Implications of conjugacy class size

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1 Introduction

The influence of the sizes of conjugacy classes on the structure of a finite group has already been considered by many authors [1], [4], [5], [6], [8], [13, 14, 15]. Baer introduced the following definition [1].

Definition Let $G$ be a finite group and let $x \in G$. The index of $x$ in $G$ is given by $[G : C_G(x)]$ and is denoted by $\text{Ind}_G(x)$.

In the same article Baer characterised all finite groups such that every element of prime power order has prime power index. He then went on to raise the question of the characterisation of those groups whose $q$-elements, for just one prime $q$, have prime power index. We address this problem. We begin by making the following definition.

Definition Let $G$ be a finite group and let $q$ be a prime such that $q$ divides $|G|$. Then $G$ is a $q$-Baer group, or equivalently has the $q$-Baer property, if every $q$-element of $G$ has prime power index.

We prove the following theorem.

Theorem A Let $G$ be a $q$-Baer group for some prime $q$. Then

(a) $G$ is $q$-soluble with $q$-length 1, and

(b) there is a unique prime $p$ such that each $q$-element has $p$-power index.

Further, let $Q$ be a Sylow $q$-subgroup of $G$, then

(c) if $p = q$, $Q$ is a direct factor of $G$, or

(d) if $p \neq q$, $Q$ is abelian, $O_p(G)Q$ is normal in $G$ and $G/O_p(G)$ is soluble.

Baer’s result follows from the above theorem. More recently Chillag and Herzog have considered groups all of whose conjugacy classes have prime power size [8]. The above can also be considered as a generalisation of their result. In the same paper they considered groups whose conjugacy classes have square free size. These results have motivated the authors to address a question raised by the first author over 20 years ago [4].

N. Itô introduced the idea of a conjugate type vector of a finite group $G$ [13]. This is the vector $\{n_1, n_2, \ldots, n_r\}$ where $n_1 > n_2 > \cdots > n_r = 1$ are all the numbers that occur as sizes of conjugacy classes of $G$. The question is whether we can recognise a nilpotent group by its conjugate type vector.
To put this more precisely we introduce the product of two conjugate type vectors as follows:

\[ \{n_1, n_2, \ldots, n_r\} \times \{m_1, m_2, \ldots, m_s\} = \{n_im_j \mid 1 \leq i \leq r, 1 \leq j \leq s\}, \]

where we eliminate repeats. Thus \( \{p, 1\} \times \{p, 1\} = \{p^2, p, 1\} \). Note that if \( G \) and \( H \) are two groups with conjugate type vectors \( \vec{n} = \{n_1, n_2, \ldots, n_r\} \) and \( \vec{m} = \{m_1, m_2, \ldots, m_s\} \) respectively, then \( G \times H \) has conjugate type vector \( \vec{n} \times \vec{m} \). A nilpotent group \( G \) is the direct product of its Sylow subgroups and so its conjugate type vector is the product of conjugate type vectors which involve only one prime. The question is whether or not a group with a conjugate type vector of this shape is nilpotent. The first author proved that a group with conjugate type vector \( \{p^a, 1\} \times \{q^b, 1\} \), where \( p \) and \( q \) are primes and \( a \) and \( b \) positive integers, is nilpotent [4, Theorem 2]. In this paper we prove another special case, but this time approaching the question from the opposite direction.

**Theorem B** Let \( G \) be a finite group with conjugate type vector \( \vec{n} \) where

\[ \vec{n} = \{p_1, 1\} \times \{p_2, 1\} \times \cdots \times \{p_r, 1\} \]

and \( p_1, p_2, \ldots, p_r \) are distinct primes. Then \( G \) is nilpotent.

It is clear that to answer the question in full one must first know which conjugate type vectors it is possible for a \( p \)-group to have, we view this as another interesting question to be addressed.

In the next section we consider the structure of a group which has an element of prime power index. An early and important result of Burnside says that such a group is not simple [12, V §7] or [3, XVI §240]. We use this to obtain a generalisation of a result of Wielandt, which says that if \( x \) is a \( p \)-element of \( p \)-power index then \( x \in O_p(G) \), see [1, Lemma 6].

In the third section we prove Theorem A. Then in the final section we prove Theorem B. The proof is based on a result of Chillag and Herzog [8, Theorem 1] and before our proof we give an alternative proof of their result, in particular we avoid the use of the classification of finite simple groups.

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2 An element of prime power index

We begin by considering the structure of a group which has an element of prime power index.

Lemma 1 Let $G$ be a finite group and let $x$ be an element of prime power index. Then $x$ centralises each non-abelian chief factor of $G$.

Proof. The main theorem in Kazarin [16] says that $\langle x^G \rangle$ is a soluble subgroup of $G$. The lemma follows immediately. \qed

The following proposition is a generalisation of the result of Wielandt mentioned in the introduction. Let $x$ be a $p$-element of $p$-power index. Then, by Proposition 1, $[x^G, x^G] \subseteq O_p(G)$. This implies that $O_p(G)\langle x \rangle$ is subnormal in $G$ but then $O_p(G)\langle x \rangle$ is a subnormal $p$-subgroup and hence in $O_p(G)$. Thus $x \in O_p(G)$ as Wielandt states.

Proposition 1 Let $G$ be a finite group and let $x$ be an element of $G$ whose index is $p^n$, where $p$ is a prime and $n$ is a natural number. Then

$$[x^G, x^G] \subseteq O_p(G).$$

Proof. Let $G$ be a minimal counter-example to the proposition. Since the conditions of the proposition are inherited by quotient groups [1, Lemma 3], we can assume that $O_p(G) = 1$. So we need to show that $[x^G, x^G] = 1$.

Firstly suppose that $\langle x \rangle$ is subnormal in $G$, then $\langle x \rangle$ is contained in the Fitting subgroup $F(G)$ of $G$. But, since $F(G)$ is a $p'$-group, it follows that $x$ is central in $F(G)$. As $\langle x^G \rangle$ is in $F(G)$ the result follows. Thus we may further assume that $\langle x \rangle$ is not subnormal in $G$.

Let $N = \langle x^G \rangle$ and suppose that $N$ is a proper subgroup of $G$. By the minimality of $G$, and since $O_p(N) = 1$, we have that $[x^N, x^N] = 1$ and hence that $x$ is central in its normal closure in $N$. However, this implies that $\langle x \rangle$ is subnormal in $G$, which is a contradiction. Thus we can assume that $N = G = \langle x^G \rangle$.

Let $K$ be a minimal normal subgroup of $G$. If we consider $G/K$ we see that $G'/K = [x^G, x^G]K/K \subseteq O_p(G/K)$, by induction. Assume that $K$ is central in $G$. Then $O_p(G/K) = O_p(G)K/K$ and in this case $O_p(G/K) = 1$. So $G' \leq Z(G)$ and hence $G$ is nilpotent and the result is true.
Thus $K$ is not central in $G$ and so is not centralised by $x$. So $K$ has order divisible by $p$. Thus $K$ is a non-abelian chief factor of $G$ since $O_p(G) = 1$. But this is false by Lemma 1. □

Let $F_1(G) = F(G)$ be the Fitting subgroup of $G$. Then define $F_n(G)$ inductively by $F_n(G)/F_{n-1}(G) = F_1(G/F_{n-1}(G))$. We can now state and prove the following theorem.

**Theorem 1** Let $G$ be a finite group. Then all elements of prime power index are in $F_2(G)$.

**Proof.** Let $x \in G$ be such that $\text{Ind}_G(x) = p^a$ for some prime $p$ and $a$ a natural number. Then $[x^G, x^G] \subseteq O_p(G) \subseteq F(G)$. So $F(G)/\langle x \rangle/F(G)$ is a subnormal nilpotent subgroup of $G/F(G)$. Thus the theorem follows. □

### 3 Groups with the $q$-Baer property

Let $q$ be a prime. In this section we consider $q$-Baer groups. Recall a $q$-Baer group is a finite group such that every $q$-element has prime power index. Note that in considering such groups we will tend to ignore direct $q'$-factors which will have no effect on this property.

**Lemma 2** Let $G$ be a $q$-Baer group. Then $G$ has $q$-length 1.

**Proof.** We can assume that $O_{q'}(G) = 1$. In particular this means that for $x$ a $q$-element of $G$ we have $[x^G, x^G] \subseteq O_q(G)$, by Proposition 1. This implies that $\langle x \rangle$ is subnormal and thus the lemma is proved. □

**Lemma 3** Let $G$ be a $q$-Baer group and suppose that all $q$-elements of $G$ have $q$-power index. Then the Sylow $q$-subgroup of $G$ is a direct factor of $G$.

**Proof.** By Wielandt’s result all $q$-elements of $G$ lie in $F(G)$ and thus, $Q$, the Sylow $q$-subgroup of $G$ is normal. If $Q$ is abelian then it is central and the lemma is proved, so assume that $Q$ is not abelian. Let $L = Z(Q)C_Q(H)$, where $H$ is a $q$-complement [12, I, §18]. It is easy to show that every non-central element of $Q$ is $Q$-conjugate to an element of $L$ and thus, by a result of Burnside [3, §26], $L = Q$. Thus all elements of $Q$ centralise $H$ and the lemma is proved. □
Theorem 2 Let $G$ be a $q$-Baer group. Then there exists only one prime $p$ such that there are $q$-elements with $p$-power index.

Proof. We proceed by induction on $|G|$. Let $M$ be a minimal normal subgroup of $G$, let $G/M$ be denoted by $\overline{G}$ and in general the image of $X \subseteq G$ in $\overline{G}$ by $\overline{X}$.

(I) We first consider the case when a Sylow $q$-subgroup of $G$ is not normal and so we can choose $M$ to be a $q'$-group. So $G$ is not a $q'$-group. By induction there is a prime, say, $p_1$ so that every $q$-element of $G$ has $p_1$-power index. If all $q$-elements of $G$ have $p_1$-power index there is nothing to prove. So let $y$ be a $q$-element of $G$ such that $y \notin Z(G)$ and such that $y$ has index prime to $p_1$. Since $\text{Ind}_{G}(y)$ divides $\text{Ind}_{G}(y)$ and these are coprime we see that $y \in Z(\overline{G})$. Let $x$ be chosen so that $x$ and $y$ are contained in the same Sylow $q$-subgroup of $G$ and so that $\overline{x}$ has index a proper power of $p_1$ in $\overline{G}$. Consequently $x$ has index a power of $p_1$ in $G$.

Consider the element $xy$. If $xy$ has index prime to $p_1$ in $G$, then $xy \in Z(G)$ and from this it would follow that $xy \in Z(G)$ which contradicts the choice of $x$. Hence $xy$ has index a power of $p_1$.

If $y$ does not centralise $M$, then by Lemma 1, $M$ is an elementary abelian $p_2$-subgroup where $p_1 \neq p_2$. However, $M$ centralises every element of index $p_1$ and so $M$ centralises both $x$ and $xy$. Hence $M$ centralises $y$ against our assumption. Therefore, $y$ centralises $M$. Let $g \in G$ then $[y, g]$ is a $q$-element of $M$ [7, Lemma 1.11], in contradiction to $M$ being a $q'$-group.

Since $G$ has $q$-length 1, by Lemma 2, the only case left to consider is when $O_{q'}(G) = 1$. We can consider the following situation.

(II) $G$ has a normal Sylow $q$-subgroup, $Q$ say. So $G$ can be written as $G = QH$ where $H$ is a $q$-complement. Note if $Q = G$ there is nothing to prove.

(IIa) Assume that $Q$ is not abelian. Then every element of $Q \setminus Z(Q)$ must have index a power of $q$. Thus, by induction and the previous lemma, $G/Z(Q)$ has a normal $q$-complement. Let $N$ be a normal subgroup of index $q$ in $G$ containing $Z(Q)$. By induction $N$ satisfies the conclusions of the theorem. However $N$ does not have a normal $q$-complement, since $O_{q'}(G) = 1$, and thus no $q$-element of $N$ has index a power of $q$ in $N$. Therefore $Q \cap N$ is abelian. However $N$ has to contain a $q$-element, say $u$, whose index in $G$ is a $q$-power. Then $u \in Z(N)$. Now let $y \in Z(Q) \setminus Z(N)$. Then $uy \notin Z(N) \cup Z(Q)$ and so cannot have prime power index. Thus we can assume that $Q$ is abelian.
(IIb) So $Q$ is an abelian normal subgroup of $G$. We copy the ideas used in [2, pages 570-1]. Let $B$ and $C$ be non-central conjugacy classes in $Q$ such that $(|B|, |C|) = 1$, if this is possible. Suppose that $|C| = p^a$ for some prime $p \neq q$ and $a \geq 1$ and that $|C| > |B|$. A standard argument shows that $BC = D$, where $D$ is a conjugacy class of $Q$. Clearly $|D| \geq |C|$, also $|D|$ divides $|B||C|$. Since $G$ is a $q$-Baer group it follows that $|D| = |C|$. Then, as above, $DB^{-1}$ is a conjugacy class of $Q$. Also $C \subseteq CBB^{-1} = DB^{-1}$, so $C = CBB^{-1}$.

The kernel of $C$ is defined as follows $\ker(C) = \{x \in G | Cx = C\}$, it is a subgroup of $G$, this is taken from [2]. Since $C(\ker(C)) = C$ it follows that $|\ker(C)|$ divides $|C|$. We have shown that $H = \langle BB^{-1} \rangle \leq \ker(C)$. This implies that $|H|$ is a power of $p$. However, since $Q$ is normal and $B \neq 1$, we know that $H$ is a $q$-group, a contradiction.

This completes the proof of the theorem. ☐

We will now use the assertion that there is only one prime $p$ without further comment. The following is a simple consequence of Theorem 2.

**Corollary 1** Let $G$ be a $q$-Baer group. Assume that the Sylow $q$-subgroups of $G$ are not abelian. Then the Sylow $q$-subgroup of $G$ is a direct factor of $G$.

**Proof.** As the Sylow $q$-subgroups are not abelian there exist $q$-elements of $q$-power index (note if $x$ is a $q$-element and $x$ centralises a Sylow $q$-subgroup $Q$ then $x \in Q$). Now the result follows from Lemma 3. ☐

**Lemma 4** Let $G$ be a $q$-Baer group satisfying $O_q'(G) = 1$. Further let $Q$ be the Sylow $q$-subgroup of $G$. Then $G/Q$ has odd order or every $q$-element has $2$-power index.

**Proof.** Let $s$ be an involution in $G$. We can assume that $q \neq 2$. So $s$ inverts some element $a$ of $Q$. But now 2 divides $[N_G(a) : C_G(a)]$ and so $[G : C_G(a)]$ is a 2-power. ☐

As a consequence of the Feit-Thompson Theorem [10] we see that if the prime $p$ is not 2 we have that $G/O_{q',q}(G)$ is soluble. However, in the remaining case we have to work much harder.

We now need some arguments concerning the representations of the extra-special group. We know that there are two types of extra-special 2-group and
that each has a unique non-linear irreducible representation over a splitting field [12, III, §13]. However, since there are 2-dimensional representations of the quaternion and the dihedral groups of order 8 over any prime field, $K$, that is of odd characteristic, any extra-special group of order $2^{2n+1}$ has a unique representation of degree $2^n$ over $K$. The next result will be needed in the proof of Theorem 3.

**Proposition 2** Let $E$ be an extra-special group of order $2^{2n+1}$ represented faithfully on a vector space $V$ of dimension $2^n$ over a field $K$ of characteristic $p > 2$. Then

(i) there exists a vector $v$ such that $C_E(v) = \langle 1 \rangle$ and

(ii) for each non-central involution $s$ of $E$ there exists a vector $w$ such that $C_E(w) = \langle s \rangle$.

**Proof.** We prove the result by induction on $n$. We note that both results are true for $n = 1$ by straightforward calculations. So now assume the result is true for $n-1$ and let us consider the case of an extra-special group $E$ of order $2^{2n+1}$. Let $Z$ be the centre of $E$ and recall that $E/Z$ has the structure of an orthogonal space [12, III §13]. The non-central involutions correspond to singular elements of the space. Furthermore the full orthogonal group acts on this space. Hence any two non-central involutions of $E$ are equivalent under the action of the automorphism group of $E$. Choose a non-central involution $s$. Then there is a dihedral subgroup $D$ of order 8 containing $s$ and a subgroup $F$ which is an extra-special group of order $2^{2n-1}$ such that $E$ is a central product of $D$ and $F$. Thus we have a direct product $F \times \langle s \rangle$ as a subgroup of index 2 in $E$. Choose the trivial representation $Id$ of $\langle s \rangle$ and the representation $X$ of degree $2^{n-1}$ of $F$. Then the $2^n$ degree representation is the induced representation obtained from $Id \times X$.

Let $U$ be the underlying vector space for the representation $Id \times X$ and let $x$ be an element of order 4 in $D$. Then we can think of the induced representation as having the underlying space $V = U \oplus Ux$. Thus any element $v$ of $V$ can be thought of as $(u, w)$, where $u, w$ are in $U$. Then the action of various elements is given by:

a) if $f \in F$ then $(u, w)f = (uf, wf)$,

b) $(u, w)x = (-w, u)$ and

c) $(u, w)s = (u, -w)$.

By induction there exists $w$ such that $wf \neq w$ for all $1 \neq f \in F$. Now choose any $y$ in $U$ not linearly dependent on $w$. Then $v = (w, y)$ satisfies (i)
for $E$. We observe that the only non-trivial element of $E$ that fixes $(w, 0)$ is $s$. Thus (ii) is satisfied for the involution $s$. However, we chose $s$ arbitrarily. Recall that there is a unique non-linear irreducible representation of $E$ so any choice of involution would lead to the same representation. Thus we have that (ii) is true for all involutions and the proposition is proved.

We now prove a proposition about permutation groups which will be helpful in the proof of the next theorem. We consider all primes $p$, although we only use the case when $p = 2$.

**Proposition 3** Let $G$ be a permutation group on a set $X$ and let $p$ be a prime. Assume that for any subset $Y$ of $X$ there is an integer $a$ so that the orbit of $Y$ under $G$ has size $p^a$. Then

(i) $G$ is a $p$-group if $p = 2$ or $p > 5$ or

(ii) $G$ is \{2, $p$\}-group if $p = 3$ or 5.

**Proof.** Let $G$ be a minimal counter-example to the proposition. By considering the transitive constituents it can be seen that $G$ is transitive, since if $G$ acts on each transitive constituent according to the conclusions of the proposition so does $G$. Let $N$ be a non-trivial proper normal subgroup of $G$, if such exists.

Assume that $N$ is not transitive. Then $G$ acts imprimitively on some partition of $X$. Let $M$ be the kernel of this action. Then $G/M$ acts on the blocks of imprimitivity. Note that for any subset of these blocks the stabiliser of this set in $G/M$ is the stabiliser of the union of the blocks. Thus both $G/M$ and $M$ satisfy the hypothesis of the proposition. Thus in case (i) both are $p$-groups and hence so is $G$. In case (ii) both are \{2, $p$\}-groups and hence so is $G$. This gives a contradiction. Thus any non-trivial normal subgroup is transitive.

Let $|X| = p^n$ and assume that $n > 1$. Note that \( \binom{p^n}{p^{n-1}} \) is divisible by only the first power of $p$. Hence there exists a subset of size $p^{n-1}$ which has an orbit of size exactly $p$. Let $N$ be the kernel of the action on this orbit. $N$ cannot act transitively and cannot be trivial. This is a contradiction.

Thus $|X| = p^n$. If $p = 2$ then $G$ is a 2-group and the proof is complete. Now assume that $p > 2$. If $G$ is not soluble then $G$ is doubly transitive by another theorem of Burnside [12, V, 21.3]. But then $G$ would have an orbit on subsets of size 2 of length $p(p - 1)/2$. This leads to a contradiction unless $p = 3$. However if $p = 3$ then $G$ has order 3 or 6 and the proof is complete.
Thus we may assume that $G$ is soluble and $p > 3$. This implies that $|G|$ divides $p(p - 1)$. Hence there is nothing left to prove if $p = 5$. It is also true that $G$ has a minimal normal regular subgroup of order $p$ so that no element fixes more than one point. Let $r$ be a prime larger than 2 and assume that $G$ has an element, say $g$, of order $r$. Then $g$ has to fix a pair of points of $X$. But then $g$ fixes both of them and so fixes more than one point. Hence $G$ has only elements of order $p$ and powers of 2. Let $s$ be an involution in $G$ and consider the orbits of subsets of size 3. There are $p(p - 1)(p - 2)/6$ such subsets and they fall into $(p - 1)(p - 2)/6$ orbits of size $p$. Now $s$ has to fix each orbit and so at least one subset of size 3. Since it fixed exactly one point each subset it fixes will consist of the fixed point and a transposition. There are at most $(p - 1)/2$ transpositions and so it can fix at most this number of orbits of subsets of size 3. However if $p > 5$ we have $1 < (p - 2)/3$ which is the final contradiction. □

The referee pointed out that for both $p = 3$ and $p = 5$ there are examples of groups satisfying the hypotheses of the above proposition which are not $p$-groups, the symmetric group of degree 3 and the dihedral group of order 10.

**Theorem 3** Let $G$ be a finite group with the $q$-Baer property such that every $q$-element has 2-power index, $q \neq 2$. Then $G/O_{q', q}(G)$ is a 2-group.

**Proof.** The proof of this theorem is by induction on $|G|$. We begin by assuming that $O_{q'}(G) = 1$. Let $Q$ be the Sylow $q$-subgroup and let $H$ be a complement to $Q$ and assume that $H$ is not a 2-group.

Let $M$ be a minimal normal subgroup of $G$. Assume that $M < Q$. By induction on $G/M$ we see that $[G/M : C_{G/M}(Q/M)]$ is a power of 2. Applying induction again, this time to the pre-image of $C_{G/M}(Q/M)$ in $G$, leads to a contradiction. Hence $Q$ is a minimal normal subgroup of $G$ and $H$ is a linear group acting irreducibly on $Q$ such that $[H : C_H(v)]$ is a power of 2 for any element $v$ of $Q$.

Now let $L$ be a proper normal subgroup of $H$ and apply induction to $QL$. Then we see that $L$ is a 2-group. Furthermore $H$ has a unique maximal normal subgroup, say $T$, which is a 2-group and $H/T$ is not a 2-group. We now apply Clifford Theory [12, V, §17] to $Q$ under the action of $L$. Assume that $L$ has more than one homogeneous component and write $Q = Q_1 \oplus Q_2 \oplus \cdots \oplus Q_m$, for some natural number $m$. Any element $v$ of $Q$ can be represented uniquely by $\sum v_i$, $1 \leq i \leq m$, where each $v_i \in Q_i$. Let $h \in H$
centralise $v$. Then $h$ permutes the set of $i$ such that $v_i \neq 1$. But $[H : C_H(v)]$ is a power of 2 and so for any subset $I$ of $\{1, 2, \ldots, m\}$ the subset of $H$ which normalises $\sum Q_i$ for $i \in I$ has index a power of 2. From Proposition 3 we deduce that the action of $H$ on the decomposition is as a 2-group. Thus $m = 1$ and any normal subgroup of $H$ is homogeneous. Further let $Y$ be a simple $L$ submodule of $Q$. If $h \in H$ centralises $y \in Y$ then $h$ fixes $Y$. Thus the group $Y K$ where $K$ is the stabiliser of $Y$ in $H$ satisfies the hypothesis and $[H : K]$ is a power of 2. Thus any odd order subgroup of $K$ acts trivially on $Y$. Since $L$ acts homogeneously, $L$ has trivial kernel on $Y$ so it is centralised by any odd order subgroup in $K$. Hence $[H : C_H(L)]$ is a power of 2 and this is false unless $L$ is central in $H$. So any normal subgroup of $H$ acts irreducibly on $Q$ or is central in $H$.

Now by a theorem of P. Hall [12, III, §13] we know that $T$ has the structure of a central product of an extra-special group $E$ and a group $J$, where $J$ is either cyclic, generalised quaternion, dihedral or semi-dihedral. We can assume in the last three cases that $|J| > 8$. Thus $T' = J'$ is cyclic and characteristic and since $H$ has no subgroups of index 2, $T'$ is central. But this is not true in these groups and so we see that $J$ is cyclic. Then $J$ is central in $T$ and so as above $J$ is central in $H$.

We can now complete the argument when $H/T$ is cyclic. Let $v \in Q$ so that $C_T(v) \neq 1$, such a $v$ exists as long as $T$ is not the quaternion group of order 8. It is clear that $Z(T)$ contains $T'$ and so $C_T(v)Z(T)$ is normal in $T$. Now $[H : C_H(v)]$ is a power of 2 and $C_H(v)$ normalises $C_T(v)Z(T)$. If $C_H(v)T = H$ then $C_T(v)Z(T)$ is normal in $H$ but $C_T(v)Z(T)$ is abelian (note that $C_T(v) \cap Z(T) = 1$). This is a contradiction. So $C_H(v)T \neq H$ and this cannot happen if $H/T$ has odd order. We need a separate argument for the case when $T$ is the quaternion group of order 8 but this is straightforward and we leave it to the reader.

We can now assume that $H/T$ is not cyclic, so in particular $H/T$ is not of order 2 and thus $E \neq 1$. Let $E$ have order $2^{2n+1}$. Since $E$ is normal in $H$ it acts irreducibly on $Q$. There is a unique non-linear representation of an extra-special group of order $2^{2n+1}$, it has dimension $2^n$ over any field. Thus $Q$ has dimension $2^n$.

Let $t$ be a non-central involution in $E$, then $\langle t \rangle$ is the centraliser in $E$ of an element $v$ in $Q$ by Proposition 2. So $\langle t \rangle = E \cap C_H(v)$, and $\langle t \rangle$ is in the centre of $C_H(v)$. Since $[H : C_H(v)]$ is a power of 2, each non-central involution has $2^n$ conjugates for some natural number $a$. 

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Finally using the observation that $E/Z$ is an orthogonal space we see that the number of non-central involutions in $E$ is $2 \times (2^{2n-1} \pm 2^{n-1} - 1)$ [11, §5.5, page 206]. So the number of non-central involutions in $T$ is congruent to 2 mod 4 unless $n = 1$. If $n > 1$ there is a subgroup of index 2, which is the final contradiction. If $n = 1$ then $H/T$ is cyclic.

We comment that this result is not true if the prime is not 2. Consider the group, $G$, of order $8^7$ constructed from the affine group $AGL(1,8)$ with the field automorphism of order 3 acting. Then $G$ is a 2-Baer group with every 2-element having index 7 and $O_2(G) = 1$ but $G/O_2(G)$ not a 7-group.

The proof of the main theorem follows quickly from these results.

**Theorem 4** Let $G$ be a $q$-Baer group and let $p$ be the prime such that there are $q$-elements with $p$-power index. Let $Q$ be a Sylow $q$-subgroup of $G$. If $p = q$ then $Q$ is a direct factor of $G$. If $p \neq q$ then
\begin{enumerate}[i)]  \item $QO_p(G)$ is normal in $G$ and  \item $G/O_q'(G)$ is soluble.  \end{enumerate}

**Proof.** If $p = q$ the result follows from Corollary 1.

Now suppose $p \neq q$ and assume that $O_p(G) = 1$. Let $x$ be an element of $Q$ then, by Proposition 1, $\langle x \rangle$ is subnormal in $G$. Hence $Q$ is normal in $G$ as required. Assume further that $O_q'(G) = 1$. To prove the result we only need to show that in this case $G/Q$ is soluble. If $p \neq 2$ the result follows from Lemma 4 and if $p = 2$ the result follows from Theorem 3. 

The above theorem along with Lemma 2 and Theorem 2 give Theorem A. Consider the following lemma.

**Lemma 5** Let $G$ be a $q$-Baer group and a $p$-Baer group for primes $p \neq q$. Suppose all $q$-elements have $p$-power index then all $p$-elements have $q$-power index.

**Proof.** Suppose the lemma is not true, then all $p$-elements have $r$-power index for some prime $r \neq q$. Let $P$ be a Sylow $p$-subgroup of $G$. If $r = p$ then $P$ is a direct factor by Corollary 1, a contradiction, so $r \neq p$. Let $x \in P$ then since $\text{Ind}_G(x)$ is prime to $p$ and $q$ and $O_p(G) \leq G$, by the previous theorem, we have that $C_G(x) \supseteq QO_p(G)$. This holds for all $x \in P$ contradicting the fact that $q$-elements have $p$-power index.

So in particular, if every element of prime power order has prime power
index and for a specific prime $q$ all $q$-elements have $p$-power index, then $PQ$ is a direct factor of $G$, where $P$ is a Sylow $p$-subgroup and $Q$ is a Sylow $q$-subgroup. Thus Baer’s characterisation of such groups [1], follows from Theorem A.

## 4 Conjugacy classes of square free size

We thank the referee for the proof of the next lemma which is considerably simpler than the original.

**Lemma 6** Suppose that $G$ is a simple non-abelian group of even order. Then there is an element of $G$ which has conjugacy class of length divisible by 4.

**Proof.** Suppose that $G$ is a simple non-abelian group of even order and let $z$ be an involution in $G$. Assume that $x$ is an arbitrary element of $G$. Then assuming that the lemma is false, $[G : C_G(x)]$ is either odd or is equal to $2n$ where $n$ is odd. In the first case $z$ is conjugate to an element of $C_G(x)$. In the second case, either $z$ is conjugate to an element of $C_G(x)$ or $z$ operates as a fixed point free permutation on the $2n$ cosets of $C_G(x)$ in $G$. However, $z$ is an involution and so must, in the latter case, be a product of $n$ transpositions. That is $z$ acts as an odd permutation and we have a contradiction to the simplicity of $G$. Hence in both cases $z$ is conjugate to an element of $C_G(z)$. Using Burnside we conclude that $z \in Z(G)$, a contradiction. □

The previous lemma enables us to prove the next proposition, which is Proposition 5 of [8], without the aid of the classification of finite simple groups.

**Proposition 4** Let $G$ be a finite group with conjugate type vector $\pi = \{n_1, n_2, \ldots, n_r\}$. Suppose $n_i$ is not divisible by 4, for $1 \leq i \leq r$. Then $G$ is soluble.

**Proof.** Note if a prime $p$ does not divide the size of any conjugacy class of $G$ then the Sylow $p$-subgroup of $G$ is an abelian direct factor [4, Lemma 1]. So, using the Feit-Thompson Theorem [10], we can assume that $n_i$ is even for at least one $n_i$, $1 \leq i \leq r$. Let $G$ be a minimal counter-example to the lemma. We can assume that $G$ is simple since if $N$ was a normal subgroup of $G$ we would have, by minimality or the Feit-Thompson Theorem, that $N$ and $G/N$ were soluble and hence $G$ soluble. So, by the previous lemma, the
proof is complete. □

Replacing the first paragraph of the proof of Lemma 1.2 in [8] by the above yields a classification-free proof of Chillag and Herzog’s Theorem 1.

**Theorem B** Let $G$ be a finite group with conjugate type vector

$$\bar{n} = \{p_1, 1\} \times \cdots \times \{p_r, 1\}$$

where $p_1 < \cdots < p_r$ are primes. Then $G$ is nilpotent.

**Proof.** We prove this result by induction on $r$. Note that for $r = 1$ the result is true by a result of Ito [13] and for $r = 2$ the result follows from [4, Theorem 2].

Let $L = \{x \in G : \text{Ind}_G(x) = 1 \text{ or } p_1\}$. We show that $L$ is a normal subgroup of $G$. Let $C_1, C_2, \ldots, C_r$ be the distinct conjugacy classes of $G$. Assume that $C_i$ and $C_j$ are classes with $p_1$-elements, if they exist. Then

$$C_iC_j = \sum_{s=1}^{r} a_{ijs}C_s; \text{ where } a_{ijs} \in \mathbb{Z} \quad (s = 1, \ldots, r).$$

So

$$p_1^2 = \sum_{s=1}^{r} a_{ijs}|C_s|.$$  

By [9, 87.4]

$$a_{ijs} = \frac{n_s}{n_j}a_{i^*s},$$

where $n_s = [G : C_s]$, $n_j = [G : C_j]$ and $C_i^*$ is the conjugacy class of elements which are the inverse of those in $C_i$. If $|C_s| = p_m$ ($m = 2, \ldots, r$), we see that

$$a_{ijs} = \frac{p_1}{p_m}a_{i^*s}.$$  

So $p_1$ divides $a_{ijs}$ and thus $p_1p_m$ divides $a_{ijs}|C_s|$, which is greater than $p_1^2$ unless $a_{ijs} = 0$. Similarly if $|C_s|$ is divisible by at least two primes, then $a_{ijs}|C_s| > p_1^2$ unless $a_{ijs} = 0$. Thus $L$ is a subgroup and it is clearly normal as is $C_G(L)$.

Assume first that there exists a $p_1$-element $x$ of index $p_1$. Then there exists a normal $p_1$-complement, $K$ [4, Theorem 1]. Furthermore $K < C_G(x)$. But now every $p_1'$-element $y$ of $C_G(x)$ has index in $C_G(x)$ prime to $p_1$, since
$C_G(xy) = C_G(x) \cap C_G(y)$. So the Sylow $p_1$-subgroup of $C_G(x)$ is a direct factor of $C_G(x)$ [4, Lemma 1]. So $C_G(x) = K \times P_x$, where $P_x$ is a Sylow $p_1$-subgroup of $C_G(x)$. Let $u$ be an element of index prime to $p_1$. Then $u \in C_G(x)$ and from the structure of $C_G(x)$ it can be seen that $u \in K$. Hence $K$ satisfies the hypothesis of the theorem with respect to the primes $p_2, \ldots, p_r$ and so, by induction, is nilpotent. So $C_G(x) = K \times P_x$ is nilpotent. Note that since $C_G(x)$ is of index $p_1$ and $p_1$ is the smallest prime divisor of $|G|$, it follows that $C_G(x)$ is normal.

Now assume that there is no $p_1$-element of index $p_1$. Consider the following filtration of $G$, $1 \leq Z(G) \leq F(G) \leq G$, where $Z(G)$ denotes the centre of $G$ and $F(G)$ the Fitting subgroup of $G$. By [8, Theorem 1] we know that $|G/F(G)|$ is square free and cyclic. As $p_1$ is the smallest prime and we are assuming that there does not exist a $p_1$-element of index $p_1$ we have that $F(G) \cap P_1$ is central. Also since $|P_1/(F(G) \cap P_1)| = p_1$, we have that $P_1$ is abelian. Note also that $|G/C_G(L)| = p_1$, since under our present hypothesis there is a unique subgroup of index $p_1$.

Let $u \in G$ with $\text{Ind}_{G}(u) = p_1 m$ where $(p_1, m) = 1$. If $u \in C_G(L)$ then $\text{Ind}_{C_G(L)}(u) = m$. If $u$ is not in $C_G(L)$ then the order of $u$ is divisible by $p_1$. So we can write $u$ as $u = u_1 u_2$ with $[u_1, u_2] = 1$ and $|u_1|$ a power of $p_1$ and $|u_2|$ coprime to $p_1$. Note $u_1 \in Z(P_1)$ for some Sylow $p_1$-subgroup $P_1$ of $G$, in particular $P_1 = \langle u_1, P_1 \cap F(G) \rangle$. However $u_2 \in C_G(P_1)$ so $C_G(u_1 u_2) \geq P_1$ which is a contradiction. Hence for all $m$ with $(m, p_1) = 1$ there exists an element $u \in C_G(L)$ with $\text{Ind}_{C_G(L)}(u) = m$, namely $u \in G$ with $\text{Ind}_{G}(u) = p_1 m$. Thus by induction $C_G(L)$ is nilpotent.

So now in all cases $|G/F(G)| = p_1$ and $G = F(G)/\langle x \rangle$ with $x^{p_1} \in F(G)$. For $P_1$, a Sylow $p_1$-subgroup of $G$, either there exists a $p_1$-element of index $p_i$ or $P_1$ is abelian and $\text{Ind}_{G}(x) = p_1$. In the first case there exists a normal $p_1$-complement [4, Theorem 5], so $P_1$ is a direct factor and by induction the proof is complete. However $\text{Ind}_{G}(x) = p_1$ for at most one $p_i$ and we are done by induction since the case $r = 2$ has been dealt with [4, Theorem 2].

References


