GENERALIZATION OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY USING A DIFFERENTIAL OPERATOR

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In this paper, we introduce the generalized class \( S^p_{n,j}(p,q;\alpha;A,B) \) of analytic and p-valent functions with negative coefficients defined by the operator \( D^p_n f(z) \). We give some properties of functions in this class and obtain numerous sharp results including (for example) coefficient estimates, distortion theorem, radii of starlikeness, convexity, close-to-convexity and modified-Hadamard products of functions belonging to this class. Finally, several applications involving an integral operator and certain fractional calculus operators are also considered.

1. Introduction

Let \( T(j,p) \) denote the class of functions of the form:

\[
f(z) = z^p - \sum_{k=j+p}^{\infty} a_k z^k \quad (a_k \geq 0; \ p, j \in \mathbb{N} = \{1, 2, 3, \ldots\}),
\]

which are analytic and p-valent in the open unit disc \( U = \{z \in \mathbb{C} : |z| < 1\} \). A function \( f(z) \in T(j,p) \) is said to be p-valent starlike of order \( \alpha \) if it satisfies the inequality:

\[
|f(z)| \leq \frac{1}{\left(1 - |z|\right)^{\alpha}}, \quad |z| < 1.
\]

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\[\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p, \; p \in \mathbb{N}; \; z \in U). \tag{1.2}\]

We denote by \(T^*_j(p, \alpha)\) the class of all \(p\)-valent starlike functions of order \(\alpha\). Also a function \(f(z) \in T(j, p)\) is said to be \(p\)-valent convex of order \(\alpha\) if it satisfies the inequality:

\[\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; \; z \in U). \tag{1.3}\]

We denote by \(C_j(p, \alpha)\) the class of all \(p\)-valently convex functions of order \(\alpha\), and we note that (see for example Duren [10] and Goodman [11])

\[f(z) \in C_j(p, \alpha) \iff \frac{zf'(z)}{p} \in T^*_j(p, \alpha) \quad (0 \leq \alpha < p); \tag{1.4}\]

the classes \(T^*_j(p, \alpha)\) and \(C_j(p, \alpha)\) were studied by Owa [17].

For each \(f(z) \in T(j, p)\), we have (see [8])

\[f^{(q)}(z) = \delta(p, q)z^{p-q} - \sum_{k=j+p}^{\infty} \delta(k, q)a_k z^{k-q} \tag{1.5}\]

\[(q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \; p > q),\]

where

\[\delta(p, q) = \frac{p!}{(p - q)!} = \begin{cases} 1, & (q = 0), \\ p(p-1)\ldots(p-q+1), & (q \neq 0). \end{cases} \tag{1.6}\]
Also for a function \( f(z) \in T(j, p) \), we define the differential operator \( D_p^n f^{(q)}(z) \) by:

\[
\begin{align*}
D_p^0 f^{(q)}(z) &= f^{(q)}(z), \\
D_p^1 f^{(q)}(z) &= D f^{(q)}(z) = \frac{z}{p-q} \left( f^{(q)}(z) \right)' = \frac{z}{p-q} f^{(q+1)}(z) \\
&= \delta(p,q)z^{p-q} - \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right) \delta(k,q)a_k z^{k-q}, \\
D_p^2 f^{(q)}(z) &= D \left( D_p^1 f^{(q)}(z) \right) \\
&= \delta(p,q)z^{p-q} - \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^2 \delta(k,q)a_k z^{k-q}, \\
\end{align*}
\]

and (in general)

\[
D_p^n f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=j+p}^{\infty} \left( \frac{k-q}{p-q} \right)^n \delta(k,q)a_k z^{k-q} \tag{1.7}
\]

\((p, j \in \mathbb{N}; n, q \in \mathbb{N}_0, p > q)\). The operator \( D_p^n f^{(q)}(z) \) was introduced and studied by Aouf [4,5].

We note that:

(i) Putting \( q = 0 \) in (1.7), we have the operator \( D_p^n \) which was introduced and studied by Kamali and Orhan [13] and Aouf and Mostafa [6];

(ii) Putting \( q = 0 \) and \( p = 1 \) in (1.7), we have the operator \( D_1^n = D^n \) which was introduced by Salagean [19].

**Definition 1.** For \(-1 \leq A < B \leq 1\) and \(0 \leq \alpha < p - q\), let \( S_j^n(p,q,\alpha; A, B) \) be the class of functions \( f(z) \in T(j, p) \) which

\[
\frac{z(D_p^n f^{(q)}(z))'}{D_p^n f^{(q)}(z)} < (p - q - \alpha) \frac{1 + A z}{1 + B z} + \alpha,
\]
or
\[
\frac{z \left( D_n^p f^{(q)}(z) \right)'}{D_n^p f^{(q)}(z)} < \frac{(p-q) + [(p-q)B + (A-B)(p-q-\alpha)] z}{1 + Bz} \quad (z \in U),
\]
that is, that
\[
S_n^j(p, q; \alpha; A, B) = \{ f(z) \in T(j, p) : \}
\]
\[
\begin{align*}
& \quad \frac{z \left( D_n^p f^{(q)}(z) \right)'}{D_n^p f^{(q)}(z)} - (p-q) < 1, \quad z \in U \quad (1.9) \\
& B \frac{z \left( D_n^p f^{(q)}(z) \right)'}{D_n^p f^{(q)}(z)} - [B(p-q) + (A-B)(p-q-\alpha)] \end{align*}
\]
\[
(0 \leq \alpha < p-q; \quad -1 \leq A < B \leq 1; \quad p, j \in \mathbb{N}; \quad q, n \in \mathbb{N}_0; \quad p > q).
\]

Specializing the parameters \( n, j, q, A \) and \( B \) we obtain the following subclasses studied by various authors.

(i) \( S_n^0(p, q, \alpha; -1, 1) = S_j(n, p, q, \alpha) \) \((0 \leq \alpha < p-q, \quad p, j \in \mathbb{N}, \quad q, n \in \mathbb{N}_0, \quad p > q)\) (Aouf [4]);

(ii) \( S_j^0(p, q, \alpha; -\beta, \beta) = S_j(p, q, \alpha, \beta) \) and \( S_j^1(p, q, \alpha; -\beta, \beta) = C_j(p, q, \alpha, \beta) \) \((0 \leq \alpha < p-q, \quad 0 < \beta \leq 1, \quad p, j \in \mathbb{N}, \quad q, n \in \mathbb{N}_0, \quad p > q)\) (Aouf [3]);

(iii) \( S_j^0(p, q, \alpha; -1, 1) = S_j(p, q, \alpha) \) and \( S_j^1(p, q, \alpha; -1, 1) = C_j(p, q, \alpha) \) (Chen et al. [8]);
(iv) $S^0_j(p, 0, \alpha; -\beta, \beta) = T^*_j(p, \alpha, \beta)$ and $S^0_j(p, 1, \alpha; -\beta, \beta) = C_j(p, \alpha, \beta)(0 \leq \alpha < p, 0 < \beta \leq 1, p, j \in \mathbb{N})$ (Aouf [2]);

(v) $S^0_1(p, 0, \alpha; A, B) = T^*_p(A, B, \alpha)$ and $S^1_1(p, 0, \alpha; A, B) = C_p(A, B, \alpha)(-1 \leq A < B \leq 1, 0 \leq \alpha < p, p \in \mathbb{N})$ (Aouf [1]);

(vi) $S^0_1(p, 0, \alpha; -\beta, \beta) = S^*_0(p, \alpha, \beta)$ and $S^1_1(p, 0, \alpha; -\beta, \beta) = C^*_0(p, \alpha, \beta)(0 \leq \alpha < p, 0 < \beta \leq 1, p \in \mathbb{N})$ (Hossen [12]);

(vii) $S^0_1(p, 0, \alpha; -1, 1) = T^*(p, \alpha)$ and $S^1_1(p, 0, \alpha; -1, 1) = C(p, \alpha)$ (Owa [17] and Salagean et al. [20]);

(viii) $S^0_1(p, 0, \alpha; -1, 1) = \begin{cases} T^*_j(p, \alpha), \\ T_\alpha(p, j), \end{cases}$ (Owa [18]),

\[ (0 \leq \alpha < p, p, j \in \mathbb{N}); \]

(ix) $S^1_1(p, 0, \alpha; -1, 1) = \begin{cases} C_j(p, \alpha), \\ CT_\alpha(p, j), \end{cases}$ (Owa [18]),

\[ (0 \leq \alpha < p, p, j \in \mathbb{N}). \]

Also we note that

(i) $S^p_1(p, q, \alpha; -\beta, \beta) = S^p_j(p, q, \alpha, \beta)$

\[ \begin{cases} f(z) \in T(j, p) : \frac{z(D^p_q f^{(q)}(z))'}{D^p_q f^{(q)}(z)} - (p-q) \left< \beta, 0 < \beta \leq 1 \right. \end{cases}, \]

\[ 0 < \beta \leq 1 \]

\[ (1.10) \]
In our present paper, we shall make use of the familiar integral operator $J_{c,p}$ defined by (cf. [7], [14] and [15]; see also [24])

\[
(J_{c,p}f)(z) = \frac{c + p}{zc} \int_0^z t^{c-1} f(t) dt \quad (f \in T(j,p); \quad c > -p; \quad p \in \mathbb{N}),
\]

as well as the fractional calculus operator $D_z^\mu$ for which it is well known that (see for details [16] and [22]; see also Section 5 below)

\[
D_z^\mu \{z^\rho\} = \frac{\Gamma(\rho + 1)}{\Gamma(\rho + 1 - \mu)} z^{\rho - \mu} \quad (\rho > -1; \quad \mu \in \mathbb{R}),
\]

in terms of gamma functions.

2. Coefficient Estimates

Unless otherwise mentioned, we assume in the reminder of this paper that $\delta(p,q)$ is defined by (1.6), $-1 \leq A < B \leq 1, 0 \leq \alpha < p - q, p, j \in \mathbb{N}, q, n \in \mathbb{N}_0, p > q$ and $z \in U.$

**Theorem 1.** Let the function $f(z)$ be given by (1.1). Then $f(z) \in S_j^n(p,q,\alpha;A,B)$ if and only if

\[
\sum_{k=j+p}^{\infty} \left| (1 + B)(k - p) + (B - A)(p - q - \alpha) \right| \left( \frac{k - q}{p - q} \right)^n \delta(k,q) a_k 
\leq (B - A)(p - q - \alpha) \delta(p,q). \tag{2.1}
\]

**Proof.** Assume that the inequality (2.1) holds true. We find from (1.1) and (2.1) that
\[ \left| z(D^n_q f(z))' - (p - q)(D^n_p f(z))' \right| \\
- \left| Bz(D^n_q f(z))' - [B(p - q) + (A - B)(p - q - \alpha)](D^n_p f(z))' \right| \\
= \left| \sum_{k=j+p}^{\infty} (k - p) \left( \frac{k-q}{p-q} \right)^n \delta(k, q) a_k z^{k-q} \right| \\
- \left| \sum_{k=j+p}^{\infty} [(A - B)(p - q - \alpha) - B(k - p)] \left( \frac{k-q}{p-q} \right)^n \delta(k, q) a_k z^{k-q} \right| \\
\leq \sum_{k=j+p}^{\infty} (k - p) \left( \frac{k-q}{p-q} \right)^n \delta(k, q) a_k z^{k-q} \\
\leq 0. \\
\]

Hence, by the maximum modulus theorem, we have
\[
\frac{z(D^n_p f(q)(z))'}{D^n_p f(q)(z)} - (p - q) < 1.
\]

Thus \( f(z) \in S^n_j(p, q, \alpha; A, B) \).

Conversely, let \( f(z) \in S^n_j(p, q, \alpha; A, B) \) be given by (1.1). Then from (1.1) and (1.9), we find that

\[
\frac{z(D^n_p f(q)(z))'}{D^n_p f(q)(z)} - (p - q) < 1.
\]

Now, since \( \Re(z) \leq |z| \) for all \( z \), we have

\[
\text{Re}\left\{ \frac{\sum_{k=j+p}^{\infty} (k-p) \left( \frac{k-q}{p-q} \right)^n \delta(k,q) a_k z^k}{-(A-B)(p-q-\alpha)\delta(p,q)z^{p-q} + \sum_{k=j+p}^{\infty} [(A-B)(p-q-\alpha) - B(k-p)] \left( \frac{k-q}{p-q} \right)^n \delta(k,q) a_k z^k} \right\} \leq 1,
\]

(2.2)
choose values of \( z \) on the real axis so that \( \frac{z(D_n^p f^{(q)}(z))'}{D_n^p f^{(q)}(z)} \) is real. Upon clearing the denominator in (2.2) and letting \( z \rightarrow 1^- \) through real values, we have

\[
\sum_{k=j+p}^{\infty} (k - p) \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k \\
\leq - (A - B)(p - q - \alpha) \delta(p, q) \\
+ \sum_{k=j+p}^{\infty} [(A - B)(p - q - \alpha) - B(k - p)] \left( \frac{k - q}{p - q} \right)^n \delta(k, q) a_k.
\]

This gives the required condition.

**Corollary 1.** Let the function \( f(z) \) defined by (1.1) be in the class \( S_j^n(p, q, \alpha; A, B) \). Then

\[
a_k \leq \frac{(B - A)(p - q - \alpha) \delta(p, q)}{[1 + B](k - p) + (B - A)(p - q - \alpha)] \left( \frac{k - q}{p - q} \right)^n \delta(k, q)
\]

(2.3)

\((k \geq j + p)\),

the result is sharp for the function \( f(z) \) given by

\[
f(z) = z^p - \frac{(B - A)(p - q - \alpha) \delta(p, q)}{[1 + B](k - p) + (B - A)(p - q - \alpha)] \left( \frac{k - q}{p - q} \right)^n \delta(k, q) z^k
\]

(2.4)

\((k \geq j + p)\).

3. Distortion Theorem
Theorem 2. If the function \( f(z) \) defined by (1.1) is in the class \( S^p_j(p, q, \alpha; A, B) \). Then

\[
\left\{ \delta(p, m) - \frac{(B - A)(p - q - \alpha)(j + p - q)!\delta(p, q)}{(j + p - m)!(j(1 + B) + (B - A)(p - q - \alpha))\left(\frac{j + p - q}{p - q}\right)^n}\right\} |z|^{p-m} \leq |f^{(m)}(z)| \leq \left\{ \delta(p, m) + \frac{(B - A)(p - q - \alpha)(j + p - q)!\delta(p, q)}{(j + p - m)!(j(1 + B) + (B - A)(p - q - \alpha))\left(\frac{j + p - q}{p - q}\right)^n}\right\} |z|^{p-m}
\]

(3.1)

\((m \in \mathbb{N}_0; p > \text{max}\{q, m\})\).

The result is sharp for the function \( f(z) \) given by

\[
f(z) = z^p - \frac{(B - A)(p - q - \alpha)\delta(p, q)}{[j(1 + B) + (B - A)(p - q - \alpha)]\left(\frac{j + p - q}{p - q}\right)^n}\delta(j + p, q)z^{j+p}.
\]

(3.2)

Proof. In view of Theorem 1, we have

\[
\left[\frac{[j(1 + B) + (B - A)(p - q - \alpha)]\left(\frac{j + p - q}{p - q}\right)^n\delta(j + p, q)}{(B - A)(p - q - \alpha)\delta(p, q)(j + p)!}\right] \sum_{k=j+p}^{\infty} k!a_k
\]

\[
\leq \sum_{k=j+p}^{\infty} \frac{[(1 + B)(k - p) + (B - A)(p - q - \alpha)]\left(\frac{k - q}{p - q}\right)^n\delta(k, q)a_k}{(B - A)(p - q - \alpha)\delta(p, q)} \leq 1,
\]
which readily yields

$$\sum_{k=j+p}^{\infty} k!a_k \leq \frac{(B - A)(p - q - \alpha)(j + p - q)!\delta(p, q)}{|j(1 + B) + (B - A)(p - q - \alpha)| \left(\frac{j + p - q}{p - q}\right)^n}.$$  (3.3)

Now, by differentiating both sides of (1.1) \(m\)-times, we have

$$f^{(m)}(z) = \delta(p, m)z^{p-m} - \sum_{k=j+p}^{\infty} \delta(k, m)a_k z^{k-m}$$  (3.4)

\((k \geq j + p; \ m \in \mathbb{N}_0; \ p > \max\{q, m\})\),

and Theorem 2 would follow from (3.3) and (3.4).

4. Radii of Starlikeness, Convexity and Close-to-Convexity

**Theorem 3.** Let the function \(f(z)\) defined by (1.1) be in the class \(S^n_j(p, q, \alpha; A, B)\). Then

(i) \(f(z)\) is \(p\)-valently starlike of order \(\varphi(0 \leq \varphi < p)\) in \(|z| < r_1\),

where

$$r_1 = \inf_k \left\{ \frac{[(k-p)(1+B)+(B-A)(p-q-\alpha)] \left(\frac{k-q}{p-q}\right)^n \delta(k, q) \left(\frac{p-\varphi}{k-\varphi}\right)^{1/k-p}}{(B - A)(p - q - \alpha)\delta(p, q)} \right\}^{1/k-p}$$  (4.1)

\((k \geq j + p);\)
(ii) $f(z)$ is $p$-valently starlike of order $\varphi (0 \leq \varphi < p)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left\{ \frac{[(k-p)(1+B)+(B-A)(p-q-\alpha)] \left( \frac{k-q}{p-q} \right)^n \delta(k,q) p(p-\varphi)}{(B-A)(p-q-\alpha)\delta(p,q)} \right\}^{\frac{1}{k-p}}$$

$(k \geq j+p);$ (4.2)

(iii) $f(z)$ is $p$-valently starlike of order $\varphi (0 \leq \varphi < p)$ in $|z| < r_3$, where

$$r_3 = \inf_k \left\{ \frac{[(k-p)(1+B)+(B-A)(p-q-\alpha)] \left( \frac{k-q}{p-q} \right)^n \delta(k,q) \left( \frac{p-\varphi}{k} \right)}{(B-A)(p-q-\alpha)\delta(p,q)} \right\}^{\frac{1}{k-p}}$$

$(k \geq j+p);$ (4.3)

Each of these results is sharp for the function $f(z)$ given by (2.4).

**Proof.** It is sufficient to show that

$$\left| \frac{z f'(z)}{f(z)} - p \right| \leq p - \varphi \quad (|z| < r_1; \ 0 \leq \varphi < p; \ p \in \mathbb{N}),$$

or

$$\left| \frac{z f'(z)}{f(z)} - p \right| = \left| 1 - \sum_{k=j+p}^{\infty} a_k z^{k-p} \right| \leq \left| 1 - \sum_{k=j+p}^{\infty} a_k |z|^{k-p} \right|. \quad (4.5)$$
Inequality (4.4) holds true, when
\[ \sum_{k=j+p}^{\infty} (k-p)a_k |z|^{k-p} \leq p - \varphi, \]
thus
\[ \sum_{k=j+p}^{\infty} \left( \frac{k - \varphi}{p - \varphi} \right) a_k |z|^{k-p} \leq 1, \]
(4.6)

using inequality (2.1), then (4.6) holds true if
\[
\left( \frac{k - \varphi}{p - \varphi} \right) a_k |z|^{k-p} \leq \frac{[(1+B)(k-p)+(B-A)(p-q-\alpha)] \left( \frac{k-q}{p-q} \right)^n \delta(k,q)a_k}{(B-A)(p-q-\alpha)\delta(p,q)}
\]
\[
(k \geq j+p),
\]
(4.7)
or
\[
|z| \leq \left\{ \frac{[(1+B)(k-p)+(B-A)(p-q-\alpha)] \left( \frac{k-q}{p-q} \right)^n \delta(k,q) \left( \frac{p-q}{k-\varphi} \right)^{\frac{1}{p-q}}}{(B-A)(p-q-\alpha)\delta(p,q)} \right\}^{\frac{1}{k-p}}
\]
\[
(k \geq j+p),
\]
(4.8)
or
\[
r_1 = \inf_k \left\{ \frac{[(1+B)(k-p)+(B-A)(p-q-\alpha)] \left( \frac{k-q}{p-q} \right)^n \delta(k,q) \left( \frac{p-q}{k-\varphi} \right)^{\frac{1}{p-q}}}{(B-A)(p-q-\alpha)\delta(p,q)} \right\}^{\frac{1}{k-p}}
\]
\[
(k \geq j+p).
\]
(4.9)
This completes the proof of (4.1).
To prove (4.2) and (4.3) it is sufficient to note that
\[
\left| \frac{z f''(z)}{f'(z)} - p \right| \leq p - \varphi \ (|z| < r_2; \ 0 \leq \varphi < p) \tag{4.10}
\]
and
\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \varphi \ (|z| < r_3; \ 0 \leq \varphi < p), \tag{4.11}
\]
respectively.

5. Modified-Hadamard Products

For the functions
\[
f_{\nu}(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,\nu} z^k \quad (a_{k,\nu} \geq 0; \nu = 1, 2), \tag{5.1}
\]
we denote by \((f_1 * f_2)(z)\) the modified Hadamard product (or convolution) of the functions \(f_1(z)\) and \(f_2(z)\), that is,
\[
(f_1 * f_2)(z) = z^p - \sum_{k=j+p}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z). \tag{5.2}
\]

**Theorem 4.** Let the functions \(f_{\nu}(z) \ (\nu = 1, 2)\) defined by (5.1) be in the class \(S_{\nu}^p(p, q, \alpha; A, B)\). Then \((f_1 * f_2)(z) \in S_{\gamma}^p(p, q, \gamma; A, B)\), where
\[
\gamma = (p-q)
\]
and
\[
\frac{(B - A)(1 + B)j(p - q - \alpha)^2 \delta(p, q)}{[(1+B)j+(B-A)(p-q-\alpha)]^2 \left( \frac{j+p-q}{p-q} \right)^2 \delta(j + p, q) - (B - A)^2 (p-q-\alpha)^2 \delta(p, q)}.
\tag{5.3}
\]
The result is sharp for the functions \( f_\nu(z) \) \((\nu = 1, 2)\) given by

\[
f_\nu(z) = z^p - \frac{(B - A)(p - q - \alpha)\delta(p,q)}{[(1 + B)(p - q - \alpha)]^{\frac{1 + p - q}{p - q}}} z^{j + p} \delta(j + p, q)
\]

\((\nu = 1, 2)\).

**Proof.** Employing the technique used earlier by Schild and Silverman [21], we need to find the largest \( \gamma \) such that

\[
\sum_{k=j+p}^{\infty} \frac{[(1 + B)(k - p) + (B - A)(p - q - \gamma)] \left( \frac{k - q}{p - q} \right)^n \delta(k, q)}{(B - A)(p - q - \gamma) \delta(p, q)} a_{k,1}a_{k,2} \leq 1
\]

\((5.5)\)

\((f_\nu(z) \in S_j^n(p, q; A, B); \nu = 1, 2)\).

Since \( f_\nu(z) \in S_j^n(p, q; A, B) \) \((\nu = 1, 2)\), we readily see that

\[
\sum_{k=j+p}^{\infty} \frac{[(1 + B)(k - p) + (B - A)(p - q - \gamma)] \left( \frac{k - q}{p - q} \right)^n \delta(k, q)}{(B - A)(p - q - \alpha) \delta(p, q)} a_{k,\nu}
\]

\(\leq 1 \quad (\nu = 1, 2)\).

\((3.11)\)

Therefore, by the Cauchy-Schwarz inequality, we obtain

\[
\sum_{k=j+p}^{\infty} \frac{[(1 + B)(k - p) + (B - A)(p - q - \gamma)] \left( \frac{k - q}{p - q} \right)^n \delta(k, q)}{(B - A)(p - q - \alpha) \delta(p, q)} \sqrt{a_{k,1}a_{k,2}} \leq 1.
\]

\((5.7)\)

Now from (5.5) and (5.7) we only need to show that
\[
\frac{(1 + B)(k - p) + (B - A)(p - q - \gamma)}{(p - q - \gamma)} a_{k,1}a_{k,2}
\]

\[
\leq \frac{(1 + B)(k - p) + (B - A)(p - q - \alpha)}{(p - q - \alpha)} \sqrt{a_{k,1}a_{k,2}} \quad (k \geq j + p), \quad (5.8)
\]

or, equivalently, that

\[
\sqrt{a_{k,1}a_{k,2}} \leq \frac{[(1 + B)(k - p) + (B - A)(p - q - \alpha)](p - q - \gamma)}{[(1 + B)(k - p) + (B - A)(p - q - \gamma)](p - q - \alpha)}. \quad (5.9)
\]

Hence, in the light of inequality (5.7), it is sufficient to prove that

\[
\frac{(B - A)(p - q - \alpha)\delta(p, q)}{[(1 + B)(k - p) + (B - A)(p - q - \alpha)] \left(\frac{k - q}{p - q}\right)^\alpha \delta(k, q)} \leq \frac{[(1 + B)(k - p) + (B - A)(p - q - \gamma)](p - q - \gamma)}{[(1 + B)(k - p) + (B - A)(p - q - \gamma)](p - q - \alpha)} \quad (k \geq j + p). \quad (5.10)
\]

It follows from (5.10) that,

\[
\gamma \leq (p - q)
\]

\[
- \frac{(B - A)(1 + B)(k - p)(p - q - \alpha)^2\delta(p, q)}{\left(\frac{k - q}{p - q}\right)^\alpha [(1+B)(k-p)+(B-A)(p-q-\alpha)]^2 \delta(k,q) - (B-A)^2(p-q-\alpha)^2 \delta(p,q)} \quad (k \geq j + p). \quad (5.11)
\]
Now, defining the function $G(k)$ by

$$G(k) = (p-q) \frac{(B-A)(1+B)(k-p)(p-q-\alpha)^2\delta(p,q)}{\left(\frac{k-q}{p-q}\right)^n \left[(1+B)(k-p)+(B-A)(p-q-\alpha)\right]^2 \delta(k,q)-(B-A)^2(p-q-\alpha)^2\delta(p,q)}$$

\[(k \geq j+p)\]  \hspace{1cm} (5.12)

We see that $G(k)$ is an increasing function of $k$. Therefore, we conclude that

$$\gamma = G(j+p) = (p-q) \frac{(B-A)(1+B)j(p-q-\alpha)^2\delta(p,q)}{\left(\frac{j+p-q}{p-q}\right)^n \left[(1+B)j+(B-A)(p-q-\alpha)\right]^2 \delta(j+p,q)-(B-A)^2(p-q-\alpha)^2\delta(p,q)}$$

\[(j \geq j+p)\]  \hspace{1cm} (5.13)

which evidently completes the proof of Theorem 4.

Putting (i) $n = 0$, $A = -1$ and $B = 1$ and (ii) $n = 1$, $A = -1$ and $B = 1$ in Theorem 4, we obtain the following two corollaries which correct the results obtained by Chen et al. [8, Theorems 5 and 6].

**Corollary 2.** Let the functions $f_{\nu}(z)$ ($\nu = 1, 2$) defined by (5.1) be in the class $S^0_j(p,q,\alpha; -1, 1) = S_j(p,q,\alpha)$. Then $(f_1*f_2)(z) \in S_j(p,q,\gamma)$, where

$$\gamma = (p-q) - \frac{j(p-q-\alpha)^2\delta(p,q)}{(j+p-q-\alpha)^2\delta(j+p,q)-(p-q-\alpha)^2\delta(p,q)}. \hspace{1cm} (5.14)$$
The result is sharp.

**Corollary 3.** Let the functions $f_\nu(z)$ ($\nu = 1, 2$) defined by (5.1) be in the class $S^1_{j}(p, q, \alpha; -1, 1) = C_2(p, q, \alpha)$. Then $(f_1 * f_2)(z) \in C_2(p, q, \gamma)$, where

$$
\gamma = (p - q) - \frac{j(p - q - \alpha)^2 \delta(p, q + 1)}{(j + p - q - \alpha)^2 \delta(j + p, q + 1) - (p - q - \alpha)^2 \delta(p, q + 1)}. \quad (5.15)
$$

The result is sharp.

Using arguments similar to those in the proof of Theorem 4, we obtain the following results.

**Theorem 5.** Let a function $f_1(z)$ defined by (5.1) be in the class $S^n_{j}(p, q, \alpha; A, B)$. Suppose also that a function $f_2(z)$ defined by (5.1) be in the class $S^n_{j}(p, q, \tau; A, B)$. Then $(f_1 * f_2)(z) \in S_j(p, q, \zeta; A, B)$, where

$$
\zeta = (p - q) - \frac{(B - A)(1 + B)j(p - q - \alpha)(p - q - \tau)\delta(p, q)}{[(1 + B)j + (B - A)(p - q - \tau)\delta(j + p, q) - (B - A)^2(p - q - \alpha)(p - q - \tau)\delta(p, q)\delta(j + p, q)]}. \quad (5.16)
$$

The result is the best possible for the functions

$$
f_1(z) = z^p - \frac{(B - A)(p - q - \alpha)\delta(p, q)}{[(1 + B)j + (B - A)(p - q - \alpha)\delta(j + p, q)]} z^{j + p} \quad (5.17)
$$
and
\[ f_2(z) = z^p - \frac{(B - A)(p - q - \tau) \delta(p, q)}{[(1 + B)j + (B - A)(p - q - \tau)] \left( \frac{i + p - q}{p - q} \right)^n \delta(j + p, q)} z^{j+p}. \]

(5.18)

**Theorem 6.** Let the functions \( f_\nu(z) \) (\( \nu = 1, 2 \)) defined by (5.1) be in the class \( S^n_j(p, q, \alpha; A, B) \). Then the function
\[ h(z) = z^p - \sum_{k=j+p}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k, \]
belongs to the class \( S^n_j(p, q, \xi; A, B) \), where
\[ \xi = (p - q) - \]
\[ \frac{2(B-A)(1+B)j(p - q - \alpha)^2 \delta(p, q)}{[(1+B)j+(B-A)(p-q-\alpha)]^2 \left( \frac{i + p - q}{p - q} \right)^n \delta(j+p,q) - 2(B-A)^2(p-q-\alpha)^2 \delta(p,q)}. \]
(5.19)

The result is the sharp for the functions \( f_\nu(z) \) (\( \nu = 1, 2 \)) defined by (5.4).

6. Applications of Fractional Calculus

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [9], [16], [23] and [24]; see also the various references cited therein). For our present investigation, we recall the following definitions.
**Definition 2.** The fractional integral of order $\mu$ is defined, for a function $f(z)$, by

$$D_{z}^{\mu}f(z) = \frac{1}{\Gamma(\mu)} \int_{0}^{z} \frac{f(\mu)}{(z-\zeta)^{1-\mu}} d\zeta \quad (\mu > 0), \quad (6.1)$$

where the function $f(z)$ is analytic in a simply-connected domain of the complex $z$-plane containing the origin and the multiplicity of $(z - \zeta)^{-\mu-1}$ is removed by requiring $\log(z - \zeta)$ to be real when $z - \zeta > 0$.

**Definition 3.** The fractional derivative of order $\mu$ is defined, for a function $f(z)$, by

$$D_{z}^{-\mu}f(z) = \frac{1}{\Gamma(1-\mu)} \int_{0}^{z} \frac{f(\mu)}{(z-\zeta)^{\mu}} d\zeta \quad (0 \leq \mu < 1), \quad (6.2)$$

where the function $f(z)$ is constrained, and the multiplicity of $(z - \zeta)^{-\mu}$ is removed, as above.

**Definition 4.** Under the hypotheses of definition 2, the fractional derivative of order $\mu$ is defined, for a function $f(z)$, by

$$D_{z}^{n+\mu}f(z) = \frac{d^n}{dz^n} \{D_{z}^{\mu}f(z)\} \quad (0 \leq \mu < 1; n \in \mathbb{N}_0). \quad (6.3)$$

In this section we shall investigate the growth and distortion properties of functions in the class $S_{j}^{\mu}(p, q, \alpha; A, B)$, involving the operators $J_{c,p}$ and $D_{z}^{\mu}$. In order to derive our results, we need the following lemma given by Chen et al. [9].
**Lemma 1** (see Chen et al. [9]). Let the function $f(z)$ be defined by (1.1). Then

$$D_z^\mu \{(J_{c,p}f)(z)\} = \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c + p) \Gamma(k + 1)}{(c + k) \Gamma(k + 1 - \mu)} a_k z^{k-\mu}$$

(6.4)

$(\mu \in R; c > -p)$,

and

$$J_{c,p}(D_z^\mu \{f(z)\}) = \frac{(c + p) \Gamma(p + 1)}{(c + p - \mu) \Gamma(p + 1 - \mu)} z^{p-\mu} - \sum_{k=j+p}^{\infty} \frac{(c + p) \Gamma(k + 1)}{(c + k - \mu) \Gamma(k + 1 - \mu)} a_k z^{k-\mu}$$

(6.5)

$(\mu \in R; c > -p)$,

provided that no zeros appear in the denominators in (6.4) and (6.5).

**Theorem 7.** Let the function $f(z)$ defined by (1.1) be in the class $S^a_j(p, q; \alpha; A, B)$. Then

$$|D_z^\mu \{(J_{c,p}f)(z)\}| \geq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 + \mu)} \right\}$$

$$- \frac{(c + p) \Gamma(j + p + 1) (B - A) (p - q - \alpha) \delta(p, q)}{(c + j + p) \Gamma(j + p + 1 + \mu) [(1 + B) j + (B - A) (p - q - \alpha)] \left( \frac{j+p-q}{p-q} \right)^\alpha \delta(j+p, q)}$$

$|z|^{j+1}$

$|z|^{p+\mu}$ $(\mu > 0; c > -p)$

(6.6)

and

$$|D_z^\mu \{(J_{c,p}f)(z)\}| \leq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 + \mu)} \right\}$$
\[
\frac{\Gamma(j + p + 1) (B - A) (p - q - \alpha) \delta(p, q)}{\Gamma(j + p + 1 - \mu) [(1 + B)j + (B - A) (p - q - \alpha)] \left( \frac{j + p - q}{p - q} \right)^n \delta(j + p, q)}
\]

\[
|z|^j \left| \frac{z^{p - \mu}}{(c + j + p) \Gamma(j + p + 1 - \mu)} \right| (1 + B) j + (B - A) (p - q - \alpha) \delta(j + p, q)
\]

Each of the assertions (6.6) and (6.7) is sharp.

**Proof.** In view of Theorem 1, we have

\[
\left[ \frac{\Gamma(j + p + 1) (B - A) (p - q - \alpha) \delta(p, q)}{(B - A) (p - q - \alpha) \delta(p, q)} \right] \sum_{k=j+p}^{\infty} a_k
\]

\[
\leq \sum_{k=j+p}^{\infty} \left[ \frac{\Gamma(j + p + 1) (B - A) (p - q - \alpha) \delta(p, q)}{(B - A) (p - q - \alpha) \delta(p, q)} \right] \left( \frac{j + p - q}{p - q} \right)^n \delta(j + p, q) a_k
\]

\[
\leq 1,
\]

which readily yields

\[
\sum_{k=j+p}^{\infty} a_k \leq \frac{(B - A) (p - q - \alpha) \delta(p, q)}{\left[ \frac{\Gamma(j + p + 1) (B - A) (p - q - \alpha) \delta(p, q)}{(B - A) (p - q - \alpha) \delta(p, q)} \right] \left( \frac{j + p - q}{p - q} \right)^n \delta(j + p, q)}.
\]

Consider the function \( F(z) \) defined in \( U \) by

\[
F(z) = \frac{\Gamma(p + \mu + 1)}{\Gamma(p + 1)} z^{-\mu} D_z^{-\mu} \{ (J_{c,p} f)(z) \} = z^p - \sum_{k=j+p}^{\infty} \frac{(c + p) \Gamma(k + 1) \Gamma(p + 1 + \mu)}{(c + k) \Gamma(k + 1 + \mu) \Gamma(p + 1)} a_k z^k = z^p - \sum_{k=j+p}^{\infty} \phi(k) a_k z^k,
\]
where
\[
\phi(k) = \frac{(c+p)\Gamma(k+1)\Gamma(p+1+\mu)}{(c+k)\Gamma(k+1+\mu)\Gamma(p+1)}
\]
(6.10)
\[
(\mu > 0; k \geq j + p; c > -p).
\]

Since \(\phi(k)\) is a decreasing function of \(k\), when \(\mu > 0\), we get
\[
0 < \phi(k) \leq \phi(j + p) = \frac{(c+p)\Gamma(j + p + 1)\Gamma(p+1+\mu)}{(c+j+p)\Gamma(j + p + 1 + \mu)\Gamma(p+1)}
\]
(6.11)
\[
(\mu > 0; c > -p).
\]

Thus, by using (6.9) and (6.11), we deduce that
\[
|F(z)| \geq |z|^p - \phi(j + p)|z|^{j + p} \sum_{k=j+p}^{\infty} a_k \geq |z|^p
\]
(6.12)
and
\[
|F(z)| \leq |z|^p + \phi(j + p)|z|^{j + p} \sum_{k=j+p}^{\infty} a_k \leq |z|^p
\]
(6.13)
\[
(\mu > 0; c > -p),
\]
which yield the inequalities (6.6) and (6.7) of Theorem 7.
The equalities in (6.6) and (6.7) are attained for the function \( f(z) \) given by

\[
D^{-\mu}_z \{ (J_{c,p} f)(z) \} = \left\{ \begin{array}{c} \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)} \\ (c+p)\Gamma(j+p+1) (B-A) (p-q-\alpha) \delta(p,q) \\
(c+j+p)\Gamma(j+p+1+\mu) [(1+B)j+(B-A)(p-q-\alpha)] \left( \frac{i+p-q}{p-q} \right)^n \delta(j+p,q) \end{array} \right\}
\]

or, equivalently, by

\[
(J_{c,p} f)(z) = z^p \left( \frac{(c+p)(B-A)(p-q-\alpha) \delta(p,q)}{(c+j+p)[(1+B)j+(B-A)(p-q-\alpha)] \left( \frac{i+p-q}{p-q} \right)^n \delta(j+p,q)} \right)
\]

thus the proof of Theorem 7 is completed.

Using arguments similar to those in the proof of Theorem 7, we obtain the following theorem.

**Theorem 8.** Let the function \( f(z) \) defined by (1.1) be in the class \( S^\alpha_{p}(p,q,\alpha;A,B) \). Then

\[
|D^\mu_z \{ (J_{c,p} f)(z) \} | \geq \left\{ \begin{array}{c} \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} \\ (c+p)\Gamma(j+p+1) (B-A) (p-q-\alpha) \delta(p,q) \\
(c+j+p)\Gamma(j+p+1-\mu) [(1+B)j+(B-A)(p-q-\alpha)] \left( \frac{i+p-q}{p-q} \right)^n \delta(j+p,q) \end{array} \right\}
\]
\[ |z|^j \left| z^{p-\mu} (\mu > 0; c > -p) \right| \geq \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} & \\ + \frac{(c+p)\Gamma(j+p+1)(B-A)(p-q-\alpha)\delta(p,q)}{(c+j+p)\Gamma(j+p+1-\mu)[(1+B)j+(B-A)(p-q-\alpha)]} \left(\frac{j+p-q}{p-q}\right)^n \delta(j+p,q) \end{cases} \]

(6.17)

Each of the assertions (6.16) and (6.17) is sharp.

**Remark.**

(i) Putting \( A = -1 \) and \( B = 1 \) in all the above results, we obtain the corresponding results obtained by Aouf [4];

(ii) Putting (i) \( n = 0 \), \( A = -1 \) and \( B = 1 \) and (ii) \( n = 1 \), \( A = -1 \) and \( B = 1 \) in all the above results, we obtain the corresponding results obtained by Chen et al. [8];

(iii) Putting (i) \( n = 0 \), \( A = -\beta \) and \( B = \beta \) and (ii) \( n = 1 \), \( A = -\beta \) and \( B = \beta \) \((0 < \beta \leq 1)\) in all the above results, we obtain the corresponding results obtained by Aouf [3];

(iv) Putting (i) \( n = 0 \), \( q = 0 \), \( A = -\beta \) and \( B = \beta \) and (ii) \( n = 1 \), \( q = 0 \), \( A = -\beta \) and \( B = \beta \) \((0 < \beta \leq 1)\) in all the above results, we obtain the corresponding results obtained by Aouf [2];
(iv) Putting $A = -\beta$ and $B = \beta$ ($0 < \beta \leq 1$) in all the above results, we obtain the corresponding results for the class $S_{n}^{q}(p, q, \alpha, \beta)$ defined by (1.10).

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