The linearity of the transformations between inertial frames results solely from their definition. Application to Lorentz and Galilean transformations.

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Abstract
A simplified and then general demonstration is presented here showing that the transformation formulae between the space-time coordinates of two kinematically equivalent reference frames (called Galilean or inertial reference frames) are linear solely through their definition: 'Two reference frames R and R′ are kinematically equivalent when a uniform translational motion of any particle with respect to R is also a uniform translational motion with respect to R'. The basis of this demonstration is to maintain a general form for the transformation formulae between two reference frames R and R′. The acceleration in R' is then calculated, reflecting the fact that if the acceleration of a particle is zero in a reference frame R, it must also be zero in a kinematically equivalent reference frame R'. Our approach clearly separates the proof of the linearity of the transformation formulae and the demonstration of the Galilean and Lorentz transformation formulae. Neither the invariance of the speed of light in vacuum nor the assumption of space-time homogeneity are used. No assumptions are made a priori on the relative motion of these reference frames. However, it follows from our demonstration that two kinematically equivalent reference frames are necessarily driven by a relative uniform translational motion. We then show that only the Galilean transformation allows an infinite velocity. Otherwise, we come to the conclusion that there must be a speed limit U_l independent of the reference frames which, in the strict context of kinematics, is not necessarily the speed of light in vacuum. We then obtain the Lorentz transformation.

1- Definitions:

1- A kinematic problem involves only the concepts of length, time and derived quantities: speeds and accelerations.

2- Two reference frames R and R′ are kinematically equivalent when the uniform translational motion of a particle with respect to R is also a uniform translational motion with respect to R'. Usually, these reference frames are referred to as inertial or Galilean frames. This definition must be limited here. Indeed, we do not consider the concepts of mass, force, inertia, or physical equations representing, for example, electromagnetic phenomena, but solely a kinematic problem.

3- The following standard conventions will be used:
- Summation: when two indices are repeated in the same formula, for example:
  $S_i = f_{ij} g_j \ for\ i\ and\ j = 1\ to\ 3\ signifies: S_i = f_{i1} g_1 + f_{i2} g_2 + f_{i3} g_3\ for\ i = 1\ to\ 3$
  It will be specified if summation is not to be carried out.
- Kronecker symbol:
  $\delta_{ij} = 1\ if\ i = j\ and\ 0\ if\ i \neq j$

2 - Introduction

I recently retired following intensive work teaching and researching the topic of heat and mass transfer. I decided to take the opportunity to return to a research area I was particularly interested in as a student: special relativity. I then went back to some of the works of A. Einstein [1] [2] who postulates that if:

$x - ct = 0 \; implies \; x' - ct' = 0$ then $x' - ct' = \lambda(x - ct\,)$. This is of course only a sufficient condition which assumes that the transformation formulae between inertial reference frames are linear. I therefore wanted to prove that these transformation formulae are necessarily linear. The basis of this demonstration is to maintain a general form for the transformation formulae between two reference frames R and R′ driven by any relative motion: $x'_i = f_i(x_1, x_2, x_3, x_4) \; with \; i = 1 \; to \; 4$; $x_4 = t \; or \; x'_4 = t' \; being \; time$. The acceleration in R' is then calculated as a function of f_i. Reflecting the fact that if the acceleration of a particle is zero in a reference frame R, it must also be zero in a kinematically equivalent reference frame R'. The result is that the transformation formulae are necessarily linear. Our demonstration does not use the concept of space-time homogeneity, nor the invariance of the speed of light in vacuum. We then show that two kinematically equivalent reference frames are necessarily driven by a relative uniform translation motion.

During my extensive literature search with particular reference to J.H. Field [3], I was unable to find any trace of the demonstration I am proposing here, which is why I decided to publish it. The formulae expressing Lorentz transformation, which have formed the basis of the theory of special relativity since their discovery more than a century ago, are the subject of a large number of publications. For the most part, the assumptions on special relativity and isotropy of space being always fulfilled, we come across two types of demonstration:

The first one is based on the concepts of space-time homogeneity, such as for example:

J.M Lévy-Leblond [4]; Leonard J. Eisenberg [5]; Robert Resnick [6].

The second one is based on the invariance of the speed of light in vacuum, such as for example: V. Fock and N. Kemmer [7]; J. M. Lévy [8] O. Serret [9]; C. Moller [10].

A more original demonstration, also relying solely on kinematics, provided by J. H. Field [3] is essentially based on Postulate B: 'Uniqueness Postulate' which leads him to a trilinear form between the space-time coordinates of two reference frames (equations (2.1) and (2.3)). However, the trilinear form is not always a solution, as indicated by the author in his note [17] by citing the example of $\alpha + \beta + \gamma - k\alpha\beta\gamma = 0$ which although trilinear does not respect Postulate B when $\alpha = \frac{1}{k\beta}$. In addition, an infinite number of solutions can be found which comply with Postulate B without the trilinear form being verified, for example $\alpha^5 + \beta + \gamma + \beta\gamma = 0$. Moreover, the author makes it clear that the trilinear form is only a sufficient condition. Thus, the linearity of the transformation equations cannot be derived from Postulate B.

More recently, Youshan Dai and Linag Dai [11] gave a demonstration limited to a space-time of dimension (1+1) which only uses the principle of relativity.

The (simplified and then general) demonstrations of the linearity of the transformation formulae between kinematically equivalent reference frames given here are based solely on their definition, explained in 1-2. We also show that two kinematically equivalent reference frames are necessarily driven by a relative uniform translational motion. The additional assumptions outlined by J. M. Lévy-Leblond [4] are then added: Isotropy of space and group law. We deduce from this the Galilean transformation, which alone allows infinite speeds, and the Lorentz transformation, which requires the existence of a speed limit. These demonstrations (especially the simplified one) do not involve complex mathematics, which is why I think their integration into special relativity courses would be beneficial for students.

3- Linearity of transformation formulae.

3-1- Simplified demonstration of the linearity of transformation formulae in the case where the x and x' axes are colinear and parallel to their relative speed.

A reference frame R' is considered which is driven by a translational motion, not necessarily uniform at this time, with respect to a reference frame R and an event taking place at a point of coordinates x, y, z at a time t in R. We want to find out what the coordinates x', y' and z' are at a time t' of this same event in R'. For this simplified demonstration, we will choose the axes x and x', which are colinear and parallel to the velocity vector **V** of R' with respect to R. We emphasise the fact that the only component of the velocity on x: $V(t)$ is a priori a function of time. For the moment, the origins O and O' are arbitrary on the x and x' axes. Choosing this reference frame allows us to carry out a study that is easily understandable by a student, which is one of our objectives. This does not restrict the generality of this work, as I show in paragraph 3-2, where this assumption is no longer realised.

A particle in space is marked by its coordinates x, y and z at a time t with respect to R, and x', y' and z' at a time t' with respect to R'.

There is necessarily a relationship between the reference frames:

$$x' = F(x,y,z,t) \quad (3.1)$$

$$y' = H(x,y,z,t) \quad (3.2)$$

$$z' = K(x,y,z,t) \quad (3.3)$$

$$t' = G(x,y,z,t) \quad (3.4)$$

Because of the relative translational motion of R and R' of direction x (or x'), we can affirm:

1- A plane $y = y_0 = constant$ must be transformed into a plane $y' = y_0' = constant$, so $y'_0 = H(x, y_0, z, t)$ whatever x, z and t, which is only possible if y' depends only on y. Thus $y' = H(y)$ and likewise, $z' = K(z)$.

2- (3.4) can then be written: $t' = G(x, H^{-1}(y'), K^{-1}(z'), t)$. Any clock belonging to the reference frame R' and in particular to the plane y'-z' must indicate the same time t'. It follows that t' cannot depend on y' and z', thus: $t' = G(x,t)$.

3- (3.1) can be written: $x' = F(x, H^{-1}(y'), K^{-1}(z'), t)$. Since a plane y'-z' must remain a plane perpendicular to the x' axis, x' cannot depend on y' and z', thus: $x' = F(x,t)$

To summarize, equations (3.1) to (3.4) are simplified, and as shown by J.M. Lévy-Leblond [4], these equations depend on a single parameter κ linking the two reference frames:

$$x' = F(\kappa, x, t) \quad (3.5)$$

$$y' = H(y) \quad (3.6)$$

$$z' = K(z) \quad (3.7)$$

$$t' = G(\kappa, x, t) \quad (3.8)$$

The coordinates of the velocity vectors $\boldsymbol{U}$ (with respect to R) and $\boldsymbol{U}'$ (with respect to R′) of a moving particle are written as:

$$u = \frac{dx}{dt};\ v = \frac{dy}{dt};\ w = \frac{dz}{dt}\ with\ respect\ to\ R\ and\ u' = \frac{dx'}{dt'};\ v' = \frac{dy'}{dt'};\ w' = \frac{dz'}{dt'}\ with\ respect\ to\ R'$$

We will also need the coordinates of the acceleration vectors $\boldsymbol{a}$ and $\boldsymbol{a}'$:

$$\dot{u} = \frac{du}{dt};\ \dot{v} = \frac{dv}{dt};\ \dot{w} = \frac{dw}{dt}\ with\ respect\ to\ R\ and\ \dot{u}' = \frac{du'}{dt'};\ \dot{v}' = \frac{dv'}{dt'};\ \dot{w}' = \frac{dw'}{dt'}\ with\ respect\ to\ R'$$

By differentiating these relationships, κ being a constant:

$$dx' = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} dt;\quad dy' = \frac{\partial H}{\partial y} dy;\quad dz' = \frac{\partial K}{\partial z} dz;\quad dt' = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt$$

Then simplifying the writing of the partial derivatives for the convenience of the calculations (for example: $\frac{\partial F}{\partial x} = F_x$) we get the relationships between the coordinates of the velocities:

$$u' = \frac{dx'}{dt'} = \frac{\frac{dx'}{dt}}{\frac{dt'}{dt}} = \frac{uF_x+F_t}{uG_x+G_t} \tag{3.9}$$

$$v' = \frac{dy'}{dt'} = \frac{\frac{dy'}{dt}}{\frac{dt'}{dt}} = \frac{vH_y}{uG_x+G_t} \tag{3.10}$$

$$w' = \frac{dz'}{dt'} = \frac{\frac{dz'}{dt}}{\frac{dt'}{dt}} = \frac{wK_z}{uG_x+G_t} \tag{3.11}$$

It is also worth noting the important role of:

$$\frac{dt'}{dt} = uG_x + G_t \tag{3.12}$$

This relationship (3.12) excludes the case where $uG_x + G_t = 0$, otherwise, the time would be a constant in the reference frame R'. By deriving u', v', w' with respect to t', we find the coordinates of the accelerations:

$$\dot{u}' = \frac{du'}{dt'} = \frac{\frac{du'}{dt}}{\frac{dt'}{dt}} = \frac{[\dot{u}F_x+u(uF_{xx}+F_{xt})+uF_{tx}+F_{tt}](uG_x+G_t)-[\dot{u}G_x+u(uG_{xx}+G_{xt})+uG_{tx}+G_{tt}](uF_x+F_t)}{(uG_x+G_t)^3} \tag{3.13}$$

$$\dot{v}' = \frac{dv'}{dt'} = \frac{\frac{dv'}{dt}}{\frac{dt'}{dt}} = \frac{(\dot{v}H_y+v^2H_{yy})(uG_x+G_t)-vH_y[\dot{u}G_x+u(uG_{xx}+G_{xt})+uG_{tx}+G_{tt}]}{(uG_x+G_t)^3} \tag{3.14}$$

$$\dot{w}' = \frac{dw'}{dt'} = \frac{\frac{dw'}{dt}}{\frac{dt'}{dt}} = \frac{(\dot{w}H_z+w^2K_{zz})(uG_x+G_t)-wK_z[\dot{u}G_x+u(uG_{xx}+G_{xt})+uG_{tx}+G_{tt}]}{(uG_x+G_t)^3} \tag{3.15}$$

In developing:

$$\dot{u}'(uG_x + G_t)^3 = \dot{u}(F_xG_t - G_xF_t) + u^3(F_{xx}G_x - G_{xx}F_x) + u^2(F_{xx}G_t + 2F_{tx}G_x - 2G_{tx}F_x - G_{xx}F_t) + u(2F_{xt}G_t + G_xF_{tt} - 2G_{xt}F_t - F_xG_{tt}) + (F_{tt}G_t - G_{tt}F_t) \tag{3.16}$$

$$\dot{v}'(uG_x + G_t)^3 = \dot{v}H_y(uG_x + G_t) - \dot{u}vH_yG_x - vH_yG_{tt} + v^2H_{yy}G_t - 2vuH_yG_{xt} - vu^2H_yG_{xx} + uv^2H_{yy}G_x \tag{3.17}$$

$$\dot{w}'(uG_x + G_t)^3 = \dot{w}K_z(uG_x + G_t) - \dot{u}wK_zG_x - wK_zG_{tt} + w^2K_{zz}G_t - 2wuK_zG_{xt} - wu^2K_zG_{xx} + uw^2K_{zz}G_x \tag{3.18}$$

We first look at relationships (3.17) or (3.18). The nullity of the acceleration $\boldsymbol{a}$ with respect to the reference frame R does indeed lead to the nullity of the coordinates $\dot{v}'$ *and* $\dot{w}'$ of the acceleration with respect to R', whatever the velocity, $\boldsymbol{U}(u, v, w)$[1] only if:

$$H_{yy} = K_{zz} = 0 \tag{3.19}$$

and

$$G_{xx} = G_{tt} = G_{tx} = 0 \tag{3.20}$$

Equations (3.19) prove that:

$H(y) = hy + c_2$ *et* $K(z) = kz + c_3$ *where* h, k, c_2, c_3 *are functions of* κ

[1] *Note that we do not consider a geometric space with a single dimension (along x or x'), otherwise v and w would not exist and our reasoning would no longer be valid. It is further accepted that* H_y *and* K_z *cannot be zero because this would lead in R' to a translation motion parallel to the x' axis whatever the movement of the particle in R!*

It follows from equations (3.20) that G is an affine function of x and t (for students who are rightly dubious, this is demonstrated in the footnote[2]):

$G(x,t) = fx + gt + c_1 \text{ where } f, g \text{ and } c_1 \text{ are functions of } \kappa$

Taking into account (3.20) in (3.16) the nullity of the acceleration $\boldsymbol{a}$ with respect to the reference frame R does indeed lead to the nullity of the coordinate $\dot{u}'$ of the acceleration with respect to R', whatever the coordinate u of the velocity, if:

$$F_{xx} = F_{tt} = F_{tx} = 0 \qquad (3.21)$$

This also makes it possible to state that:

$F(x,t) = ax + bt + c_4 \text{ where } a, b \text{ and } c_4 \text{ are functions of } \kappa$

It is therefore shown that the equations connecting the space-time coordinates of a reference frame R to those of another reference frame R′ driven by a translational motion (not necessarily uniform a priori) in the direction of the axis x with respect to R are affine, and depend on 6 constants, which are functions of κ. We can freely decide that the origin O' is in O at the time t=0 and that the clock specific to R' is set to t'=0 at this same time. In which case, the constants c_i are zero. It follows then that:

$$x' = F(\kappa, x, t) = ax + bt \qquad (3.22)$$

$$y' = h(\kappa)y \qquad (3.23)$$

$$z' = k(\kappa)z \qquad (3.24)$$

$$t' = G(\kappa, x, t) = fx + gt \qquad (3.25)$$

If we consider the motion of the origin O' then $x'_{O'} = 0$ and $x_{O'} = \int_0^t V(\tau)\, d\tau$, the equation (3.22) leads to:

$$x'_{O'} = 0 = a\int_0^t V(\tau)\, d\tau + bt \text{ or } aV(t) + b = 0 \qquad (3.26)$$

which is only possible if $V(t) = V$ is a constant. It is thus shown that the relative motion of the two reference frames is necessarily uniform. (3.26) then takes the form:

$$x'_{O'} = 0 = aVt + bt \text{ so: } b = -aV \qquad (3.27)$$

It is then logical to use V for the parameter κ, (3.22) to (3.25) become:

$$x' = F(V, x, t) = a(V)(x - Vt) \qquad (3.28)$$

$$y' = h(V)y \qquad (3.29)$$

$$z' = k(V)z \qquad (3.30)$$

$$t' = G(V, x, t) = f(V)x + g(V)t \qquad (3.31)$$

The relationships between the velocity coordinates become:

$$u' = \frac{a(V)(u-V)}{f(V)u + g(V)} \qquad (3.32)$$

$$v' = \frac{h(V)v}{f(V)u + g(V)} \qquad (3.33)$$

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[2]*Taking the equations (3.20):*

$G_{xx} = \frac{\partial G_x}{\partial x} = 0; \text{ therefore } G_x = \frac{\partial G}{\partial x} = \lambda(t) \text{ and so } G(x,t) = x\lambda(t) + \alpha(t)$ *(1)*

$G_{tt} = \frac{\partial G_t}{\partial t} = 0; \text{ therefore } G_t = \frac{\partial G}{\partial t} = \mu(x) \text{ and so } G(x,t) = t\mu(x) + \beta(x)$ *(2)*

$\frac{d\mu}{dx} = \frac{\partial G_t}{\partial x} = G_{tx} = G_{xt} = \frac{\partial G_x}{\partial t} = \frac{d\lambda}{dt} = 0; \text{ therefore } \lambda(t) = f \text{ and } \mu(x) = g \text{ are constants}$

$G(x,t) = fx + \alpha(t) = gt + \beta(x); \text{ and so } \beta(x) - fx = \alpha(t) - gt = h \text{ which is a constant}$

$\text{it results that: } \alpha(t) = h + gt \text{ and } \beta(x) = h + fx$

Which, by plotting in equations (1) or (2) gives:

$G(x,t) = fx + gt + h \text{ where } g \text{ and } h \text{ are constants}$

$$w' = \frac{k(V)w}{f(V)u+g(V)} \quad (3.34)$$

3-2- General demonstration of the linearity of the transformation formulae in the case where the relative speed of the reference frames is arbitrary.

This time, R' is driven by a motion of speed $\boldsymbol{V(t)}$ a priori of any kind with respect to R. Let's give a more general demonstration using an index notation of the space-time coordinates:

$$x'_i = f_i(x_1, x_2, x_3, x_4) \; with \; i = 1 \; to \; 4 \quad (3.35)$$[3]

With respect to the above, x_1=x, x_2=y, x_3=z, x_4=t and the same for the primed variables. It is of course assumed that at least one of the values of $\frac{\partial f_4}{\partial x_l}$ is non-zero, otherwise time would be a constant in R'.

By differentiating (3.35) and using the usual summation convention:

$$dx'_i = dx_j \frac{\partial f_i}{\partial x_j} \; with \; i \; and \; j = 1 \; to \; 4 \quad (3.36)$$

The following is then calculated:

$$u_j = \frac{dx_j}{dx_4}; \; u'_i = \frac{dx'_i}{dx'_4} = \frac{\frac{dx'_i}{dx_4}}{\frac{dx'_4}{dx_4}} = \frac{u_j \frac{\partial f_i}{\partial x_j}}{u_l \frac{\partial f_4}{\partial x_l}} \; with \; i = 1 \; to \; 3, \; j \; and \; l = 1 \; to \; 4 \quad (3.37)$$

$$\dot{u}'_i = \frac{du'_i}{dx'_4} = \frac{\frac{du'_i}{dx_4}}{\frac{dx'_4}{dx_4}} = \frac{\frac{d}{dx_4}\left(u_j \frac{\partial f_i}{\partial x_j}\right)\left(u_l \frac{\partial f_4}{\partial x_l}\right) - \left(u_j \frac{\partial f_i}{\partial x_j}\right)\frac{d}{dx_4}\left(u_l \frac{\partial f_4}{\partial x_l}\right)}{\left(u_l \frac{\partial f_4}{\partial x_l}\right)^3} \; with \; i = 1 \; to \; 3, \; j \; and \; l = 1 \; to \; 4 \quad (3.38)$$

Thus:

$$\left(u_l \frac{\partial f_4}{\partial x_l}\right)^3 \dot{u}'_i = \left(\dot{u}_j \frac{\partial f_i}{\partial x_j} + u_j u_k \frac{\partial^2 f_i}{\partial x_j \partial x_k}\right)\left(u_l \frac{\partial f_4}{\partial x_l}\right) - \left(\dot{u}_l \frac{\partial f_4}{\partial x_l} + u_l u_k \frac{\partial^2 f_4}{\partial x_l \partial x_k}\right)\left(u_j \frac{\partial f_i}{\partial x_j}\right) \quad (3.39)$$

In developing:

$$\left(u_l \frac{\partial f_4}{\partial x_l}\right)^3 \dot{u}'_i = \dot{u}_j u_l \frac{\partial f_i}{\partial x_j}\frac{\partial f_4}{\partial x_l} - \dot{u}_l u_j \frac{\partial f_4}{\partial x_l}\frac{\partial f_i}{\partial x_j} + u_j u_k u_l \left(\frac{\partial f_4}{\partial x_l}\frac{\partial^2 f_i}{\partial x_j \partial x_k} - \frac{\partial f_i}{\partial x_j}\frac{\partial^2 f_4}{\partial x_l \partial x_k}\right) \quad (3.40)$$

If the particle acceleration is zero in relation to R, it will also be zero in relation to R', whatever the coordinates of the velocity in R may be, only if:

$$\frac{\partial f_4}{\partial x_l}\frac{\partial^2 f_i}{\partial x_j \partial x_k} - \frac{\partial f_i}{\partial x_j}\frac{\partial^2 f_4}{\partial x_l \partial x_k} = 0 \; with \; i = 1 \; to \; 3; \;\; j, k \; and \; l = 1 \; to \; 4 \quad (3.41)$$

This equation (3.41) is of course satisfied if $\frac{\partial^2 f_i}{\partial x_j \partial x_k} = 0 \; and \; \frac{\partial^2 f_4}{\partial x_l \partial x_k} = 0$, that is to say if $f_i \; and \; f_4$ are affine or linear functions of x_j, however we will show that this is the only solution.

First of all, if, for certain values of l, f_4 does not depend on x_l, $\frac{\partial f_4}{\partial x_l}$ is zero, then equation (3.41) is satisfied. The same applies if, for certain values of j, f_i does not depend on x_j. In practice, to take this into account, it will suffice to give the value 0 to a_{4j} or a_{ij} in equations (3.47).

We therefore consider cases where $\frac{\partial f_4}{\partial x_l}$ is non-zero.

Equation (3.41) can be written:

$$\frac{\partial f_4}{\partial x_l}\frac{\partial}{\partial x_k}\left(\frac{\partial f_i}{\partial x_j}\right) - \frac{\partial f_i}{\partial x_j}\frac{\partial}{\partial x_k}\left(\frac{\partial f_4}{\partial x_l}\right) = 0 \; with \; i = 1 \; to \; 3, \; j, k \; and \; l = 1 \; to \; 4 \quad (3.42)$$

That is, $\frac{\partial f_4}{\partial x_l}$ being non-zero:

[3] *By extension of the demonstration given by J. M. Lévy-Leblond [4], the functions f_i depend on 3 parameters* $\kappa_1, \kappa_2, \kappa_3$.

$\frac{\partial}{\partial x_k}\begin{bmatrix}\frac{\partial f_i}{\partial x_j}\\ \frac{\partial f_4}{\partial x_l}\end{bmatrix} = 0 \; which \; leads \; to\text{:} \; \frac{\partial f_i}{\partial x_j} = C_{4l}^{ij}\frac{\partial f_4}{\partial x_l} \; without \; summation \; on \; l \; where \; C_{4l}^{ij} \; is \; a \; constant$

By placing $C_{4l}^{ij} = \frac{a_{ij}}{a_{4l}}$ and agreeing that $a_{ij} = 0 \; if \frac{\partial f_i}{\partial x_j} = 0$ you can write:

$$\frac{1}{a_{ij}}\frac{\partial f_i}{\partial x_j} = \frac{1}{a_{4l}}\frac{\partial f_4}{\partial x_l} \; without \; summation \; on \; i, \; j \; and \; l, with \; i = 1 \; to \; 3 \; and \; j \; and \; l = 1 \; to \; 4 \qquad (3.43)$$

Explaining and without summation on i:

$$\frac{1}{a_{i1}}\frac{\partial f_i}{\partial x_1} = \frac{1}{a_{i2}}\frac{\partial f_i}{\partial x_2} = \frac{1}{a_{i3}}\frac{\partial f_i}{\partial x_3} = \frac{1}{a_{i4}}\frac{\partial f_i}{\partial x_4} = \frac{1}{a_{41}}\frac{\partial f_4}{\partial x_1} = \frac{1}{a_{42}}\frac{\partial f_4}{\partial x_2} = \frac{1}{a_{43}}\frac{\partial f_4}{\partial x_3} = \frac{1}{a_{44}}\frac{\partial f_4}{\partial x_4} \qquad (3.44)$$

Using a well-known result in mathematics concerning first-order linear partial differential equations with constant coefficients, we derive the solutions:

$$f_i(\xi_i) \; with \; \xi_i = a_{ij}x_j \; ; \; f_4(\xi_4) \; with \; \xi_4 = a_{4j}x_j \; where \; j = 1 \; to \; 4 \qquad (3.45)$$

$f_i(\xi_i) \; and \; f_4(\xi_4)$ being arbitrary functions, but in addition:

$$\frac{df_i}{d\xi_i}(\xi_i) = \frac{df_4}{d\xi_4}(\xi_4) \qquad (3.46)$$

$\xi_i \; and \; \xi_4$ are independent variables unless the particle follows a particular trajectory, which is not the case here, thus: $\frac{df_i}{d\xi_i}(\xi_i) = \frac{df_4}{d\xi_4}(\xi_4) = C \; where \; C \; is \; a \; constant$. The result is: $f_i(\xi_i) = C\xi_i + d_i$ and $f_4(\xi_4) = C\xi_4 + d_4$.

We can freely decide that the origin O' is in O at the time t=0 and that the clock specific to R' is set to t'=0 at this same time. In which case, the constants d_i are zero.

Incorporating the constant C into the coefficients a_{ij} it becomes:

$$x'_i = f_i = a_{ij}x_j \; for \; i \; and \; j = 1 \; to \; 4 \qquad (3.47)$$

Taking into account the transformation formulae (3.47), the equation (3.37),) can be written:

$$u'_i = \frac{u_j\frac{\partial f_i}{\partial x_j}}{u_l\frac{\partial f_4}{\partial x_l}} = \frac{a_{ij}u_j + a_{i4}}{a_{4l}u_l + g} \; for \; i, j, l = 1 \; to \; 3 \; and \; putting \; a_{44} = g = \frac{\partial t'}{\partial t} \; which \; plays \; a \; specific \; role \qquad ((3.48)$$

The trajectory of the origin O' of R' is considered: $\boldsymbol{u_{O'}} = \boldsymbol{V(t)} \; and \; \boldsymbol{u'_{O'}} = \boldsymbol{0},$ according to (3.48):

$$a_{ij}V_j(t) = -a_{i4} \; for \; i \; and \; j = 1 \; to \; 3 \; ; \; V_j(t) \; being \; the \; coordinates \; of \; \boldsymbol{V(t)} \qquad (3.49)$$

This relationship (3.49) seems to us fundamental insofar as it is only possible if $V_j(t) \; is \; constant \; for \; j = 1 \; to \; 3$, that is to say if the velocity vector $\boldsymbol{V(t)}$ is constant. Indeed, we can always write $V_j(t) = V_j + W_j(t) \; where \; V_j \; is \; a \; constant$, (3.49) is then written:

$$a_{ij}V_j + a_{ij}W_j(t) = -a_{i4} \; for \; i \; and \; j = 1 \; to \; 3 \qquad (3.50)$$

Only the second term of the first member depends on time, it is then necessary that:

$$a_{ij}W_j(t) = 0 \; for \; i \; and \; j = 1 \; to \; 3 \qquad (3.51)$$

The system determinant of the system (3.51) is necessarily non-zero because in (3.47), a value of time $t = x_4$ and a value of coordinates x'_1, x'_2, x'_3 corresponds to a value and only one of the coordinates x_1, x_2, x_3. Thus, the homogeneous system (3.51) has the null solution: $W_j(t) = 0 \; for \; j = 1 \; to \; 3$. It is therefore shown that two kinematically equivalent reference frames are necessarily driven by a relative uniform translational motion. It is then natural to take V_1, V_2, V_3 as parameters on which depend the coefficients a_{ij}. (3.49) is written as follows:

$$a_{ij}V_j = -a_{i4} \; for \; i \; and \; j = 1 \; to \; 3 \qquad (3.52)$$

Considering (3.52) the equation (3.48) can be written in the form:

$u'_i = \frac{a_{ij}(u_j - V_j)}{a_{4l}u_l + g} for\ i, j, l = 1\ to\ 3$ (3.53)

The equations (3.47) can be written:

$x'_i = a_{ij}(x_j - V_j t)\ and\ t' = a_{4l}x_l + gt\ for\ i, j\ and\ l = 1\ to\ 3$ (3.54)

We have thus demonstrated that the transformation between kinematically equivalent reference frames is necessarily affine (or linear). Let us define the space-time homogeneity according to J.M. Lévy-Leblond [4]: '*the transformation properties of a spatiotemporal interval* $(\Delta x, \Delta t)$ *depend only on that interval and not on the location of its end points (in the considered reference frame). In other words, the transformed interval* $(\Delta x', \Delta t')$ *obtained through an inertial transformation (5) is independent of these end points*'. It may first of all be noted that, in this sense, only geometrical and kinematic concepts are used. Here we show that space-time homogeneity results from the linearity of equations (3.47) or (3.54). Furthermore, the existence of these transformation relationships between two kinematically equivalent reference frames implies that these two reference frames are animated by a relative uniform translational motion.

4- Additional assumptions for the demonstration of the Galilean and Lorentz transformation formulae

The assumption summarised below are physically obvious but clearly explained by J.M. Lévy-Leblond [4], we have voluntarily limited them to the subject of our study: kinematics.

Hypothesis H1: Isotropy of space. All the orientations of the axes are equivalent for the description of the kinematic quantities. In particular, if a speed limit exists, its module is independent of its direction.

The following 3 hypotheses constitute the group law according to J. M. Lévy-Leblond [4].

Hypothesis H2: Identical transformation. If R' is R itself, then $a(0) = g(0) = +1\ et\ f(0) = 0$, which one can extend to $a_{ij}(0) = \delta_{ij}\ \ for\ i\ and\ j = 1\ to\ 4$

Hypothesis H3: Reverse transformation. The transformation giving the coordinates of an event in a reference frame R as a function of those of R' must be of the same functional form as that giving the coordinates of an event in the reference frame R' as a function of those of R.

Hypothesis H4: Law of composition. Take **V** as the velocity of a reference frame R' with respect to R and **V'** as the velocity of a reference frame R'' with respect to R'. The transformation of the reference frame R to R″ driven by a speed **V+V′** with respect to R must be identical to the composition of the transformations of the reference frame R to R' then R′ to R″.

5- Simplified demonstration of the Galileo and Lorentz transformation formulae in the case where the x and x' axes are collinear and parallel to the relative speed of the two reference frames

5-1- Preliminary

If the orientation of the axes x and x' is reversed, the reference frame of axis -x' being driven at a speed W with respect to that of axis -x, according to the isotropy hypothesis H1, we must have:

$-x' = a(W)(-x - Wt) = a(V)(-x + Vt)\ therefore\ a(W) = a(V)\ and\ W = -V$ (5.1)

It results in:

$a(-V) = a(V)$ (5.2)

The formulae (3.28) to (3.34) must remain of the same functional form if the inverse transformation of R' to R of velocity V' is considered (hypothesis H3). If the formulae (3.32) to (3.34) are reversed, we get:

$u = \frac{a(V')(u' - V')}{f(V')u' + g(V')} = \frac{g(V)u' + a(V)V}{-f(V)u' + a(V)}$ (5.3)

$$v = \frac{h(V')v'}{f(V')u'+g(V')} = \frac{a(V)[g(V)+f(V)V]v'}{h(V)(-f(V)u'+a(V))} \quad (5.4)$$

$$w = \frac{k(V')w'}{f(V')u'+g(V')} = \frac{a(V)[g(V)+f(V)V]w'}{k(V)(-f(V)u'+a(V))} \quad (5.5)$$

It can be shown from the H4 and H2 hypotheses that $V' = -V$. As demonstrated in a more general case in paragraph 6-1, we will therefore admit it for the time being in order to avoid repetition.

Taking into account (5.2), the identification of the coefficients in (5.3) to (5.5) leads to:

$$a(V) = g(V)\,;\ f(-V) = -f(V)\ and\ h(V) = k(V) = 1 \quad (5.6)$$

And also:

$$g^2(V) + g(V)f(V)V = 1 \quad (5.7)$$

The transformation formulae (3.28) to (3.34) can then be written:

$$x' = g(V)(x - Vt) \quad (5.8)$$

$$t' = f(V)x + g(V)t \quad (5.9)$$

$$y' = y \quad (5.10)$$

$$z' = z \quad (5.11)$$

$$u' = \frac{g(V)(u-V)}{f(V)u+g(V)} \quad (5.12)$$

$$v' = \frac{v}{f(V)u+g(V)} \quad (5.13)$$

$$w' = \frac{w}{f(V)u+g(V)} \quad (5.14)$$

The two coefficients $g(V)\ et\ f(V)$ being related by (5.7).

5-2- Galilean transformation formulae

We consider the case where u tends towards infinity, without discussing the physical validity of this hypothesis, but in terms of pure kinematics it is a possibility. If $f(V) = 0$, it follows from (5.12) that u' also tends towards infinity.

Considering (5.7) which gives $g(V) = +1$[4], the transformation equations are written:

$$x' = x - Vt \quad (5.15)$$

$$y' = y \quad (5.16)$$

$$z' = z \quad (5.17)$$

$$t' = t \quad (5.18)$$

In the sense of this demonstration, the Galilean transformation is not only a 'degenerate' case of the Lorentz transformation when the velocity V is low, but a class of solution apart: the Galilean transformation is the only one to admit infinite speeds.

5-3- Lorentz transformation formulae

If f≠0 then, according to equation (5.12), u' tends towards the finite value $\frac{g(V)}{f(V)}$ if u tends towards infinity and conversely u tends towards the finite value -$\frac{g(V)}{f(V)}$ if u' tends towards infinity. As a result, it is impossible for the speed to become infinite relative to the reference frame R or R′ because this would be contrary to our principle of 'kinematically equivalent reference frame'. It is thus necessary for us to admit that there is a limit speed of modulus U_l in R and U'_l in R′. Taking into account the assumption of isotropy (H1), the modulus of these velocities must be the same regardless of their orientation in space.

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[4]*If the velocity V is zero, R′ is R itself, in which case we necessarily have* $a(0) = g(0) = +1$ *and* $f(0) = 0$, *according to the hypothesis H2, therefore* $g(V)$ *is positive.*

Considering the modulus of the velocity of any particle which reaches this limit velocity in R. U'_l being the modulus of the corresponding velocity in R', we put $U'_l = \alpha U_l$.

$$U'^2_l = \alpha^2 U_l^2 = \frac{g^2(u-V)^2-u^2+u^2+v^2+w^2}{(fu+g)^2} = \frac{g^2(u-V)^2-u^2+U_l^2}{(fu+g)^2} \quad (5.19)$$

This leads, by developing the u polynomial, to:

$$u^2\left(f^2\alpha^2 {U_l}^2 + 1 - g^2\right) + 2u\left(fg\alpha^2 {U_l}^2 + g^2 V\right) + g^2\alpha^2 {U_l}^2 - g^2V^2 - {U_l}^2 = 0 \quad (5.20)$$

It is necessary for the 3 coefficients of this polynomial to be zero if it is desired that the identity at 0 be verified regardless of the speed u. Taking into account the fact that $g(V) > 0$, the resolution of these 3 equations gives:

$$\alpha = 1\ ;\ g = \frac{1}{\sqrt{1-\frac{V^2}{U_l^2}}}\ and\ f = -\frac{gV}{U_l^2} \quad (5.21)$$

The fact that $\alpha = 1$ shows that the speed limit U_l is also reached in R′ if it is reached in R, this is about an invariant independent of the reference frame.

The equations (3.28) to (3.34) are then written:

$$x' = g(x - Vt) \quad (5.22)$$

$$y' = y \quad (5.23)$$

$$z' = z \quad (5.24)$$

$$t' = g\left(t - \frac{V}{{U_l}^2}x\right)\ where\ g = \frac{1}{\sqrt{1-\frac{V^2}{{U_l}^2}}} \quad (5.25)$$

$$u' = \frac{u-V}{1-\frac{uV}{{U_l}^2}} \quad (5.26)$$

$$v' = \frac{v}{g\left(1-\frac{uV}{{U_l}^2}\right)} \quad (5.27)$$

$$w' = \frac{w}{g\left(1-\frac{uV}{{U_l}^2}\right)} \quad (5.28)$$

The coordinates of the accelerations are:

$$\dot{u}' = \frac{\dot{u}}{g^3\left(1-\frac{uV}{{U_l}^2}\right)^3};\quad \dot{v}' = \frac{\dot{v}}{g^2\left(1-\frac{uV}{{U_l}^2}\right)^2} + \frac{\dot{u}v\frac{V}{{U_l}^2}}{g^2\left(1-\frac{uV}{{U_l}^2}\right)^3};\ \dot{w}' = \frac{\dot{w}}{g^2\left(1-\frac{uV}{{U_l}^2}\right)^2} + \frac{\dot{u}w\frac{V}{{U_l}^2}}{g^2\left(1-\frac{uV}{{U_l}^2}\right)^3} \quad (5.29)$$

6- General demonstration of the Galilean and Lorentz transformation formulae in the case where the relative speed of the two reference frames is arbitrary

6-1- Preliminary

First, **V'** being the speed of R with respect to R', to shorten the writing of the equations, we note:

$a_{ij}(\boldsymbol{V}) = a_{ij}\ ;\ a_{ij}(\boldsymbol{V}') = a'_{ij}\ for\ i\ and\ j = 1\ to\ 4;\ g(\boldsymbol{V}) = g\ ;\ g(\boldsymbol{V}') = g'$

If we consider the motion of the origin O of R ($u_{iO} = 0$), using (3.52) and (3.53), we find:

$$-gV'_i = a_{ij}V_j = -a_{i4}\ and\ likewise\ -g'V_i = a'_{ij}V'_j = -a'_{i4} \quad (6.1)$$

In the following, we assume that the coordinate axes of the two reference frames remain parallel, which does not restrict the generality because it is always possible to switch to another coordinate system by a simple rotation as specified by V. Fock and N. Kemmer [7] and C. Moller [10] . Under these conditions,

it can in particular be affirmed that if two velocity vectors are equal $(\boldsymbol{A} = \boldsymbol{A}')$, their coordinates are equal in R and R′ $(A_i = A'_i)$.
According to the H1 isotropy hypothesis, the coordinates $(-x'_i,\ t')\ and\ (-x_i, t)$ must be linked by the transformation of the same form as equations (3.54).
$\boldsymbol{W}\ being\ the\ relative\ velocity\ of\ these\ new\ axes$ then:

$$-x'_i = a_{ij}(\boldsymbol{W})\big(-x_j - U_j t\big)\ and\ t' = -a_{4l}(\boldsymbol{W})x_l + g(\boldsymbol{W})t\ for\ i,j\ and\ l = 1\ to\ 3 \qquad (6.2)$$

By identifying in equations (3.54):

$$x'_i = a_{ij}(\boldsymbol{W})\big(x_j + W_j t\big) = a_{ij}(\boldsymbol{V})\big(x_j - V_j t\big)\ ;\ t' = -a_{4l}(\boldsymbol{W})x_l + g(\boldsymbol{W})t = a_{4l}(\boldsymbol{V})x_l + g(\boldsymbol{V})t \quad (6.3)$$

This results in the equalities:

$$\boldsymbol{W} = -\boldsymbol{V}; a_{ij}(-\boldsymbol{V}) = a_{ij}(\boldsymbol{V});\ g(-\boldsymbol{V}) = g(\boldsymbol{V}); a_{4l}(-\boldsymbol{V}) = -a_{4l}(\boldsymbol{V})\ for\ i,j\ and\ l = 1\ to\ 3 \quad (6.4)$$

According to the H4 hypothesis, the transformation from R to R’’ results from a composition of the transformations from R to R’ then from R’ to R’’. It can be verified that this leads to the following identities:

$$a_{ik}(\boldsymbol{V} + \boldsymbol{V}') = a'_{ij}a_{jk} - a'_{ij}V'_j a_{4k} \qquad (6.5)$$

$$a_{ik}(\boldsymbol{V} + \boldsymbol{V}')(V_k + V'_k) = a'_{ij}a_{jk}V_k + ga'_{ij}V'_j \qquad (6.6)$$

$$a_{4k}(\boldsymbol{V} + \boldsymbol{V}') = a'_{4j}a_{jk} + g'a_{4k} \qquad (6.7)$$

$$g(\boldsymbol{V} + \boldsymbol{V}') = gg' - a'_{4j}a_{jk}V_k \qquad (6.8)$$

Applying these 4 equations in the case where the reference frame R″ is R itself. Using the H2 hypothesis for the identical transformation, we have:

$$a_{ik}(\boldsymbol{V} + \boldsymbol{V}') = \delta_{ik}\ ;\ a_{4k}(\boldsymbol{V} + \boldsymbol{V}') = 0\ for\ i\ and\ k = 1\ to\ 3\ and\ g(\boldsymbol{V} + \boldsymbol{V}') = 1$$

Hence, the equations:

$$\delta_{ik} = a'_{ij}a_{jk} - a'_{ij}V'_j a_{4k} \qquad (6.9)$$

$$(V_i + V'_i) = a'_{ij}a_{jk}V_k + ga'_{ij}V'_j \qquad (6.10)$$

$$0 = a'_{4j}a_{jk} + g'a_{4k} \qquad (6.11)$$

$$1 = gg' - a'_{4j}a_{jk}V_k \qquad (6.12)$$

Equation (6.10), taking into account the first equation (6.1), shows that:

$$V'_i = -V_i\ or\ \boldsymbol{V}' = -\boldsymbol{V} \qquad (6.13)$$

This is an obvious result, but it is demonstrated here. Thus, taking into account (6.4):

$$a'_{ij} = a_{ij}\ ;\ a'_{4j} = -a_{4j}\ \ for\ i\ and\ j = 1\ to\ 3\ and\ g' = g \qquad (6.14)$$

We use the hypothesis concerning the inverse transformation H3 taking into account equations (6.14) and (6.4) as well as that H1 of the isotropy $\big(a_{ij} = a_{ji}\big)$. If we consider the motion of R with respect to R’ of speed $\boldsymbol{V}' = -\boldsymbol{V}$, taking into account (6.14) equations (3.54) are written:

$$x_j = a_{kj}(x'_k - V'_k t')\ and\ t = -a_{4k}x'_k + g't'\ for\ i\ and\ j = 1\ to\ 3 \qquad (6.15)$$

Using (6.15) in (3.54), taking into account the first equation (6.13) and identifying the coefficients, the following identities are obtained:

$$a_{ij}\ a_{kj} + a_{ij}\ a_{4k}V_j = \delta_{ik} \qquad (6.16)$$

$$-a_{ij}\ a_{kj}V_k + g\ a_{ij}V_j = 0 \qquad (6.17)$$

$$a_{4l}\ a_{kl} + g\ a_{4k} = 0 \qquad (6.18)$$

$$a_{4l}\ a_{kl}V_k + g^2 = 1 \qquad (6.19)$$

Finally (6.1) can be written:

$$gV_i = a_{ij}V_j = -a_{i4} \qquad (6.20)$$

It may be noted that equations (6.16) to (6.19) are equivalent to equations (6.9) to (6.12), but it was essential to use equation (6.10) first to show that $V_i' = -V_i$.

6-2 Galilean transformation formulae:

If all coefficients a_{4l} are zero, (3.54) and (3.53) are written as follows:

$$x'_i = a_{ij}(x_j - V_j t) \; and \; t' = gt \; for \; i \; and \; j = 1 \; to \; 3 \qquad (6.21)$$

$$u'_i = \frac{a_{ij}(u_j - V_j)}{g} for \; i \; and \; j = 1 \; to \; 3 \qquad (6.22)$$

In this case, since all the coefficients a_{ij} cannot be zero at the same time, if one of the coordinates u_j tends towards infinity, at least one of the coordinates u'_i also tends towards infinity; this is the Galilean transformation.

The equation (6.16) is then reduced to $a_{ij} \, a_{kj} = \delta_{ik}$. a_{ij} are the direction cosines of a rotation making it possible to pass from R to R'. Since we have imposed that the axes of R and R' remain parallel, the rotation must be the identity either $a_{ij} = \delta_{ij}$. (6.19) becomes $g^2 = 1$, or $g = 1$, g having to be positive as already seen.

Then (6.21) and (6.22) become:

$$x'_i = x_i - V_i t \; and \; t' = t \; for \; i = 1 \; to \; 3 \qquad (6.23)$$

$$u'_i = u_i - V_i \; for \; i = 1 \; to \; 3 \qquad (6.24)$$

6-3- Lorentz transformation formulae:

If in (3.53) only one of the coefficients a_{4l} is not zero, there is at least one of the coordinates u_j which, by tending alone towards infinity, causes a finite value for at least one of the coordinates u'_i, which would be contrary to our definition of 'kinematically equivalent reference frame'. Thus, for the same reasons as in the simplified demonstration, the existence of an identical speed limit U_l (as shown in 5-3) must be accepted in all kinematically equivalent reference frames, i.e.:

$U^2 = U'^2 = U_l^2$; U *and* U' *being the velocity module of* $\boldsymbol{U}$ *and* $\boldsymbol{U'}$. It results in:

$$\sum_{i=1}^{i=3}\left(a_{ij}u_j + a_{i4}\right)^2 - \left(a_{4j}u_j + g\right)^2 U_l^2 = 0 \qquad (6.25)$$

with:

$$u_1^2 + u_2^2 + u_3^2 - U_l^2 = 0 \qquad (6.26)$$

The equation (6.25) is developed to obtain:

$$C_{11}u_1^2 + C_{22}u_2^2 + C_{33}u_3^2 + 2(C_{12}u_1u_2 + C_{13}u_1u_3 + C_{23}u_2u_3) - 2aC_1u_1 - 2aC_2u_2 - 2aC_3u_3 - C = 0 \qquad (6.27)$$

Equations (6.26) and (6.27) must be satisfied simultaneously whatever the values of the speeds u_i, this requires that the 10 coefficients of these polynomials be the same[5]. The result is the following 10 equations:

$$a_{ji}a_{jk} = a_{ij}a_{kj} = \delta_{ik} + a_{4i}a_{4k}U_l^2 \; with \; i,j,k = 1 \; to \; 3 \qquad (6.28)$$

$$a_{ki}a_{k4} = a_{ik}a_{k4} = g a_{4i}U_l^2 \; with \; i \; and \; k = 1 \; to \; 3 \qquad (6.29)$$

$$\sum_{i=1}^{i=3} {a_{i4}}^2 = (g^2 - 1)U_l^2 \qquad (6.30)$$

(6.20) put in (6.30) gives:

$$g = \frac{1}{\sqrt{1 - \frac{V^2}{U_l^2}}} \qquad (6.31)$$

[5] *This can be explained to students in the following pictorial form: in a coordinate system* u_1, u_2, u_3, *the equation (6.27) represents an ellipsoid (or other quadric) that can cut the sphere of the equation (6.26) into one or more lines. As we want all the points representing the solution to be on the sphere, it is necessary that the ellipsoid be the sphere itself*

Using (6.16), (6.17), (6.19) and (6.20), we get:

$$a_{i4} = -gV_i \tag{6.32}$$

Using (6.16), (6.20) and (6.28), we get:

$$a_{4i} = \frac{a_{i4}}{U_l^2} = -\frac{gV_i}{U_l^2} \tag{6.33}$$

The coordinate axes of the two reference frames remaining parallel, if one of the coordinates V_i of the velocity is zero then $x'_i = x_i$, which then requires that $a_{ij} = \delta_{ij}\ for\ i\ and\ j = 1\ to\ 3$. This condition is achieved by introducing the dimensionless coefficients β_{ij} below:

$$a_{ij} = \delta_{ij} + \beta_{ij}w_iw_j\ \ for\ i\ and\ j = 1\ to\ 3\ where\ w_i = \frac{V_i}{V}\ with\ V = \sqrt{V_1^2 + V_2^2 + V_3^2} \tag{6.34}$$

By transferring these values into (6.20) or (6.29) we obtain:

$$\beta_{ij}w_j^2 = g - 1\ with\ i\ and\ j = 1\ to\ 3 \tag{6.35}$$

(6.28) becomes:

$$\beta_{ij}\beta_{kj}w_j^2 + 2\beta_{ik} = g^2 - 1 \tag{6.36}$$

According to the definition of the dimensionless velocities given in (6.34):

$$\sum_{j=1}^{j=3} {w_j}^2 = 1 \tag{6.37}$$

A combination of (6.35) and (6.37) gives:

$$[\beta ij - (g-1)]w_j^2 = 0 \tag{6.38}$$

(6.38) having to be verified whatever the values of the dimensionless velocities w_j, it is necessary that:

$$\beta_{ij} = g - 1 \tag{6.39}$$

It can then be checked that (6.36) is verified.

In summary:

$$a_{44} = g = \frac{1}{\sqrt{1-\frac{V^2}{U_l^2}}}\ ; a_{i4} = -gV_i\ \ ; a_{4j} = -\frac{gV_j}{U_l^2};\ a_{ij} = \delta_{ij} + (g-1)\frac{V_iV_j}{V^2}\ for\ i = 1\ to\ 3\ and\ j = 1\ to\ 3 \tag{6.40}$$

(3.47) or (3.54) then become the formulae of the Lorentz transformation:

$$x'_i = \left[\delta_{ij} + (g-1)\frac{V_iV_j}{V^2}\right]x_j - gV_it = \left[\delta_{ij} + (g-1)\frac{V_iV_j}{V^2}\right](x_j - V_jt) \tag{6.41}$$

$$t' = g\left(t - \frac{V_jx_j}{U_l^2}\right)\ with\ g = \frac{1}{\sqrt{1-\frac{V^2}{U_l^2}}} \tag{6.42}$$

These are indeed the formulae already obtained by V. Fock and N. Kemmer [7] or C. Moller [10] in particular.

7- Application to the kinematics of special relativity

The designation of the speed of light in vacuum c as the speed limit U_l makes it possible to reconcile the electromagnetic and mechanical phenomena, which is obviously the basis of the theory of special relativity developed by Poincaré and Einstein in particular. In this case, g becomes the Lorentz factor:

$$g = \gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \tag{7.1}$$

Conclusion

By expressing the fact that, if the acceleration of any particle is zero in a reference frame R, it is also zero in a kinematically equivalent reference frame R', within the meaning of our definition 1-2, we have been able to demonstrate that the transformation formulae between these reference frames are necessarily linear. It was not necessary to use the invariance of the speed of light in vacuum or the concepts of space-time homogeneity. According to our demonstration, the concept of homogeneity, in the sense that it is defined by J. M. Lévy-Leblond [4], derives from the fact that the transformation formulae between kinematically equivalent reference frames are linear, and not the reverse. The existence of these transformation formulae also proves that two kinematically equivalent reference frames are necessarily driven by a relative uniform translational motion. If we admit that infinite speeds can exist, the transformation is necessarily the Galilean transformation. Otherwise, there is a speed limit U_l which, in the strict context of kinematics, is not necessarily the speed of light. The Lorentz transformation is then the solution to the problem.

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