Remarks on Some Relationships Between the Bernoulli and Euler Polynomials

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Abstract—In a recent paper which appeared in this journal, Cheon [1] rederived several known properties and relationships involving the classical Bernoulli and Euler polynomials. The object of the present sequel to Cheon’s work [1] is to show (among other things) that the main relationship (proven in [1]) can easily be put in a much more general setting. Some analogous relationships between the Bernoulli and Euler polynomials are also considered. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

The classical Bernoulli polynomials $B_n(x)$ and the classical Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi)$$

and

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi),$$

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respectively. The corresponding Bernoulli numbers $B_n$ and Euler numbers $E_n$ are given by

$$B_n := B_n (0) = (-1)^n B_n (1) = (2^{1-n} - 1)^{-1} B_n \left( \frac{1}{2} \right)$$  \tag{3}$$

\quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; \; \mathbb{N} := \{1, 2, 3, \ldots \})

and

$$E_n := 2^n E_n \left( \frac{1}{2} \right) \quad (n \in \mathbb{N}_0)$$  \tag{4}$$

respectively.

Numerous interesting (and useful) properties and relationships involving each of these polynomials and numbers can be found in many books and tables on this subject (see, for example, [2–6]). Recently, by making use of some fairly standard techniques based upon series rearrangement, Cheon [1] rederived each of the following results (cf. [1, p. 366, Theorem 1; p. 368, Theorem 3]):

$$B_n (x + 1) = \sum_{k=0}^{n} \binom{n}{k} B_k (x) \quad (n \in \mathbb{N}_0),$$  \tag{5}$$

$$E_n (x + 1) = \sum_{k=0}^{n} \binom{n}{k} E_k (x) \quad (n \in \mathbb{N}_0),$$  \tag{6}$$

and

$$B_n (x) = \sum_{\substack{k=0 \quad (k \neq 1)}}^{n} \binom{n}{k} B_k E_{n-k} (x) \quad (n \in \mathbb{N}_0).$$  \tag{7}$$

Both (5) and (6) are well-known (rather classical) results and are obvious special cases of the following familiar addition theorems:

$$B_n (x + y) = \sum_{k=0}^{n} \binom{n}{k} B_k (x) y^{n-k} \quad (n \in \mathbb{N}_0)$$  \tag{8}$$

and

$$E_n (x + y) = \sum_{k=0}^{n} \binom{n}{k} E_k (x) y^{n-k} \quad (n \in \mathbb{N}_0),$$  \tag{9}$$

when $y = 1$. Furthermore, Cheon’s main result (7) is essentially the same as the following known relationship (cf., e.g., [3, p. 806, Entry (23.1.29)], [4, p. 29], and [6, p. 66, equation 1.6 (63)]):

$$B_n (x) = 2^{-n} \sum_{k=0}^{n} \binom{n}{k} B_{n-k} E_k (2x) \quad (n \in \mathbb{N}_0)$$  \tag{10}$$

or, equivalently,

$$2^n B_n \left( \frac{x}{2} \right) = \sum_{k=0}^{n} \binom{n}{k} B_k E_{n-k} (x) \quad (n \in \mathbb{N}_0).$$  \tag{11}$$

In Section 2 of the present sequel to Cheon’s work [1], we propose to verify the equivalence of the relationships (7) and (11). And, in Section 3, we shall consider some interesting generalizations and analogues of the equivalent relationships (7) and (11).
2. EQUIVALENCE OF RELATIONSHIPS (7) AND (11)

For both Bernoulli and Euler polynomials, the following multiplication theorems are well known (cf., e.g., [2, p. 37, equation 1.13 (11); p. 41, equations 1.14 (8) and 1.14 (9)]):

\[
B_{n}(mx) = m^{n-1} \sum_{j=0}^{m-1} B_{n}\left(x + \frac{j}{m}\right) \quad (n \in \mathbb{N}; \ m \in \mathbb{N}) \tag{12}
\]

and

\[
E_{n}(mx) = \begin{cases} 
\frac{m^{n}}{n+1} \sum_{j=0}^{m-1} \frac{(-1)^{j}}{j!} B_{n+j} \left(x + \frac{j}{m}\right) & (n \in \mathbb{N}; \ m = 1, 3, 5, \ldots), \\
-\frac{2}{n+1} m^{n} \sum_{j=0}^{m-1} (-1)^{j} B_{n+1} \left(x + \frac{j}{m}\right) & (n \in \mathbb{N}; \ m = 2, 4, 6, \ldots),
\end{cases} \tag{13}
\]

which, together, would yield the following relationships between these two polynomials when \(m = 2\) (with, of course, \(n \mapsto n - 1\) and \(x \mapsto x/2\) [3, p. 806, Entry (23.1.27)]:

\[
E_{n-1}(x) = \frac{2^{n}}{n} \left[ B_{n} \left(\frac{x + 1}{2}\right) - B_{n} \left(\frac{x}{2}\right) \right] = \frac{2}{n} \left[ B_{n}(x) - 2^{n} B_{n} \left(\frac{x}{2}\right) \right] \quad (n \in \mathbb{N}). \tag{14}
\]

Since \(B_{1} = -1/2\), by separating the second \((k = 1)\) term of the sum in (11), we readily find from (11) that

\[
2^{n} B_{n} \left(\frac{x}{2}\right) = \sum_{k=0}^{n} \binom{n}{k} B_{k} E_{n-k}(x) - \frac{n}{2} E_{n-1}(x) \quad (n \in \mathbb{N}), \tag{15}
\]

which, in light of the second relationship in (14), immediately yields (7). And, by simply reversing these steps, we can easily deduce (11) from (7).

3. GENERALIZATIONS AND ANALOGUES OF THE EQUIVALENT RELATIONSHIPS (7) AND (11)

For a real or complex parameter \(\alpha\), the *generalized Bernoulli polynomials* \(B_{n}^{(\alpha)}(x)\) and the *generalized Euler polynomials* \(E_{n}^{(\alpha)}(x)\), each of degree \(n\) in \(x\) as well as in \(\alpha\), are defined by means of the following generating functions (see, for details, [6, Section 1.6], [7, p. 253 et seq.], and [8, Section 2.8]):

\[
\left( \frac{t}{e^{t} - 1}\right)^{\alpha} e^{x t} = \sum_{n=0}^{\infty} B_{n}^{(\alpha)}(x) \frac{t^{n}}{n!} \quad (|t| < 2\pi; \ 1^{\alpha} := 1) \tag{16}
\]

and

\[
\left( \frac{2}{e^{t} + 1}\right)^{\alpha} e^{x t} = \sum_{n=0}^{\infty} E_{n}^{(\alpha)}(x) \frac{t^{n}}{n!} \quad (|t| < \pi; \ 1^{\alpha} := 1), \tag{17}
\]

respectively. Clearly, we have

\[
B_{n}^{(1)}(x) = B_{n}(x) \quad \text{and} \quad E_{n}^{(1)}(x) = E_{n}(x) \quad (n \in \mathbb{N}). \tag{18}
\]
Moreover, it is easily observed from (16) and (17) that
\[ B_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)}(x) B_{n-k}^{(\beta)}(y) \quad (n \in \mathbb{N}_0) \] (19)
and
\[ E_n^{(\alpha+\beta)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(\alpha)}(x) E_{n-k}^{(\beta)}(y) \quad (n \in \mathbb{N}_0) \] (20)
respectively. In fact, several further addition theorems analogous to the well-known (rather classical) results (19) and (20) were considered, two decades ago, by Srivastava et al. [9] (see also [6, p. 62, equations 1.6 (26) and 1.6 (27); p. 66, equation 1.6 (68)]).

From the generating functions (16) and (17), it follows also that
\[ B_n^{(\alpha)}(x+1) - B_n^{(\alpha)}(x) = n B_{n-1}^{(\alpha-1)}(x) \quad (n \in \mathbb{N}_0) \] (21)
and
\[ E_n^{(\alpha)}(x+1) + E_n^{(\alpha)}(x) = 2 E_n^{(\alpha-1)}(x) \quad (n \in \mathbb{N}_0), \] (22)
respectively. Furthermore, since
\[ B_n^{(0)}(x) = E_n^{(0)}(x) = x^n \quad (n \in \mathbb{N}_0), \] (23)
upon setting \( \beta = 0 \) in the addition theorems (19) and (20) and interchanging \( x \) and \( y \), we obtain
\[ B_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)}(y) x^{n-k} \quad (n \in \mathbb{N}_0) \] (24)
and
\[ E_n^{(\alpha)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(\alpha)}(y) x^{n-k} \quad (n \in \mathbb{N}_0), \] (25)
respectively. Obviously, the familiar addition theorems (8) and (9) correspond to the special cases of (24) and (25), respectively, when \( \alpha = 1 \).

Next, by combining (21) and (24) (with \( x = 1 \) and \( y \rightarrow x \)), we find that
\[ B_n^{(\alpha-1)}(x) = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k^{(\alpha)}(x) \quad (n \in \mathbb{N}_0), \] (26)
which, in the special case when \( \alpha = 1 \), immediately yields the following familiar expansion (cf., e.g., [4, p. 26]):
\[ x^n = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k} B_k(x) \quad (n \in \mathbb{N}_0) \] (27)
in series of the classical Bernoulli polynomials \( \{B_n(x)\}_{n=0}^{\infty} \). In precisely the same manner, the addition theorem (25) in conjunction with (22) would lead us to
\[ E_n^{(\alpha-1)}(x) = \frac{1}{2} \left[ E_n^{(\alpha)}(x) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(\alpha)}(x) \right] \quad (n \in \mathbb{N}_0) \] (28)
and
\[ x^n = \frac{1}{2} \left[ E_n(x) + \sum_{k=0}^{n} \binom{n}{k} E_k(x) \right] \quad (n \in \mathbb{N}_0). \] (29)
In view of (23), this last familiar expansion (29) (cf., e.g., [4, p. 30]) in series of the classical Euler polynomials \( \{E_n(x)\}_{n=0}^{\infty} \) is indeed an immediate consequence of (28) when \( \alpha = 1 \).

Making use of some of the above known formulas and identities, we now prove an interesting generalization of the equivalent relationships (7) and (11), which is given by Theorem 1 below.
THEOREM 1. The following relationship:

\[ B_n^{(\alpha)}(x + y) = \sum_{k=0}^{n} \binom{n}{k} \left[ B_k^{(\alpha)}(y) + \frac{k}{2} B_k^{(\alpha-1)}(y) \right] E_{n-k}(x) \]  

(\alpha \in \mathbb{C}; \ n \in \mathbb{N}_0)  

holds true between the generalized Bernoulli polynomials and the classical Euler polynomials.

PROOF. First of all, upon suitably substituting from (29) into the right-hand side of (24), we get

\[ B_n^{(\alpha)}(x + y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)}(y) \left[ E_{n-k}(x) + \sum_{j=0}^{n-k} \binom{n-k}{j} E_j(x) \right] \]

\[ + \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)}(y) \sum_{j=0}^{n-k} \binom{n-k}{j} E_j(x), \]  

which, by inverting the order of summation, yields

\[ B_n^{(\alpha)}(x + y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)}(y) E_{n-k}(x) + \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} E_j(x) \sum_{k=0}^{n-j} \binom{n-j}{k} B_k^{(\alpha)}(y). \]  

The innermost sum in (32) can be evaluated by means of (24) itself with, of course,

\[ x = 1 \quad \text{and} \quad n \mapsto n - j \quad (0 \leq j \leq n; \ n, j \in \mathbb{N}_0). \]

We thus find from (32) that

\[ B_n^{(\alpha)}(x + y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} B_k^{(\alpha)}(y) E_{n-k}(x) + \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} E_j(x) B_{n-j}^{(\alpha)}(y + 1) E_j(x), \]  

or, equivalently, that

\[ B_n^{(\alpha)}(x + y) = \frac{1}{2} \sum_{k=0}^{n} \binom{n}{k} \left[ B_k^{(\alpha)}(y) + B_k^{(\alpha)}(y + 1) \right] E_{n-k}(x), \]  

which, in light of the recurrence relation (21), leads us at once to relationship (30) asserted by Theorem 1.

REMARK 1. In terms of the generalized Bernoulli numbers \( \{B_n^{(\alpha)}\}_{n=0}^{\infty} \), by setting \( y = 0 \) in Theorem 1, we obtain the following special case:

\[ B_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \left( B_k^{(\alpha)}(y) + \frac{k}{2} B_k^{(\alpha-1)}(y) \right) E_{n-k}(x) \quad (\alpha \in \mathbb{C}; \ n \in \mathbb{N}_0). \]  

Since, by definition (16),

\[ B_1^{(\alpha)} = -\frac{\alpha}{2} \quad \text{and} \quad B_n^{(0)} = \delta_{n,0} \quad (n \in \mathbb{N}_0), \]

a further special case of (35) when \( \alpha = 1 \) would yield the equivalent relationships (7) and (11), \( \delta_{m,n} \) being the Kronecker delta.

REMARK 2. Alternatively, in view of (23), assertion (30) of Theorem 1 gives us the following (presumably new) relationship between the classical Bernoulli and the classical Euler polynomials when \( \alpha = 1 \):

\[ B_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} \left[ B_k(y) + \frac{k}{2} y^{k-1} \right] E_{n-k}(x), \]  

which, by letting \( y \to 0 \), immediately yields the equivalent relationships (7) and (11) once again.

Finally, by appealing instead to (25) and (27), our demonstration of Theorem 1 can be applied \( \textit{mutatis mutandis} \) in order to derive an interesting analogue of Theorem 1, which is given by Theorem 2 below.
Theorem 2. The following relationship:

\[ E_n^{(\alpha)}(x + y) = \sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} \left[ E_{k+1}^{(\alpha-1)}(y) - E_{k+1}^{(\alpha)}(y) \right] B_{n-k}(x) \quad (\alpha \in \mathbb{C}, \ n \in \mathbb{N}_0) \]  

holds true between the generalized Euler polynomials and the classical Bernoulli polynomials.

Remark 3. In light of (23), a special case of assertion (38) of Theorem 2 when \( \alpha = 1 \) gives us the following relationship:

\[ E_n(x + y) = \sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} \left[ y^{k+1} - E_{k+1}(y) \right] B_{n-k}(x), \]  

which, upon setting \( y = 0 \), immediately yields

\[ E_n(x) = -\sum_{k=0}^{n} \frac{2}{k+1} \binom{n}{k} E_{k+1}(0) B_{n-k}(x) \quad (n \in \mathbb{N}_0). \]  

This last relationship (40) between the classical Euler and the classical Bernoulli polynomials is evidently analogous to the equivalent relationships (7) and (11). In fact, since [5, p. 29]

\[ E_n(0) = (-1)^n E_n(1) = \frac{2(1 - 2^{n+1})}{n+1} B_{n+1} \quad (n \in \mathbb{N}), \]  

relationship (40) can easily be rewritten in the following equivalent form:

\[ E_{n-2}(x) = 2 \binom{n}{2}^{-1} \sum_{k=0}^{n-2} \binom{n}{k} (2^{n-k} - 1) B_{n-k}B_k(x) \quad (n \in \mathbb{N} \setminus \{1\}), \]  

which incidentally is a known result recorded by (for example) Abramowitz and Stegun [3, p. 806, Entry (23.1.28)] (see also [4, p. 29]).

References