Point Spectra of Some Mutually Exciting Point Processes

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SUMMARY

The point spectral matrix is obtained for a class of mutually exciting point processes. The solution makes use of methods similar to those used in solving Wiener–Hopf integral equations.

POINT PROCESS; SPECTRUM; COVARIANCE DENSITY

1. INTRODUCTION

IN contagious processes (e.g. measles, hijacking, etc.) the occurrence of events increases the probability of further events occurring in the near future. Also several series of events may interact with each other, for example one might consider notifications of some disease in a number of adjacent regions which would interact through infectives or carriers moving between the regions. In this paper we postulate a model for such processes and derive a general expression for the point spectral matrices. These theoretical spectra are useful for comparison with spectra estimated from data and thus provide a means of evaluating the fit of such a model in the manner of Bartlett (1963). The model studied was put forward in an earlier paper (Hawkes, 1971) but the solution was obtained only in special cases. In this paper an elegant solution is obtained for the general case.

Consider a stationary k-variate point process $\mathbf{N}(t)$, where $N_i(t)$ represents the cumulative number of events in the ith process up to time t, with intensity vector

$$\boldsymbol{\lambda} = \mathscr{E}\{d\mathbf{N}(t)\}/dt$$

and covariance density matrix

$$\boldsymbol{\mu}(\tau) = \mathscr{E}\{d\mathbf{N}(t+\tau)\, d\mathbf{N}^{\mathrm{T}}(t)\}/(dt)^2 - \boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}}$$

which does not depend on t. We note that

$$\boldsymbol{\mu}(-\tau) = \boldsymbol{\mu}^{\mathrm{T}}(\tau). \tag{1}$$

If events cannot occur multiply then $\mathscr{E}[\{dN_i(t)\}^2] = \mathscr{E}\{dN_i(t)\}$ so that we may write the complete covariance density as

$$\boldsymbol{\mu}^{(c)}(\tau) = \mathbf{D}\,\delta(\tau) + \boldsymbol{\mu}(\tau), \tag{2}$$

where $\mathbf{D} = \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_k)$, $\delta(\tau)$ is the Dirac delta function and $\boldsymbol{\mu}(\tau)$ is continuous at the origin. Then the point spectral density matrix

$$\mathbf{f}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, \boldsymbol{\mu}^{(c)}(\tau)\, d\tau$$

$$= \frac{1}{2\pi}(\mathbf{D} + \mathbf{M}(\omega)). \tag{3}$$

We use the convention that for a matrix function denoted by a Greek symbol, the matrix of Fourier transforms (without the factor 2π) is denoted by the corresponding Roman capital. For example

$$\mathbf{M}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau}\, \boldsymbol{\mu}(\tau)\, d\tau.$$

Hawkes (1971) introduced a class of mutually exciting processes in which the intensity vector at any instant t depends linearly on the history of the process up to that instant. Thus we suppose the increment $\Delta N_i(t)$ over the interval $(t, t+\delta t)$ is such that

$$P\{\Delta N_i(t) = 1 \,|\, \mathbf{N}(s), s \leqslant t\} = \Lambda_i(t)\, \delta t + o(\delta t) \tag{4}$$

and the probability of more than one event of any kind is $o(\delta t)$. $\boldsymbol{\Lambda}(t)$ is assumed to be a random variable depending on the past so that

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\upsilon} + \int_{-\infty}^{t} \boldsymbol{\gamma}(t-u)\, d\mathbf{N}(u). \tag{5}$$

We assume $\gamma_{ij}(u) = 0,\ u < 0.$ (6)

In Hawkes (1971) a method of finding $\boldsymbol{\mu}(\tau)$ and $\mathbf{f}(\omega)$ was given in the rather special case of exponential decay in which

$$\boldsymbol{\gamma}(u) = \sum_{m=1}^{n} \mathbf{A}_m\, e^{-\beta_m u}, \quad u > 0.$$

In this paper we obtain a simple solution for general $\boldsymbol{\gamma}(u)$.

2. MUTUALLY EXCITING PROCESSES

Assuming stationarity we have from (5)

$$\boldsymbol{\lambda} = \mathscr{E}\{\boldsymbol{\Lambda}(t)\} = \boldsymbol{\upsilon} + \int_{-\infty}^{\infty} \boldsymbol{\gamma}(u)\, \boldsymbol{\lambda}\, du$$

so that

$$\boldsymbol{\lambda} = \{\mathbf{I} - \mathbf{G}(0)\}^{-1}\, \boldsymbol{\upsilon}$$

provided this has positive elements. Now for $\tau > 0$ we may write, by virtue of (4) and (5),

$$\begin{aligned}\boldsymbol{\mu}(\tau) &= \mathscr{E}\left[\left\{\boldsymbol{\upsilon} + \int_{-\infty}^{t+\tau} \boldsymbol{\gamma}(t+\tau-u)\, d\mathbf{N}(u)\right\} \frac{d\mathbf{N}^{\mathrm{T}}(t)}{dt}\right] - \boldsymbol{\lambda}\boldsymbol{\lambda}^{\mathrm{T}} \\ &= \int_{-\infty}^{\tau} \boldsymbol{\gamma}(\tau-u)\, \boldsymbol{\mu}^{(c)}(u)\, du,\end{aligned}$$

so that by (2)

$$\boldsymbol{\mu}(\tau) = \boldsymbol{\gamma}(\tau)\, \mathbf{D} + \int_{-\infty}^{\infty} \boldsymbol{\gamma}(\tau-u)\, \boldsymbol{\mu}(u)\, du, \quad \text{for } \tau > 0. \tag{7}$$

This is the fundamental integral equation we have to solve subject to condition (1). This is similar to the Wiener–Hopf integral equation but with a rather different condition. The method of solution is similar. We introduce a supplementary matrix

$$\boldsymbol{\beta}(\tau) = \boldsymbol{\gamma}(\tau)\,\mathbf{D} + \int_{-\infty}^{\infty} \boldsymbol{\gamma}(\tau-u)\,\boldsymbol{\mu}(u)\,du - \boldsymbol{\mu}(\tau), \quad -\infty < \tau < \infty. \tag{8}$$

We observe immediately that $\boldsymbol{\beta}(\tau) = \mathbf{0}$ for $\tau > 0$ so that the matrix of Fourier transforms $\mathbf{B}(\omega)$ is regular in the upper half plane $\mathrm{Im}\,(\omega) > 0$, by which we mean that each element of the matrix is regular in this region. Similarly from (6) it follows that $\mathbf{G}(\omega)$ is regular in the lower half-plane $\mathrm{Im}\,(\omega) < 0$. In deriving our results we shall assume slightly stronger conditions that $\boldsymbol{\gamma}$ and $\boldsymbol{\beta}$ both decay exponentially so that we can find $\eta > 0$ such that

$$\begin{aligned} \gamma_{ij}(u) &< A\,e^{-\eta u}, \\ |\beta_{ij}(u)| &< B\,e^{\eta u} \end{aligned} \tag{9}$$

for some constants A, B. Then it follows from Noble (1958, p. 12) that $\mathbf{G}(\omega)$ is regular for $\mathrm{Im}\,(\omega) < \eta$ and it can be shown that $\mathbf{G}(\omega) \to 0$ as $|\omega| \to \infty$ in this region. Similarly $\mathbf{B}(\omega)$ is regular in the half plane $\mathrm{Im}\,(\omega) > -\eta$ and $\mathbf{B}(\omega) \to 0$ as $|\omega| \to \infty$ in this region.

On taking Fourier transforms of equation (8) we have

$$\mathbf{B}(\omega) = \mathbf{G}(\omega)\,\mathbf{D} + \mathbf{G}(\omega)\,\mathbf{M}(\omega) - \mathbf{M}(\omega)$$

or

$$\mathbf{M}(\omega) = \{\mathbf{I} - \mathbf{G}(\omega)\}^{-1}\{\mathbf{G}(\omega)\,\mathbf{D} - \mathbf{B}(\omega)\}. \tag{10}$$

From (1) it follows that $\mathbf{M}(-\omega) = \mathbf{M}^{\mathrm{T}}(\omega)$ so that

$$\{\mathbf{I} - \mathbf{G}(-\omega)\}^{-1}\{\mathbf{G}(-\omega)\,\mathbf{D} - \mathbf{B}(-\omega)\} = \{\mathbf{G}(\omega)\,\mathbf{D} - \mathbf{B}(\omega)\}^{\mathrm{T}}\{\mathbf{I} - \mathbf{G}^{\mathrm{T}}(\omega)\}^{-1}$$

or

$$\{\mathbf{I} - \mathbf{G}(-\omega)\}\,\mathbf{B}^{\mathrm{T}}(\omega) + \mathbf{G}(-\omega)\,\mathbf{D} = \mathbf{B}(-\omega)\{\mathbf{I} - \mathbf{G}^{\mathrm{T}}(\omega)\} + \mathbf{D}\mathbf{G}^{\mathrm{T}}(\omega), \tag{11}$$

at least for values of ω for which $\mathbf{M}(\omega)$ exists.

By virtue of this equality we can define a matrix function $\mathbf{H}(\omega)$ equal to the left-hand side of (11) for $\mathrm{Im}\,(\omega) > -\eta$ and equal to the right-hand side for $\mathrm{Im}\,(\omega) < \eta$. From the remarks following (9) we see that $\mathbf{H}(\omega)$ is regular in the whole complex plane and $\mathbf{H}(\omega) \to 0$ as $|\omega| \to \infty$. Hence from Liouville's theorem, Copson (1935, p. 70), it follows that $\mathbf{H}(\omega)$ is identically zero and in particular the right-hand side of (11) is zero for $\mathrm{Im}(\omega) < \eta$ so that

$$\mathbf{B}(\omega) = -\mathbf{D}\mathbf{G}^{\mathrm{T}}(-\omega)\{\mathbf{I} - \mathbf{G}^{\mathrm{T}}(-\omega)\}^{-1}.$$

Substituting in (10) we have

$$\mathbf{M}(\omega) = \{\mathbf{I} - \mathbf{G}(\omega)\}^{-1}\{\mathbf{G}(\omega)\,\mathbf{D} + \mathbf{D}\mathbf{G}^{\mathrm{T}}(-\omega) - \mathbf{G}(\omega)\,\mathbf{D}\mathbf{G}^{\mathrm{T}}(-\omega)\}\{\mathbf{I} - \mathbf{G}^{\mathrm{T}}(-\omega)\}^{-1} \tag{12}$$

in the strip $-\eta < \mathrm{Im}\,(\omega) < \eta$.

Substituting this into (3) we have finally the spectral density matrix

$$\mathbf{f}(\omega) = \frac{1}{2\pi}\{\mathbf{I} - \mathbf{G}(\omega)\}^{-1}\,\mathbf{D}\{\mathbf{I} - \mathbf{G}^{\mathrm{T}}(-\omega)\}^{-1}. \tag{13}$$

If the conditions (9) are not satisfied the derivation given here is not valid. We note, however, that the above result still provides a solution satisfying (1), (7) and (8) and one would expect it to be unique.

In the simple case of exponential decay in which the $\gamma_{ij}(v) = \alpha_{ij} e^{-\beta_{ij} v}$, $v>0$ ($i,j = 1$ to k), the form of (13) is such that on inversion the covariance densities will in general have the form of a weighted sum of exponentials

$$\mu_{ij}(\tau) = \sum_r c_{ij}(\tau)\, e^{-\eta_r \tau}, \quad \tau > 0,$$

where $i\eta_r$ are the values of ω for which $|\mathbf{I} - \mathbf{G}(\omega)| = 0$. If η_r has multiplicity m_r then the constant $c_{ij}(\tau)$ is replaced by a polynomial in τ of degree $m_r - 1$. In the case of just two point processes, $k = 2$, some explicit results are given in Hawkes (1971).

3. SELF-EXCITING PROCESS

In the univariate case, the so-called self-exciting process, the above results may be written more simply as

$$M(\omega) = \frac{\lambda\{G(\omega) + G(-\omega) - G(\omega)\, G(-\omega)\}}{\{1 - G(\omega)\}\{1 - G(-\omega)\}}, \tag{14}$$

and the spectral density function

$$f(\omega) = \frac{\lambda}{2\pi\{1 - G(\omega)\}\{1 - G(-\omega)\}}$$

or for real ω

$$f(\omega) = \frac{\lambda}{2\pi|1 - G(\omega)|^2}. \tag{15}$$

As a special case we take the simple exponential decay $\gamma(u) = \alpha e^{-\beta u}$, $u>0$ and $a<\beta$. Then

$$M(\omega) = \lambda\alpha(2\beta\alpha)/\{(\beta - \alpha - i\omega)(\beta - \alpha + i\omega)\}$$

which is the transform of

$$\mu(\tau) = \frac{\lambda\alpha(2\beta - \alpha)}{2(\beta - \alpha)} e^{-(\beta-\alpha)|\tau|}$$

and

$$f(\omega) = \frac{\lambda(\beta^2 + \omega^2)}{2\pi\{(\beta - \alpha)^2 + \omega^2\}}, \tag{16}$$

in agreement with equations (16) and (17) of Hawkes (1971). It was remarked in that paper that the same spectrum could be obtained from a doubly stochastic process, a characteristic of which is that $f(\omega) \geqslant \lambda/2\pi$, see Bartlett (1963). However, we can see from (15) that this is not necessarily the case for all self-exciting processes. To see this we consider any function $\gamma(u)$, not identically zero, such that

$$\gamma(u) = 0, \quad h(n - \tfrac{1}{4}) < u < h(n + \tfrac{1}{4}), \quad n = 0, 1, 2, \ldots.$$

Then

$$\mathrm{Re}\{G(\omega)\} = \int_0^\infty \gamma(u) \cos(\omega u)\, du < 0 \quad \text{for } \omega = 2\pi/h$$

so that $|1 - G(\omega)| > 1$ and thus $f(2\pi/h) < \lambda/2\pi$.

4. PROCESS WITH EXCITOR OF KNOWN SPECTRUM

Here we assume two vector point processes with $\mathbf{N}_2(t)$ having a known spectrum

$$\mathbf{f}_{22}(\omega) = \frac{1}{2\pi}\{\mathbf{D}_{22}+\mathbf{M}_{22}(\omega)\}$$

and helps to excite the process $\mathbf{N}_1(t)$ in such a way that

$$P\{\Delta N_i(t) = 1 \mid \mathbf{N}_1(s), -\infty < s \leqslant t \text{ and } \mathbf{N}_2(s), -\infty < s < \infty\} = \Lambda_i(t)\,\delta t + o(\delta t), \tag{17}$$

where

$$\boldsymbol{\Lambda}_1(t) = \boldsymbol{\upsilon} + \int_{-\infty}^{t} \boldsymbol{\gamma}_1(t-u)\,d\mathbf{N}_1(u) + \int_{-\infty}^{t} \boldsymbol{\gamma}_2(t-u)\,d\mathbf{N}_2(u), \tag{18}$$

if $N_i(t)$ is an element of $\mathbf{N}_1(t)$.

We note particularly that the whole history of the $\mathbf{N}_2(t)$ process occurs in the conditioning event.

Then by the same argument as before we have

$$\boldsymbol{\mu}_{11}(\tau) = \boldsymbol{\gamma}_1(\tau)\,\mathbf{D}_{11} + \int_{-\infty}^{\infty} \boldsymbol{\gamma}_1(\tau-u)\,\boldsymbol{\mu}_{11}(u)\,du + \int_{-\infty}^{\infty} \boldsymbol{\gamma}_2(\tau-u)\,\boldsymbol{\mu}_{21}(u)\,du \quad \text{for } \tau > 0, \tag{19}$$

where

$$\boldsymbol{\mu}_{ij}(\tau) = \mathscr{E}\{d\mathbf{N}_i(t+\tau)\,d\mathbf{N}_j^{\mathrm{T}}(t)\}/dt^2 - \boldsymbol{\lambda}_1\,\boldsymbol{\lambda}_2^{\mathrm{T}}.$$

However, the similar equation

$$\boldsymbol{\mu}_{12}(\tau) = \int_{-\infty}^{\infty} \boldsymbol{\gamma}_1(\tau-u)\,\boldsymbol{\mu}_{12}(u)\,\boldsymbol{\mu}_{12}(u)\,du + \int_{-\infty}^{\infty} \boldsymbol{\gamma}_2(\tau-u)\,\boldsymbol{\mu}_{22}^{(c)}(u)\,du \tag{20}$$

holds for all values of τ because of the nature of the conditioning event in (17). This leads to the transformed equation

$$\mathbf{M}_{12}(\omega) = \mathbf{G}_1(\omega)\,\mathbf{M}_{12}(\omega) + \mathbf{G}_2(\omega)\,\{\mathbf{D}_{22}+\mathbf{M}_{22}(\omega)\}$$

or

$$\mathbf{M}_{12}(\omega) = \{\mathbf{I}-\mathbf{G}_1(\omega)\}^{-1}\,\mathbf{G}_2(\omega)\,\{\mathbf{D}_{22}+\mathbf{M}_{22}(\omega)\}. \tag{21}$$

The cross spectrum $\mathbf{f}_{12}(\omega) = (1/2\pi)\,\mathbf{M}_{12}(\omega)$.

As before we introduce a supplementary matrix

$$\boldsymbol{\beta}(\tau) = \boldsymbol{\gamma}_1(\tau)\,\mathbf{D}_{11} + \int_{-\infty}^{\infty} \boldsymbol{\gamma}_1(\tau-u)\,\boldsymbol{\mu}_{11}(u)\,du + \int_{-\infty}^{\infty} \boldsymbol{\gamma}_2(\tau-u)\,\boldsymbol{\mu}_{21}(u)\,du - \boldsymbol{\mu}_{11}(\tau) \tag{22}$$

from which we obtain

$$\mathbf{B}(\omega) = \mathbf{G}_1(\omega)\,\mathbf{D}_{11} + \mathbf{G}_1(\omega)\,\mathbf{M}_{11}(\omega) + \mathbf{G}_2(\omega)\,\mathbf{M}_{21}(\omega) - \mathbf{M}_{11}(\omega)$$

which together with the condition $\mathbf{M}_{12}(-\omega) = \mathbf{M}_{21}^{\mathrm{T}}(\omega)$ gives

$$\mathbf{M}_{11}(\omega) = \{\mathbf{I}-\mathbf{G}_1(\omega)\}^{-1} \times [\mathbf{G}_1(\omega)\,\mathbf{D}_{11} - \mathbf{B}(\omega) + \mathbf{G}_2(\omega)\,\{\mathbf{D}_{22}+\mathbf{M}_{22}(\omega)\}\,\mathbf{G}_2^{\mathrm{T}}(-\omega)\,\{\mathbf{I}-\mathbf{G}_1^{\mathrm{T}}(-\omega)\}^{-1}]. \tag{23}$$

The condition $\mathbf{M}_{11}(-\omega) = \mathbf{M}_{11}^{\mathrm{T}}(\omega)$ leads to equation (11) with $\mathbf{D}$ replaced by $\mathbf{D}_{1_1}$ and $\mathbf{G}(\omega)$ replaced by $\mathbf{G}_1(\omega)$. Precisely the same analytic argument follows therefore and we get

$$\mathbf{B}(\omega) = -\mathbf{D}_{11}\mathbf{G}_1^{\mathrm{T}}(-\omega)\{\mathbf{I}-\mathbf{G}_1^{\mathrm{T}}(-\omega)\}^{-1},$$

which we substitute in (23) to obtain

$$\mathbf{M}_{11}(\omega) = \{\mathbf{I}-\mathbf{G}_1(\omega)\}^{-1}[\mathbf{G}_1(\omega)\mathbf{D}_{11}+\mathbf{D}_{11}\mathbf{G}_1^{\mathrm{T}}(-\omega)-\mathbf{G}_1(\omega)\mathbf{D}_{11}\mathbf{G}_1^{\mathrm{T}}(-\omega) + \mathbf{G}_2(\omega)\{\mathbf{D}_{22}+\mathbf{M}_{22}(\omega)\}\mathbf{G}_2^{\mathrm{T}}(-\omega)]\{\mathbf{I}-\mathbf{G}_1^{\mathrm{T}}(-\omega)\}^{-1}. \tag{24}$$

Then

$$\mathbf{f}_{11}(\omega) = \frac{1}{2\pi}\{\mathbf{D}_1+\mathbf{M}_{11}(\omega)\}$$

$$= \frac{1}{2\pi}\{\mathbf{I}-\mathbf{G}_1(\omega)\}^{-1}[\mathbf{D}_{11}+\mathbf{G}_2(\omega)\{\mathbf{D}_{22}+\mathbf{M}_{22}(\omega)\}\mathbf{G}_2^{\mathrm{T}}(-\omega)]\{\mathbf{I}-\mathbf{G}_1^{\mathrm{T}}(-\omega)\}^{-1}. \tag{25}$$

We observe that these results agree with those obtained in Section 2 when $\mathbf{G}_2(\omega) = 0$.

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