 6, 4, 3)$. Hanani [5, Proof of Lemma 3] and Mills [11, Proof of Lemma 7] have independently constructed an $H(6, 3, 4, 3)$ design so, again by Lemma 2.2, an $H(6, 6, 4, 3)$ design exists. We may now apply Lemma 2.3 to give the result.

We turn now to the construction of $H(m, 6, 4, 3)$ designs with $m$ odd.

In previous papers [8, 10] Hartman has defined various kinds of pairing structures to aid in the construction of Steiner quadruple systems (three-wise balanced designs with block size 4). Here we illustrate the use of pairings in the construction of $H$-frames.

For $x \in \mathbb{Z}_n$ we define $|x|$ by

$$|x| = x \quad \text{if} \quad 0 \leq x \leq n/2$$

$$= -x \quad \text{if} \quad n/2 < x < n.$$
Let $A$ be a set of edges of the complete graph with vertex set $\mathbb{Z}_n$. We define $L(A)$, the set of chord lengths of $A$, by

$$L(A) = \{|x - y| : [x, y] \in A\}.$$ 

A regular graph $(V, E)$ of degree $k$ has a one-factorization if the edge set $E$ can be partitioned in $k$ parts $E = F_1 \cup F_2 \cup \cdots \cup F_k$ so that each $F_i$ is a partition of the vertex set $V$ into pairs. The parts $F_i$ are called one-factors, and a partial one-factor is any set of vertex disjoint edges.

The following Lemma is proved by Stern and Lenz in [16].

**Lemma 2.5.** Let $G$ be a graph with vertex set $\mathbb{Z}_{2n}$ and let $L$ be a set of integers in the range $1, 2, \ldots, n$, such that $[a, b]$ is an edge of $G$ if and only if $|a - b| \in L$. Then $G$ has a one-factorization if and only if $2n/\gcd(j, 2n)$ is even for some $j \in L$.

For non-negative integers $n$ and $s$ we define a $B$-pairing $B(n, s)$ to be an ordered 4-tuple $(D, R_0, R_1, R_2)$ with the following properties:

1. **(B1)** The set $D$ is a subset of $\mathbb{Z}_{6n}$ of cardinality $6s$.
2. **(B2)** The sets $R_0, R_1, R_2$ are partial one-factors of the complete graph with vertex set $\mathbb{Z}_{6n}$, each containing the same number, say $r$, of edges. The vertices covered by $R_i$ will be denoted by $VR_i$.
3. **(B3)** Partitioning: The set $\mathbb{Z}_{6n}$ is partitioned by the sets $D, VR_0, VR_1, VR_2$ so

$$\mathbb{Z}_{6n} = D \cup VR_0 \cup VR_1 \cup VR_2$$

and hence $6n = 6s + 6r$, so $r = n - s$.
4. **(B4)** Forbidden Edge Lengths: No partial one-factor $R_i$ contains an edge with length divisible by $n$ so

$$jn \notin L(R_i), \quad j = 1, 2, 3, \quad i = 0, 1, 2.$$ 

5. **(B5)** Distinct Edge Lengths: The edges within each partial one-factor $R_i$ are all of different lengths, so

$$|L(R_i)| = r = n - s.$$ 

6. **(B6)** One-factorization: For each $i = 0, 1, 2$ define $\Gamma_i$ to be the graph with vertex set $\mathbb{Z}_{6n}$ and edge set containing the edge $[a, b]$ if and only if

$$|a - b| \notin L(R_i) \cup \{n, 2n, 3n\}. $$

This condition requires that each of the graphs $\Gamma_0, \Gamma_1$ and $\Gamma_2$ have a one-factorization.
We first show that existence of a $B$-pairing implies existence of an $H$-frame design.

**Theorem 2.6.** If there exists a $B(n, s)$ then there exists an $H(3n + s, 6, 4, 3)$ with three holes of size $n + s$ which intersect on holes of size $s$.

**Proof.** Let $(D, R_0, R_1, R_2)$ be a $B(n, s)$ pairing. We construct an $H$-frame $(X, G, B, F)$ as follows:

Let $X = (\mathbb{Z}_{6n} \times \mathbb{Z}_3) \cup \{\infty_0, \infty_1, ..., \infty_{6s-1}\}$.

We define the groups

$$G(i, j) = \{(kn + i, j): k = 0, 1, 2, 3, 4, 5\}, \quad 0 \leq i < n, \quad 0 \leq j < 3$$

$$G(\infty, j) = \{\infty_{ks + j}: k = 0, 1, ..., 5\}, \quad 0 \leq j < s$$

and the frames

$$\phi_1 = \{G(\infty, j): 0 \leq j < s\}, \quad \phi_{2+i} = \phi_i \cup \{G(j, i): 0 \leq k < n\}, \quad i = 0, 1, 2.$$ The block-set $B$ consists of all blocks of the following three forms.

1. $[(a, 0)(b, 1)(c, 2)],$ where $a + b + c \equiv d \pmod{6n}$, $d$ is the $j$th member of $D$ and $0 \leq j < 6s$.

2. $[(a+q, i)(a+t, i)(b, i+1)(c, i+2)],$ where $a+b+c \equiv 0 \pmod{6n}$ and $[q, t]$ runs through all edges in $R_i$ and $i = 0, 1, 2$.

Property (B4) guarantees that blocks of type (2) are indeed transverses of $G$.

The final set of blocks is formed as follows: Let $F'_1 | F'_2 | ... | F'_{4n + 2s - 6}$ be a one-factorization of the graphs $\Gamma_i$ defined as in property (B6) of $B$-pairings, then

3. $[(a, i)(b, i)(c, i+1)(d, i+1)],$ where $[a, b]$ ranges over all edges in $F'_j$ and $[c, d]$ ranges over all edges in $F'_{j+1}$, $i \in \mathbb{Z}_3$ and $j = 1, 2, ..., 4n+2s-6$ is the final set of blocks.

Blocks of type (3) are transverses since $n \not\equiv (a - b)$ for any edge $[a, b]$ of $\Gamma_i$. Note that

$$|B| = (6s)(6n)^2 + 3(n-s)(6n)^2 + 3(4n+2s-6)(3n)^2$$

$$\frac{1}{4} 6^3 \left[ \binom{3n+s}{3} - 3 \binom{n+s}{3} + 2 \binom{s}{3} \right]$$

which is the right number of blocks. If $T$ is any three-element transverse of $G$ which is not the transverse of some group then $T$ has one of the three following forms.
(1) $T=\{\infty,(a,j)(b,k)\}$ with $j \neq k$. The fourth member of a block containing this transverse is the element $(c, m)$, where $\{m\} = \mathbb{Z}_3 \setminus \{j, k\}$ and $c$ is the solution to $a + b + c \equiv d \pmod{6n}$, where $d$ is the $i$th member of $D$.

(2) $T=\{(a,0)(b,1)(c,2)\}$. If $a + b + c \in D$ then the fourth member of a block containing $T$ is $\infty_j$, where $a + b + c$ is the $j$th member of $D$.

If $a + b + c \notin D$ then $a + b + c \in VR_i$ for some $i$, by the partitioning axiom (B3). Say $[d, a+b+c] \in VR_0$, for definiteness. Then $(d - b - c, 0)$ is the fourth member of a block containing $T$.

(3) $T=\{(a,i)(b,i)(c,j)\}$, $i \neq j$, and $a \neq a$. If $|a-b| \in L(R_i)$ then a fourth member of a block containing $T$ is of the form $(d, k)$, where $\{k\} = \mathbb{Z}_3 \setminus \{i, j\}$, and $d$ is determined as follows: By the distinct-edge-lengths axiom (B5) a unique edge $[x, y] \in R_i$ has the property $|x - y| = |a - b|$ and in fact there is some ordering of $x, y$ such that $x + \alpha = a$ and $y + \alpha = b$, then $d = x - a - c = y - b - c$.

If $|a - b| \notin L(R_i)$ then $[a, b] \in \Gamma_i$ so $[a, b]$ is in some one-factor $F^i_k$ of $\Gamma_i$. The vertex $c$ is in a unique edge $[c, d]$ of the one-factor $F^i_k$ and $\Gamma_j$ and this determines that $(d, j)$ is the fourth member of a block containing $T$. This completes the proof. 1

We now show the existence of $B$-pairings by direct construction.

**Theorem 2.7.** For all integers $n \geq 2$ there exists a $B(n, 0)$.

**Proof.** For $n$ even let $D = \emptyset$, $R_0 = \{\{n-j, n+j-1\}: 1 \leq j \leq n\}$, $R_1 = 2n + R_0$, $R_2 = 4n + R_0$

where

$$aX + b = \{[ax + b, ay + b]: [x, y] \in X\}.$$ 

Note that the $VR_0 = \{j: 0 \leq j < 2n\}$ so the partitioning axiom (B3) follows. Axioms (B4) and (B5) follow from the fact that $L(R_i) = \{1, 3, 5, ..., 2n - 1\}$. The one-factorization axiom follows from Lemma 2.5 since, for example, all edges of length $2n + 1$ are present in each $\Gamma_i$. 1

For $n$ odd let $D = \emptyset$

$$R_0 = \{[n-1, 3n-2]\} \cup \{[n-1-j, n-1+j]: 1 \leq j \leq n-1\}$$
$$R_1 = \{[5n-3, 6n-1]\} \cup \{[3n-2-j, 3n-2+j]: 1 \leq j \leq n-1\}$$
$$R_2 = \{[6n-3, 6n-2]\} \cup \{[5n-3-j, 5n-3+j]: 1 \leq j \leq n-1\}.$$
The partitioning axiom (B3) is immediate. Axioms (B4) and (B5) follow since if

\[ L = \{2, 4, 6, ..., 2n - 2\} \]

then \( L(R_0) = L \cup \{2n - 1\} \), \( L(R_1) = L \cup \{n + 2\} \), \( L(R_2) = L \cup \{1\} \). The one-factorization axiom (B6) again follows from Lemma 2.5, since edges of length \(2n + 1\) are present in each \( I_i \).

We can use these construction of \( B(n, 0)'s \) to obtain a \( B(n, s) \) by removing \( s \) edges from each \( R_i \) and placing their vertices in \( D \). This procedure guarantees that axioms (B1)–(B5) will hold. The one-factorization axiom (B6) will also hold, since the edges of length \(2n + 1\) remain in the graphs \( I_i \).

We note that a \( B(1, 0) \) cannot exist since \( R_i \neq \emptyset \) but no chord lengths are permitted by axiom (B4), and a \( B(1, 1) \) exists trivially with \( D = \mathbb{Z}_6 \), \( R_0 = R_1 = R_2 = \emptyset \).

These arguments imply the truth of:

**THEOREM 2.8.** For all \( n \geq s \geq 0 \) with \((n, s) \neq (1, 0)\) there exists a \( B(n, s) \).

Combining Theorems 2.6 and 2.8 we have:

**THEOREM 2.9.** For all integers \( n \geq s \geq 0 \) with the exception of \((n, s) = (1, 0)\) there exists an \( H(3n + s, 6, 4, 3) \) frame having three holes of size \( n + s \) which intersect on a common hole of size \( s \).

We turn now to the use of composition methods for the construction of \( H(m, 2, 4, 3) \) frames. These imply the existence of \( H(m, 6, 4, 3) \) frames by Lemma 2.2.

We begin by giving direct construction of some small \( H \) frames and related designs.

**DESIGN 1.A.** An \( H(7, 2, 4, 3) \) design. Let \( X = \mathbb{Z}_7 \times \mathbb{Z}_2 \), \( G = \{\{(i, 0)\} \in \mathbb{Z}_7\} \). Consider the group of permutations of \( X \) generated by \( \pi_1 \) and \( \pi_2 \), where \( \pi_1(x, i) = (x + 1, i) \) and \( \pi_2(x, i) = (3x, i + 1) \). Define \( B \) to be the \( \langle \pi_1, \pi_2 \rangle \)-orbits of the quadruples

\[
[(0, 0)(1, 0)(2, 0)(4, 0)] \]
\[
[(0, 0)(1, 1)(2, 1)(4, 1)] \]
\[
[(5, 0)(6, 0)(1, 1)(2, 1)]
\]

and

\[
[(4, 0)(5, 0)(2, 1)(3, 1)].
\]
The reader may check, using the methods described in [9], that \((X, B, G)\) is in fact an \(H(7, 2, 4, 3)\) design.

**DESIGN 1.B.** An \(H(11, 2, 4, 3)\) frame-design. Let \(X = (\mathbb{Z}_6 \times \mathbb{Z}_3) \cup \{\infty\} \times \mathbb{Z}_4\) by the point set. Define the following 11 groups

\[
G(i, j) = \{(i, j)(i + 3, j)\}, \quad i = 0, 1, 2, \quad j \in \mathbb{Z}_3
\]

\[
G(\infty, j) = \{(\infty, j)(\infty, j + 2)\}, \quad j = 0, 1
\]

and take \(G\) to the set containing them. Define the following four holes

\[
F_1 = \{G(\infty, j): j = 0, 1\}
\]

\[
F_{2+j} = F_1 \cup G(i, j): i = 0, 1, 2, \quad j = 0, 1, 2
\]

and take \(F\) to be the set containing them.

Finally, we define \(B\) to the set of all blocks of the following forms:

1. \([(\infty, j)(a, 0)(b, 1)(c, 2)], \) where \(a + b + c = k \mod 6\) and \((j, k)\) ranges over the pairs \((0, 0)(1, 2)(2, 3)(3, 4)\);

2. \([(a, i)(a + 3b, i + 1)(1 - 2a - 3b, i + 2)(5 - 2a - 3b, i + 2)], \) where \(a \in \mathbb{Z}_6, i \in \mathbb{Z}_3, b = 0, 1;\)

3. \([(a, i)(a + 2, i)(a + 3b, i + 1)(a + 3b + 2, i + 1)], \) where \(a \in \mathbb{Z}_6, i \in \mathbb{Z}_3, b = 0, 1;\)

4. \([(a, i)(a + 1, i)(a + 2b, i + 1)(a + 2b + 1, i + 1)], \) where \(a \in \mathbb{Z}_6, i \in \mathbb{Z}_3, b = 0, 1, 2.\)

The reader may check this design using the methods described in [10]. Note that the holes are all of sizes 2 or 5, and that they intersect in a pencil-like manner, using pencil with its geometrical connotations. An \(H(11, 2, 4, 3)\) design was first constructed by Mills.

**DESIGN 2.** An \(H(4, 6, 4, 3)\) design which exists by Theorem 2.4.

Designs 3.A and 3.B given below are both examples of special frames introduced by Hanani [5, Definition 5]. For the purposes of this paper we require only the two examples given below, so we omit any definition of these structures.

**DESIGN 3.A.** Let the point set be

\[
X = (\mathbb{Z}_6 \times \mathbb{Z}_3) \cup (\{\infty\} \times \mathbb{Z}_2).
\]

Define \(G(i, j)\) as in Design 1.B, let \(G(\infty, 0)\) be the group \(\{\infty\} \times \mathbb{Z}_2,\) and
take $\mathcal{G}$ to be the set containing all these groups. Define four holes

$$F_1 = \{G(\infty, 0)\},$$

$$F_{2+j} = F_1 \cup \{G(i, j): i = 1, 2\}, \quad j = 0, 1, 2.$$ 

The design has two block sets $\mathcal{B}$ and $\bar{\mathcal{B}}$ defined below. Let $\mathcal{B}$ be the set of all blocks of the form

$$[(a, i)(a + 1, i)(b, i + 1)(c, i + 2)]$$

where $a + b + c \equiv 2i \pmod{6}$ and $i = 0, 1, 2$.

Let $\bar{\mathcal{B}}$ be the set of all blocks of the following forms:

1. $[(\infty, j)(a, 0)(b, 1)(c, 2)],$ where $a + b + c \equiv 3j \pmod{6}$ and $j = 0, 1$;
2. $[(a, i)(a + 2, i)(a + 3b + 1, i + 1)(a + 3b + 1, i + 2)],$ where $a \in \mathbb{Z}_6$, $i \in \mathbb{Z}_3$ and $b = 0, 1$;
3. $[(a, i)(a + 2, i)(a + 3, i + k)(a + 5, i - k)],$ where $a \in \mathbb{Z}_6$, $i \in \mathbb{Z}_3$ and $k = 1, 2$;
4. $[(a, i)(a + 2, i)(a, i + 1)(a + 2, i + 1)],$ where $a \in \mathbb{Z}_6$, $i \in \mathbb{Z}_3$.

The reader may check that the blocks in $\mathcal{B}$ contain all the 3-subsets of $X$ of the forms given below and no others:

$$(T_0)(a, 0)(b, 1)(c, 2), \quad a, b, c \in \mathbb{Z}_6$$

and

$$(T_1)(a, i)(a + 1, i)(b, j), \quad a, b \in \mathbb{Z}_6, i, j \in \mathbb{Z}_3, i \neq j.$$ 

The blocks in $\bar{\mathcal{B}}$ contain all 3-subsets of the form $(T_0)$ and all 3-subsets of the following forms and no others:

$$(T_2)(a, i)(a + 2, i)(b, j), \quad a, b \in \mathbb{Z}_6, i, j \in \mathbb{Z}_3, i \neq j$$

$$(T_3)(\infty, i)(a, j)(b, k), \quad i \in \mathbb{Z}_2, a, b \in \mathbb{Z}_6, j, k \in \mathbb{Z}_3, j \neq k.$$ 

**DESIGN 3.B.** Let the point set be

$$X = (\mathbb{Z}_6 \times \mathbb{Z}_3) \cup (\infty \times \mathbb{Z}_{10}).$$

Define $G(i, j)$ as in Design 1.B, let

$$G(\infty, j) = \{\infty, j)(\infty, j + 5)\}, \quad j = 0, 1, 2, 3, 4$$

and take $\mathcal{G}$ to be the set containing all these groups. Define four holes
$F_1 = \{G(\infty, j): j = 0, 1, \ldots, 4\}$ and $F_{2+j} = F_1 \cup \{G(i, j): i = 0, 1, 2, j = 0, 1, 2\}$. Let $B$ be the set of all blocks of the forms (1), (2) (3) and (4) defined in Design 1.B. Let $\tilde{B}$ be the set of all blocks of the form

$$[(\infty, j)(a, 0)(b, 1)(c, 2)], \quad \text{where } a + b + c \equiv k \pmod{6}$$

and $(j, k)$ ranges over the pairs $(4, 0)(5, 1)(6, 2)(7, 3)(8, 4)(9, 5)$.

The blocks in $B$ contain all those triples of the forms $(T_0)$, $(T_1)$, $(T_2)$ as well as those of the form

$$(T_4)(\infty, i)(a, j)(b, k), \quad i = 0, 1, 2, 3, a, b \in \mathbb{Z}_6, j, k \in \mathbb{Z}_3, j \neq k$$

and no others. The blocks $\tilde{B}$ contain all triples of the forms $(T_0)$ and

$$(T_5)(\infty, i)(a, j)(b, k), \quad i = 4, 5, \ldots, 9, a, b \in \mathbb{Z}_6, j, k \in \mathbb{Z}_3, j \neq k$$

and no others.

We now illustrate the use of these designs by proving:

**Theorem 2.10.** For any integer $m \equiv 0$ or 2 (mod 6) there exists an $H(3m + \epsilon, 2, 4, 3)$ frame with $\epsilon = 1$ or 5, containing only trivial holes or holes with five groups.

**Proof.** Hanani [4] has shown that there exists a Steiner quadruple system on $m + 2$ points if and only if $m \equiv 0$ or 2 (mod 6). Let $Z, \mu (\{x, \tilde{x}\}$ be the point set and let $Q$ be the block set of such a Steiner quadruple system. We now construct an $H(3m + \epsilon, 2, 4, 3)$ frame. Let the point set be $X = (Z_6 \times Z_m) \cup (\{\infty\} \cup Z_{2\epsilon})$.

Let

$$G(i, j) = \{(i, j)(i + 3, j)\}, \quad i = 0, 1, 2, j \in \mathbb{Z}_m$$

and

$$G(\infty, j) = \{(\infty, j)(\infty, j + \epsilon)\}, \quad j = 0, 1, \ldots, \epsilon - 1$$

and take $G$ to be the set of all these groups.

We construct the block set $B$ by taking copies of Designs 1.A, 1.B, 2, 3.A and 3.B on the points of $X$ in the following manner:

If $[a, \tilde{x}, x, y]$ is a block in $Q$ then:

**Case $\epsilon = 1$.** Write the blocks of Design 1.A—an $H(7, 2, 4, 3)$ design on the points of the set

$$(Z_6 \times \{x, y\}) \cup (\{\infty\} \times Z_2)$$

taking care to preserve the groups $G(i, x)$, $G(1, y)$ and $G(\infty, 0)$. 
Case $e = 5$. Write the blocks of Design 1.B—an $H(11, 2, 4, 3)$ frame on the points of the set

$$(Z_6 \times \{x, y\}) \cup (\{\infty\} \times Z_{10})$$

taking care to preserve the groups and mapping the holes in Design 1.B to the holes

$$F_0 = \{G(\infty, 0), G(\infty, 1)\}$$

$$F_1 = \{G(\infty, j): j = 0, 1, \ldots, 4\}$$

$$F_{2+x} = \{G(i, x): i = 0, 1, 2\} \cup \{G(\infty, 0), G(\infty, 1)\}$$

$$F_{2+y} = \{G(i, j): i = 0, 1, 2\} \cup \{G(\infty, 0), G(\infty, 1)\}$$

in the new design.

If $[x, y, z, t]$ is a block in $Q$ not containing $a$ or $\bar{a}$ then write the blocks of Design 2 on the points $Z_0 \times \{x, y, z, t\}$, mapping the groups of Design onto the sets $Z_6 \times \{i\}$.

If $[a, x, y, z]$ is a block on $Q$ not containing $\bar{a}$ write the blocks of Design 3.A (if $e = 1$) or Design 3.B (if $e = 5$) on the points $(Z_6 \times \{x, y, z\}) \cup (\{\infty\} \times Z_{2e})$ taking care to preserve groups and holes in the natural manner.

If $[\bar{a}, x, y, z]$ is a block in $Q$ not containing $a$ then write the blocks of Design 3.A or 3.B as above.

The set of all blocks constructed in this manner from all blocks in $Q$ is the block set of an $H(3m + e, 2, 4, 3)$ frame.

We can now give a proof of:

**Theorem 1.** An $H(m, 6, 4, 3)$ design exists for all positive integers $m \neq 3$, with the possible exceptions of $m = 9, 27$ or 81.

**Proof.** If $m$ is even then this is just Theorem 2.4. If $m \equiv 1, 5, 7$ or 11 (mod 18) then by Theorem 2.10 there exists an $H(m, 2, 4, 3)$ frame containing only holes of size 5 or trivial holes. This implies the existence of an $H(m, 6, 4, 3)$ by Lemma 2.1, Lemma 2.2 and the existence of an $H(5, 6, 4, 3)$ constructed by Mills in [11]. If $m \equiv 13$ or 17 (mod 18) then by Theorem 2.9 there exists an $H(m, 6, 4, 3)$ frame containing holes of size $n + s$ and $s$, where $3n + s = m$. Taking $s = 1$ or 2 according as $m \equiv 13$ or 17 (mod 18) we have $n + s = 5$ or 1 (mod 6), and so by induction an $H(n + s, 6, 4, 3)$ design exists and hence by Lemma 2.1 an $H(m, 6, 4, 3)$ design exists.

If $m \equiv 3$ (mod 6) then by Theorem 2.9 there exists an $H(m, 6, 4, 3)$ frame containing holes of size $n$, where $m = 3n$ and $n \equiv 1$ (mod 2). Again using induction and Lemma 2.1, it suffices to exhibit an $H(243, 6, 4, 3)$ design.
The inverse plane of order 16 is a three-wise balanced design with 257 points and blocks of size 17 which contains two disjoint blocks. Delete 12 points from one block and two points from a disjoint block. This leaves a three-wise balanced design on 243 points with blocks of sizes 5, 13, 14, 15, 16 and 17. By Lemma 2.3 this gives an \( H(243, 6, 4, 3) \) design.

By similar methods we obtain:

**Theorem 2.** If an \( H(9, 6, 4, 3) \) design exists then for all odd \( m \geq 5 \) an \( H(m, 6, 4, 3) \) frame exists containing exactly one hole of size 5, 7 or 9 (and every other group in precisely one hole of size 2).

**Proof.** If \( m \equiv 1 \) or \( 7 \mod (mod 18) \) then the proof of Theorem 2.10 shows the existence of an \( H(m, 2, 4, 3) \) frame with exactly one hole of size 7 (by omitting one copy of design 1A). Lemma 2.2 then gives the result.

If \( m \equiv 5 \) or \( 11 \mod (mod 18) \) then the proof of Theorem 2.10 constructs an \( H(m, 2, 4, 3) \) frame with at least one hole of size 5, using Lemmas 2.2 and 2.1 to replace all but one of the holes with an \( H(5, 6, 4, 3) \) gives the result. If \( m \equiv 13 \) or \( 17 \mod (mod 18) \) or \( m \equiv 3 \mod (mod 6) \) then we proceed by induction as in the proof of Theorem 1, noting that we can replace two of the holes of size \( n + s \) by \( H \) designs and the third by an \( H \) frame having one hole of size 5, 7 or 9. 

We remark that 13 = 3 \times 4 + 1 and 15 = 3 \times 5 so that an \( H(m, 6, 4, 3) \) frame for \( m = 13 \) or 15 exists with a single hole of size 5. We also note the existence of an \( H(243, 6, 4, 3) \) frame with a single hole of size 5 (Proof of Theorem 1), so that holes of size 9 are only used when \( m = 9, 27 \) and 81. This means that for all odd \( m \neq 9, 27, 81 \), an \( H(m, 6, 4, 3) \) frame exists with a single hole of size 5 or 7; and this is the main ingredient in a proof of Theorem 8.

In order to prove the existence of \( H \) frames of the types referred to in Theorem 3 we need to construct three-wise balanced designs with certain properties. We begin with a result on partition of integers which will be used to show the existence of these designs.

**Lemma 2.11.** Let \( E = \{f_1, f_2, \ldots, f_e\} \) be a finite set of integers satisfying \( f_{i+1} - f_i > 2 \) for \( i = 1, 2, \ldots, e-1 \), \( f_1 - m > 1 \) and \( m - f_e > 1 \). Let \( p \geq 2 \) and let \( n \) be an integer with \( 2m \leq n \leq pM \); then there exists a partition of \( n \) with at most \( p \) parts each part lying in the range \([m, M]\) \( \setminus E \).

**Proof (Outline).** For some \( 2 \leq t \leq p \), \( n \) lies in the range \([tm, tM]\). Take \( t \) parts of sizes \([n/t]\) and \([n/t]\) whose sum is \( n \). If some of these parts lie in the forbidden set \( E \) then perturb these by taking alternatively parts of size \( f_i + 1 \) and \( f_i - 1 \) until at most one part lies in the forbidden set. This last forbidden part may be removed by a similar perturbation using one of the permitted parts.
We use this lemma to construct three-wise balanced designs containing a block of a fixed size \( a \), and no blocks of a forbidden range of sizes.

**Theorem 2.12.** Let \( E = \{ f_1, f_2, \ldots, f_e \} \) be a set of integers satisfying \( f_1 = 3, f_{i+1} - f_i > 2 \) for \( i = 1, 2, \ldots, e - 1 \), and let \( a > 2 \) be an integer not in \( E \). If \( q \) is a prime power with \( q \geq \max\{a - 1, f_e + 7\} \) then for all integers \( v \) satisfying \( q^2 - q + a - (q - 3)/(q - f_e - 2)/2 \) \( \leq v < q^2 - q + a \) there exists a three-wise balanced design on \( v \) points containing a block of size \( a \) and no block size \( f_i \) for \( i = 1, 2, \ldots, e \).

**Proof.** For every prime power \( q \) there exists an inverse plane of order \( q \) which is a three-wise balanced design on \( q^2 + 1 \) points with each block of size \( q + 1 \). Removing \( q + 1 - a \) points from a single block \( B_0 \) leaves a three-wise balanced design on \( q^2 - q - a \) points with a block of size \( a \) and other blocks of sizes \( q + 1, q \) and \( q - 1 \). Let \( \alpha \) and \( \beta \) be two of the remaining points of \( B_0 \) and let \( B_0, B_1, B_2, \ldots, B_q \) be the set of blocks containing \( \alpha \) and \( \beta \). Except for \( \alpha \) and \( \beta \) these blocks are disjoint and cover the points of the design.

Let \( x_1, x_2, \ldots, x_e \), be a partition of \( q^2 - q + a - v \) into parts of size at most \( q - 3 \) and with no parts of size \( q + 1 - f_i \) for \( i = 1, 2, \ldots, e \). Such a partition exists by Lemma 2.11, with \( 2 \leq s \leq \lceil (q - f_e - 2)/2 \rceil \). Removing \( x_i \) points from block \( B_i \) leaves a three-wise balanced design on \( v \) points containing one block of size \( a \) and blocks of sizes \( q + 1 - x_i \) not in \( E \) for each \( i \). Every other block has at least \( q - 1 - 2s \) points since any block of the design intersects \( B_i \) in at most two points, and \( q - 1 - 2s > f_e \), so no block of the design has size \( f_i \), for any \( i \).

In the case that \( q \equiv f_e \mod 2 \) we need only that \( q \geq \max\{a - 1, f_e + 6\} \) to guarantee that \( s \geq 2 \).

**Theorem 2.13.** For all \( v \geq 8737 \) there exists a three-wise balanced design on \( v \) points containing a block of size 83, and no block of size 3, 9, 27 or 81.

**Proof.** Let \( f_1(q) = q^2 - q - (q - 3)/(q - 83)/2 \) and let \( f_2(q) = q^2 - q \). Theorem 2.6 establishes the existence of a design of the required type for all \( f_1(q) + 83 \leq v \leq f_2(q) + 83 \), provided that \( q \geq 88 \) is a prime power. It remains to show that all integers \( v \geq 8737 = f_1(97) + 83 \) fall into an interval of this type for some prime power \( q \). It is easily verified that with \( q_1 = 97, q_2 = 101, q_3 = 103, q_4 = 107, q_5 = 109, q_6 = 113 \) and \( q_7 = 121 \), we have \( f_1(q_{i+1}) \leq f_2(q_i) \) for \( i = 1, 2, \ldots, 6 \). Furthermore \( f_1(1.1q) \leq f_2(q) \) for all \( q \geq 117 \). It is shown in [7] that for \( q \geq 116 \), there is always a prime between \( q \) and \( 1.1q \), and this completes the proof.

Theorem 3 now follows from the above, Lemma 2.3 and Theorem 1.
3. Construction of $H(m, 6, 3, 2)$-Frames

In this section we describe methods for constructing families of designs balanced for pairs ($t = 2$). We could discuss them in terms of $H$-frames for consistency with the preceding sections, however the accepted terminology for the designs constructed is group divisible designs (GDDs). The techniques used are not new, being due to Hanani [6] and Wilson [18]; see Wilson's paper for definitions and terminology.

Let $f$ be an odd number. Our aim in this section is to construct $H(2n + f, 6, 3, 2)$ frames having $n + 1$ pair wise disjoint holes, $n$ of size 2 and one of size $f$, or equivalently, group divisible designs having $n$ groups of size 12, one group of size $6f$ and blocks of size 3. A necessary condition for existence of such a design is given below.

**Lemma 3.1.** If $(X, G, A)$ is GDD having $n$ groups of size 12, one group of size $12k + 6$ and blocks of size 3, then

$$n > k + 2 \quad \text{and} \quad (n, k) \neq (2, 0).$$

**Proof.** Counting the number of blocks which intersect the $(12k + 6)$-group, we get $(12k + 6)(12n/2)$. But each of these blocks contains one pair of points from distinct 12-groups, hence $12^2n(n - 1)/2 \geq (12k + 6)(12n/2)$ with equality if $n = 2$. This yields the result. \[\]

We shall show that this condition is also sufficient when $k \leq 4$, using constructions which imply the existence of these designs for all $n \geq \lceil 12k + 2/8 \rceil$.

We begin by constructing these group divisible designs which we use later in the constructions.

**Design 3.2.** A GDD with three groups of size 4, one group of size 8 and blocks of size 3.

$$X = Z_{12} \cup \{\infty_1, \infty_2, \ldots, \infty_8\}$$

$$G = \{[i, i + 3, i + 6, i + 9]: i = 0, 1, 2\} \cup \{\{\infty_1, \infty_2, \ldots, \infty_8\}\}.$$  

The graph with vertex set $Z_{12}$ and all edges with lengths in $\{1, 2, 4, 5\}$ has a one-factorization by Lemma 2.5. Completing these one-factors by $\infty_1, \infty_2, \ldots, \infty_8$ gives all the blocks of the design.

**Design 3.3.** A GDD with three groups of size 4, one group of size 6 and blocks of size 3.

$$X = Z_{12} \cup \{\infty_1, \ldots, \infty_6\}$$

$$G = \{(i, i + 3, i + 6, i + 9): i = 0, 1, 2\} \cup \{\{\infty_1, \ldots, \infty_6\}\}.$$
Take the four blocks \([j, j + 4, j + 8], j = 0, 1, 2, 3\). The remaining blocks come from completing a one-factorization of the graph on \(Z_{12}\) with chord lengths \(\{1, 2, 5\}\), again using Lemma 2.5.

**DESIGN 3.4.** A GDD with three groups of size 4, one group of size 2 and blocks of size 3.

\[
X = Z_{12} \cup \{\infty_1, \infty_2\}
\]

\[
G = \{[i, i + 3, i + 6, i + 9]: i = 0, 1, 2\} \cup \{\{\infty_1, \infty_3\}\}
\]

Take the 12 blocks \([j, j + 1, j + 5], j \in Z_{12}\), and complete a one-factorization of the graph on \(Z_{12}\) with chord lengths \(Z\).

**LEMMA 3.5.** For all \(n \equiv 0 \text{ or } 1 \pmod{3}\) there exists a group of divisible design containing \(n\) groups of size 4 and block size 3.

A proof is given in [6], using the existence of Steiner triple systems with \(2n + 1\) points and Lemma 2.2.

**LEMMA 3.6.** If there exists a pair of orthogonal Latin squares of side \(n, n \equiv 0 \text{ or } 1 \pmod{3}\) and \(n > \lceil (12k + 6)/8 \rceil\), then there exists a GDD with \(n\) groups of size 12, one of size \(12k + 6\) and block size 3, for all \(k > 0\).

**Proof.** There is standard equivalence between a pair of orthogonal Latin squares of side \(n\) and a GDD with four groups of size \(n\) and block size 4. Delete \(n - \lceil (12k + 6)/8 \rceil\) points from one of the groups. Since at least one point has been deleted the resulting design may be considered as a group divisible design with \(n\) groups of size 3, one group of size \(s = \lceil (12k + 6)/8 \rceil\) and blocks of sizes 3, 4 and \(n\). All blocks which meet the \(s\)-group are of size 4. We now apply Wilson's fundamental construction [16] weighting each point in a 3-group with weight 4; and (if \(k\) is even) weighting one point with weight 6 and \(s - 1\) points with weight 8 or (if \(k\) is odd) weighting three points with weight 6 and \(s - 3\) points with weight 8. The result then follows using Lemma 3.5, Designs 3.2 and 3.3 and Wilson's construction.

**LEMMA 3.7.** If there exist a pair of orthogonal Latin squares of side \(n, n \equiv 0 \text{ or } 2 \pmod{3}\), \(n > \lceil (12k + 2)/8 \rceil\) then there exists a GDD with \(n\) groups of size 12 and one group of size \(12k + 6\).

**Proof.** As in the previous theorem, we begin with a group divisible design having four groups of size \(n\) and block size 4. Add a new point, say \(\infty\), to the groups and delete \(n - \lceil (12k + 2)/8 \rceil\) of the old points from a single group. Since at least one point has been deleted we obtain a group
divisible design having \( n \) groups of size 3 and one group of size 
\[ s = \lceil (12k + 2)/8 \rceil + 1, \]
and blocks of sizes 3, 4 and \( n + 1 \). The group of size \( s \) contains \( \infty \), blocks containing \( \infty \) are of size \( n + 1 \), and all other blocks 
meeting this group are of size 4. Again we use Wilson's construction 
weighting each point in a 3-group and the point \( \infty \) with weight 4; one 
point of the \( s \)-group with weight 6, and (if \( k \) is even and non-zero) \( s - 3 \) 
points with weight 8 and one other point of weight 4 or (if \( k \) is odd) \( s - 2 \) 
points with weight 8. The result then follows if \( k \neq 0 \) from Wilson's con-
struction, designs 3.2 and 3.3 and Lemma 3.5. If \( k = 0 \), weight the second 
point of the \( s \) group with weight 2 and use Design 3.4.

\[ \]

**Corollary 3.8.** For all \( n > \lceil (12k + 2)/8 \rceil, \) \( k \geq 0 \) and \( n \neq 2 \) or 6, there 
exists a GDD with \( n \) groups of size 12, one group of size \( 12k + 6 \) and blocks 
of size 3.

**Proof.** The result follows from the previous lemmas and the existence of 
a pair of orthogonal Latin squares of side \( n \) for all \( n \geq 3, \) \( n \neq 6 \) [1].

**Corollary 3.9.** For all \( n > 62 \) there exists a GDD with \( n \) groups of size 
12, one group of size 498 and block size 3.

Proof is immediate from corollary 3.8 with \( k = 41 \). This corollary is just a 
restatement of Theorem 5.

**Lemma 3.10.** There exists a GDD with six groups of size 12 and one 
group of size \( s \) for every even \( s \) in the range \( 12 \leq s \leq 60 \).

**Proof.** Since there exist five mutually orthogonal Latin squares of side 
12 [3], there exists a group divisible design having seven groups of size 12. 
Write \( s \) in the form \( s = 5a + 3b + c \), where \( a + b + c = 12 \) and \( a, \) \( b, \) \( c \) are 
nonnegative integers. We apply Wilson's fundamental construction 
weighting \( c \) points in a group with weight 1, \( b \) points in the same group 
with weight 3, and the remaining points of the group with weight 5. Any 
block containing seven points of weight 1 is replaced by a Fano plane. Any 
block containing a point of weight 3 is replaced by the affine plane of order 
3 with one block deleted. A block containing a point of weight 5 is replaced 
by the standard 5-point completion of a one-factorization of \( K_6 \), with its 5-
point block deleted.

**Theorem 3.11.** For \( 0 \leq k \leq 4 \) any \( n \geq k + 2 \) with \( (n, k) \neq (2, 0) \) there 
exists a group-divisible design comprising \( n \) groups of size 12, one group of 
size \( 12k + 6 \) and blocks of size 3.

**Proof.** By Corollary 3.8 and Lemma 3.10, it is sufficient to exhibit such
designs with \((n, k) = (4, 2) (5, 3)(6, 0)\) and \((7, 4)\). In these four cases we exhibit solutions with point set \(X = \mathbb{Z}_{12} \cup \{\infty ; i \in \mathbb{Z}_{12} + 6\}\) and groups

\[G_i = \{0, n, 2n, \ldots, 11n\} + i, \quad i - 0 < 1, \ldots, n - 1\]

and

\[G_n = \{\infty ; i \in \mathbb{Z}_{12} + 6\}.

**Case 1.** \(n = 4, k = 2\).

Take 48 blocks of the form \([0, 1, 3] + i, i \in \mathbb{Z}_{48}\). Let \(G\) be the graph with vertex set \(\mathbb{Z}_{48}\) and edge set \(\{[i, i + m] ; i \in \mathbb{Z}_{48}, m \in \{5, 7, 9, \ldots, 23\} \cup \{6, 10, 14, 18, 22\}\}\). \(G\) has a one-factorization by Lemma 2.5 which can be completed by the members of \(G_4\).

**Case 2.** \(n = 5, k = 3\).

Take 60 blocks of the form \([0, 12, 16] + i, i \in \mathbb{Z}_{60}\). The graph with vertex set \(\mathbb{Z}_{60}\) and edge set \(\{[i, i + m] ; i \in \mathbb{Z}_{60}, m \in \{1-3, 6-9, 11, 13, 14, 17-19, 21-24, 26-29\}\}\) is 42-regular and has a one-factorization by Lemma 2.5. Completing this one-factorization by the members of \(G_5\) constructs the remaining blocks of the design.

**Case 3.** \(n = 6, k = 0\).

Take 9 \(\times\) 72 blocks of the forms

\[\begin{align*}
[0 & 13 23] + i, \\
[0 & 14 22] + i, \\
[0 & 15 21] + i, \\
[0 & 16 20] + i, \\
[0 & 17 19] + i, \\
[0 & 24 34] + i, \\
[0 & 26 33] + i, \\
[0 & 27 32] + i, \\
[0 & 28 31] + i, \\
i \in \mathbb{Z}_{72}.
\end{align*}\]

The graph with chord lengths 1, 23 and 35 has a one-factorization by Lemma 2.5 and the remaining blocks come from completing the one-factors.

**Case 4.** \(n = 7, k = 4\).

Take 3 \(\times\) 84 blocks of the form \([0, 4, 36] + i, [0, 8, 20] + i, [0, 16, 40] + i, i \in \mathbb{Z}_{48}\). The graph with chord lengths \(m \equiv 0 \pmod{4}\) and \(m \equiv 0 \pmod{7}\) has a one-factorization by Lemma 2.5; completing the one-factors constructs the remaining blocks of the design.

This theorem with \(k = 2, 3, 4\) is just Theorem 4.

The cyclic methods given in the proof above may perhaps by generalized to prove the following:

**Conjecture 3.12.** For any \(f \geq 1\) and \(n > \lceil f/2 \rceil\) and \((n, f) \neq (2, 1)\) there exists a group divisible design with \(n\) groups of size 12, one group of size \(6f\) and blocks of size 3.
4. Concluding Remarks

Theorems 6 and 8 provide strong motivation for the study of the numbers $C(3, 4, 31)$ and $C(3, 4, 43)$. A brute force attack on these numbers by computer would be extremely expensive; however, the use of a hill-climbing technique may be successful in constructing minimal configurations.

On a deeper level we believe that the techniques used in Section 2, although quite specific to block size $k = 4$, may give clues to the eventual construction of Steiner systems $S(3, k, v)$ with $k > 4$. We note that an $H(m, 2, 4, 3)$ design implies the existence of a Steiner quadruple system on $2m$ points by adding $(\binom{m}{2})$ blocks of the form $G_i \cup G_j$, $i < j$, where $G_1, G_2, \ldots, G_m$ are the groups of the $H$-design. So the results of Section 2, particularly Theorem 2.10, can be interpreted as new constructions for Steiner quadruple systems, despite the fact that an $H(5, 2, 4, 3)$ design does not exist.

REFERENCES

