Distributed Joint Resource Allocation in Primary and Cognitive Wireless Networks

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Abstract—We develop a game theoretic framework for distributed resource allocation in the uplink of a cognitive radio network where secondary users (SUs) share the bandwidth with primary users (PUs). Each PU has a fixed data rate and applies power control to achieve its target SINR, while each SU jointly adjusts its data rate and transmit power by maximizing its utility. Since the SUs’ total interference on the primary network is kept below a given threshold, there exist couplings between SUs’ transmit power levels. Hence, the SUs’ game belongs to the generalized Nash equilibrium (GNE) problems. We separately analyze the proposed resource allocation algorithm for SUs and the one utilized by PUs, and derive the condition under which the PUs’ power control algorithm converges for SUs’ fixed transmit power levels. We also derive the sufficient condition under which the SUs’ joint data rate and power control algorithm converges for fixed transmit power levels of PUs. We then derive the sufficient condition for convergence of the algorithms when PUs and SUs simultaneously apply their resource allocation schemes. Simulations confirm our analysis and demonstrate that the proposed framework is energy efficient and provides PUs and SUs with their quality of service requirements.

Index Terms—Cognitive radio network, distributed joint resource allocation, game theory, generalized Nash equilibrium, pricing.

I. INTRODUCTION

Scarcity of available spectrum makes its efficient use very desirable. Cognitive radio [1] is an important concept for improving the efficiency of using the available bandwidth, in which, the primary network is licensed to use the bandwidth, and the secondary network may dynamically utilize the same bandwidth so far as its interference is acceptable to the primary network [2]–[7].

Resources in cognitive radio networks can be allocated via a centralized approach [8], [9], or by a distributed one [10]. In the centralized approach, resources are allocated to secondary users (SUs) by their base station via solving the resource allocation problem. However, many resource allocation problems in communications networks are NP-hard [11], meaning that when the number of users is increased, solving such problems via a centralized approach becomes exponentially prohibitive, and the scalability issue would arise.

To overcome this, distributed approaches have been proposed and considered.

Distributed approaches for resource allocation include formulation of the problem as a non-cooperative game [12], and game theory has been used for efficient allocation of resources to each user in wireless CDMA networks [13]–[17]. In such games, each user iteratively chooses a strategy vector from its strategy space to maximize its utility. The outcome of the game, when it exists, is called the Nash equilibrium (NE), at which no user can increase its utility by unilaterally changing its strategy. In cognitive radio networks, SUs’ transmit power levels are constrained in such a way that their interference on primary users (PUs) stay below a tolerable threshold.

The problem of allocating transmit power to SUs in cognitive radio networks has already been considered in the literature [18]–[20]. In [18], a transmit power control game for SUs is proposed, and their algorithm converges to a point near the NE of the game. In [19], a power control game is proposed, in which PUs and SUs are players of the same game, and the SUs’ interference on PUs is constrained. The authors in [20] formulate the problem of power control in the downlink of a cognitive radio network by a multi-leader multi-follower game, where base stations are leaders and SUs are followers. However, such schemes as in [18]–[20], which only consider a single resource, may be inefficient and in some cases would result in no solution. In contrast, joint allocation of resources can lead to higher efficiencies, since it has more degrees of freedom as compared to allocating a single-resource.

Distributed joint allocation of transmit power and data rate in cognitive radio networks is considered in [4], [21], where in [21], an energy efficient approach using the utility function in [13] is proposed. It was shown that for SUs fixed data rates, there exists a NE in the power allocation game; and for SUs fixed transmit power levels, there exists a NE in the data rate allocation game. Similarly, in [4], a logarithmic utility function is used, and a joint data rate and power allocation scheme is proposed. In neither of the above references, no analysis for existence of a NE in the joint rate and power control game is provided. Indeed, when the utility is not jointly quasi-concave on transmit power and data rate strategy spaces, the joint allocation game may admit no NE and the distributed algorithm may not converge.

In this paper, we propose a framework for distributed resource allocation in the uplink of a cognitive radio network where SUs simultaneously share the bandwidth with PUs (the underlay paradigm). In the primary network, each PU has a fixed data rate and applies power control to achieve its target SINR, while each SU jointly adjusts its data rate and
its transmit power by maximizing its utility over its strategy space. The SUs’ utility function is the same as in [17], which takes into account channel conditions in aiming for higher data rates and acceptable SINRs. In our proposed scheme, the transmit power of each SU is allocated in such a way that the SUs’ total interference on PUs is below a given threshold. This introduces couplings between transmit power levels of SUs, meaning that in addition to the impact of each SU on utilities of other SUs, their strategies are inter-dependent as well. Hence, the game belongs to the generalized Nash equilibrium (GNE) problems, which in general is hard to analyze.

We separately analyze the proposed resource allocation algorithm for SUs and the one utilized by PUs, and derive the condition for convergence of the PUs’ power control algorithm when SUs’ transmit power levels are fixed. We also derive the sufficient condition for convergence of the SUs’ joint data rate and power control algorithm when PUs’ transmit power levels are fixed. We show that the SUs’ game is equivalent to a variational inequality problem, and further show that this problem always has one solution. Moreover, we obtain the sufficient condition for GNE’s uniqueness. In addition, we obtain a simple and practical sufficient condition for GNE’s uniqueness in terms of pricing in the utility function of each user. We also develop a distributed algorithm that converges to this unique GNE, and show that under this sufficient condition and when the convergence point of the distributed algorithm lies within the strategy space of SUs, at the GNE of the game, each SU will achieve a predefined SINR. This means that one can define different QoS levels for different SUs based on their required SINRs, and at the GNE of the game, SUs will achieve their predefined QoS levels.

Although SUs and PUs apply different algorithms, they affect each other through the interference they produce. This makes the analysis of the framework difficult, and in general, although the respective resource allocation schemes of PUs and SUs may converge when applied in solo, when both algorithms are simultaneously applied, they may not converge. We derive the sufficient condition for convergence of the algorithms when PUs and SUs simultaneously apply their respective resource allocation algorithms.

This paper is organized as follows. The system model is introduced in Section II, and the primary network resource allocation scheme is introduced and analyzed in Section III. The joint resource allocation scheme for the secondary network is presented and analyzed in Section IV. Convergence of the proposed resource allocation framework is in Section V, followed by practical considerations in Section VI. Simulation results are presented in Section VII, and conclusions are in Section VIII.

II. System Model

The direct sequence CDMA-based (DS-CDMA) primary network has a single cell with a set of PUs \( \mathcal{N} = \{1, \ldots, N\} \), and the DS-CDMA-based secondary network has a single cell with a set of SUs \( \mathcal{M} = \{1, \ldots, M\} \). Both networks use conventional matched filters in their receivers, and utilize pseudo-random signature sequences. PUs and SUs are randomly spread in the coverage area of their respective networks.

The channel gain from SU \( j \) to its base station is \( h_{j}^{S} \), from SU \( j \) to primary base station is \( h_{j}^{SP} \), from PU \( i \) to its base station is \( h_{i}^{P} \), and from PU \( i \) to secondary base station is \( h_{i}^{PS} \). SUs share the bandwidth \( W \) with PUs.

In this network, PU \( i \) transmits at a given rate \( R_{i}^{P} \) with power \( p_{i}^{P} \), and SU \( j \) transmits at a variable rate \( \gamma_{j}^{S} \) with power \( p_{j}^{S} \). The SINR (bit energy to interference density ratio) for PU \( i \) at its base station, as in [13], [22], [23], is

\[
\gamma_{i}^{P} = \frac{W h_{i}^{P} p_{i}^{P}}{\sum_{n \neq i, n \in \mathcal{N}} h_{n}^{P} p_{n}^{P} + \sum_{m \in \mathcal{M}} h_{m}^{SP} p_{m}^{S} + N_{0}^{P}},
\]

where \( N_{0}^{P} \) denotes noise power at the PUs’ base station, and \( \frac{W}{\gamma_{i}^{P}} \) is the processing gain of PU \( i \). Similarly, the SINR (bit energy to interference density ratio) for SU \( j \) at its base station is

\[
\gamma_{j}^{S} = \frac{W h_{j}^{S} p_{j}^{S}}{r_{j}^{S} \sum_{m \neq j, m \in \mathcal{M}} h_{m}^{SP} p_{m}^{S} + \sum_{n \in \mathcal{N}} h_{n}^{PS} p_{n}^{P} + N_{0}^{S}},
\]

where \( N_{0}^{S} \) denotes noise power at the SUs’ base station.

We define the normalized interference plus noise for PU \( i \) as the ratio of interference for this user at its base station (\( \sum_{n \neq i, n \in \mathcal{N}} h_{n}^{P} p_{n}^{P} + \sum_{m \in \mathcal{M}} h_{m}^{SP} p_{m}^{S} + N_{0}^{P} \)) to its direct channel gain (\( h_{i}^{P} \)) as

\[
I_{i}^{P} = \sum_{n \neq i, n \in \mathcal{N}} \frac{h_{n}^{P} p_{n}^{P}}{h_{i}^{P}} + \sum_{m \in \mathcal{M}} \frac{h_{m}^{SP} p_{m}^{S}}{h_{i}^{P}} + \frac{N_{0}^{P}}{h_{i}^{P}},
\]

and similarly, the normalized interference plus noise for SU \( j \) at its base station is

\[
I_{j}^{S} = \sum_{m \neq j, m \in \mathcal{M}} \frac{h_{m}^{SP} p_{m}^{S}}{h_{j}^{S}} + \sum_{n \in \mathcal{N}} \frac{h_{n}^{PS} p_{n}^{P}}{h_{j}^{S}} + \frac{N_{0}^{S}}{h_{j}^{S}}.
\]

Note that the normalized interference plus noise for SU \( j \) is affected by its direct channel gain and by the interference from other users; and hence it can be regarded as a measure for its channel condition. The normalized interference plus noise \( I_{j}^{S} \) is a function of PUs’ and SUs’ transmit power levels, i.e., it is \( I_{j}^{S}(p_{i}^{P}, p_{j}^{S}) \) where \( p_{i}^{P} = [p_{1}^{P}, \ldots, p_{N}^{P}] \) and \( p_{j}^{S} = [p_{1}^{S}, \ldots, p_{M}^{S}] \), but for simplicity in notations, we do not write its arguments. As such, the SINR for SU \( j \) is

\[
\gamma_{j}^{S} = \frac{W p_{j}^{S}}{r_{j}^{S} I_{j}^{S}}.
\]

For a given modulation type, the data rate of PU \( i \) is fixed at \( R_{i}^{P} \) and its target SINR at \( \gamma_{i}^{P} \). This means that in the primary network, only power control is applied with a view to meeting the target SINR of each PU, i.e., \( \gamma_{i}^{P} \geq \gamma_{i}^{P} \), while minimizing the total transmit power of all PUs. We may restrict the transmit power of PU \( i \) to \( p_{i}^{P} \in P_{i}^{P} = \{p_{i}^{P} : 0 \leq p_{i}^{P} \leq \hat{p}_{i}^{P}\} \), where \( \hat{p}_{i}^{P} \) is
the primary network aims to solve the following problem
\[
\min_{p^P} \sum_{n \in \mathcal{N}} U^P_n \tag{6}
\]
Subject to: \( \gamma^P_n \geq \delta^P_n \ \forall n \in \mathcal{N}, \)
where \( P^P = P^P_1 \times P^P_2 \times \cdots \times P^P_N. \)

We now analyze the primary network’s resource allocation problem (6), and note that SUs’ transmissions appear as variable interference to each PU. Here, we assume that SUs’ transmit power levels are fixed but arbitrary; and later in Section V consider variable transmit power levels for SUs.

To proceed, we first define the followings. The normalized channel gain matrix \( H^P \) is defined by
\[
[H^P]_{i,n} = \begin{cases} 
0 & \text{if } i = n, \\
\frac{h^P_{i,n}}{h^N_{i}} & \text{if } i \neq n.
\end{cases}
\]
The diagonal matrix \( D^P \) is defined by
\[
[D^P]_{i,n} = \begin{cases} 
\frac{h^P_{i,n}}{h^N_{i}} p^P_n & \text{if } i = n, \\
0 & \text{if } i \neq n.
\end{cases}
\]
The vector \( v^P \) is defined by
\[
[v^P]_i = \sum_{m \in M} \frac{h^S_m p^S_m}{h^N_i} + \frac{N_0}{h^N_i},
\]
with the above definitions, we have the following lemma.

**Lemma 1.** Assume the matrix \( H^P \) is irreducible and the transmit power of each SU is arbitrarily fixed. The SINR vector of PUs is feasible iff \( \rho(D^P H^P) < 1 \), where \( \rho(\cdot) \) is the spectral radius function. In addition, the distributed power update function
\[
(p^P(k+1) = T^P(p^P(k), p^S(k)),
\]
where the update function for PU \( i \) is
\[
T^P_i(p^P, p^S) = \frac{h^P_i}{\mu} \gamma^P_i p^P_i (p^P + p^S),
\]
converges to the unique solution of the power control problem (6) given by
\[
p^P = (I - D^P H^P)^{-1} D^P v^P.
\]
The primary network applies the distributed power control algorithm in (10), which is known as the distributed target SINR tracking algorithm [24]. In general, convergence of this algorithm in the presence of SUs depends on actions of SUs, but as stated in Lemma 1, for any arbitrary fixed transmit power of each SU, this algorithm converges. The impact of SUs are discussed in Section V.

**IV. RESOURCE ALLOCATION IN SECONDARY NETWORK**

**A. Problem Statement**

In the secondary network, both the data rate and the transmit power of each SU are variable, i.e., a joint data rate and power control scheme is utilized by the secondary network. Each SU has a utility function that is a function of both its data rate and transmit power, and aims to maximize its utility over its data rate and power strategy spaces. As in [17], we choose the utility of SU \( j \) as
\[
U_j = \log(\alpha_2 I_j^S r_j^S + \alpha_1 p_j^S) - \frac{\lambda}{2} \frac{\alpha_2}{\alpha_1} (r_j^S)^2 + \frac{\alpha_1}{\alpha_2} r_j^S (p_j^S)^2,
\]
which has interesting and desirable properties as described below. In this utility, \( \lambda \) is the pricing factor, \( I_j^S \) is the normalized interference plus noise for SU \( j \), and \( \alpha_1 \) and \( \alpha_2 \) are constant parameters. The role of \( \alpha_1 \) and \( \alpha_2 \) is later discussed in Theorem 4. The utility function in (13) can be written as
\[
U_j = \log(\alpha_2 I_j^S r_j^S) + \log(1 + \frac{\alpha_2 p_j^S}{\alpha_1 I_j^S r_j^S}) - \frac{\lambda}{2} \frac{\alpha_2}{\alpha_1} (r_j^S)^2 + \frac{\alpha_1}{\alpha_2} r_j^S (p_j^S)^2.
\]
Note that the first term is a function of the SU’s data rate, and the second term is a function of the SU’s SINR (as defined in (2) and (5)) for a given data rate. This means that via this utility, each user aims to achieve a higher data rate and a higher SINR. The third term is the pricing to control the SU’s selfish behavior. We assume that the transmit power levels and data rates of SUs are bounded, i.e., \( p_j^S \leq p_j^S \), \( r_j^S \leq r_j^S \). Federal Communications Commission (FCC) has terminated its proceeding on interference temperature model as the sole criteria for permitting SUs to utilize the spectrum licensed to the primary network in the underlay paradigm; and has favored the argument that “interference from both transmitters and noise sources, that is present at a receiver at any instance of time” should be considered instead [25]. This means that for the time being, cognitive radio networks can only utilize the licensed spectrum if the license holder is agreeable to leasing the spectrum to the cognitive radio network, and by taking into account the actual noise power, provides SUs with values of the acceptable aggregate interference, denoted by \( Q \).

In our model, cooperation between the primary and the cognitive secondary network is assumed. In such cases, the SUs’ total interference on the primary network should be below the given threshold \( Q \). This can be achieved by upper bounding the transmit power of each SU so that
\[
\sum_{m \in M} h^S_m P_m^S \leq Q.
\]
The joint rate and power control game for SUs is denoted by \( G = (M, \{ (p^S_j, p^S_{-j}), (r^S_j, r^S_{-j}) \}, \{ u_j \}), \) where \( M = \{ 1, \cdots, M \} \) is the set of SUs, \( R^S_j \) is the data rate strategy set, \( P^S_j(p^S_{-j}) = \{ p^S_j : p^S_j \in P^S_j, \sum_{m \in M} h^S_m P_m^S \leq Q \} \) is the power strategy set, and \( u_j \) is the utility function for SU \( j \). In this game, each SU \( j \) tries to solve the following problem
\[
\max_{(p^S_j, p^S_{-j}), (r^S_j, r^S_{-j})} u_j(p^S_j, p^S_{-j}, r^S_j, r^S_{-j}), \quad \forall j \in M.
\]
Since each SU’s strategy in (15) depends on the chosen strategies of other SUs, the game belongs to the generalized...
Nash equilibrium (GNE) problems\(^1\). In order to analyze the GNEP, we start by writing the partial Lagrangian function of the problem (15) as \( L_j = u_j - \omega_j \left( \sum_{m \in \mathcal{M}} h_{m}^{s_{j}^{p_{m}}} - Q \right) \), where \( \omega_j \) is the Lagrange multiplier corresponding to the power constraint (14). Now we write the partial Karush-Kuhn-Tucker (KKT) conditions [26] of the problem (15) for each SU as

\[
\frac{\partial L_j}{\partial p_{j}} = \frac{\partial u_j}{\partial p_{j}} - \omega_j h_{j}^{s_{j}^{p_{j}}} = 0, \quad \forall j \in \mathcal{M},
\]

\[
0 \leq \omega_j \perp \sum_{m \in \mathcal{M}} h_{m}^{s_{j}^{p_{m}}} \leq Q, \quad \forall j \in \mathcal{M},
\]

\[
p_{j}^{*} \in \mathcal{P}_{j}^{s_{j}}, \quad \forall j \in \mathcal{M},
\]

where in (17), the complementarity condition \( 0 \leq a \perp b \geq 0 \) means \( a \geq 0, b \geq 0, \) and \( a \times b = 0. \)

The solution to (15) for which we have \( \omega_1 = \omega_2 = \ldots = \omega_{\mathcal{M}} = \lambda \), if it exists, is called the variational solution [27]. Using the KKT conditions (16), (17), and (18), and setting \( \omega_j = \lambda \) for all SUs in (16) and (17), it is easy to see that variational solutions are solutions to the following new game denoted by \( \mathcal{G} \), in which each SU \( j \) tries to solve the following problem

\[
\max_{(p_{j}^{s_{j}}, r_{j}^{s_{j}})} u_j(p_{j}^{s_{j}}, p_{j}^{s_{j}}, r_{j}^{s_{j}}), \quad \forall j \in \mathcal{M},
\]

where

\[
u_j = \log(\alpha_2 I_{j}^{s_{j}} + \alpha_1 p_j^{s_{j}}) - \frac{\lambda}{2} \left( \frac{\alpha_1}{\alpha_2} I_{j}^{s_{j}} r_{j}^{s_{j}} \right)^2 + \frac{\alpha_1}{\alpha_2} \frac{1}{I_{j}^{s_{j}}} (p_j^{s_{j}})^2 - \lambda \perp h_{j}^{s_{j}^{p_{j}^{s_{j}}}},
\]

and \( \lambda \) is chosen such that the following complementarity condition holds

\[
0 \leq \lambda \perp \sum_{m \in \mathcal{M}} h_{m}^{s_{j}^{p_{m}}} \leq Q.
\]

From now on, we only consider the game \( \mathcal{G} \), which is easier than (15) to deal with, meaning that we assume resources are allocated to SUs via solving the problem (19) while satisfying (21).

### B. Problem Analysis

Similar to the primary network, here we assume that PUs’ transmit power levels are arbitrarily fixed. Later, in Section V, we will consider variable PUs’ transmit power levels. To analyze the game \( \mathcal{G} \), first we show that it is equivalent to a variational inequality problem, and then analyze the latter. Define \( \mathcal{X} \subset \mathbb{R}^{2M} \) such that

\[
\mathcal{X} = \{ [p_{1}^{s_{1}}, \ldots, p_{M}^{s_{M}}]^{T} : r_{j}^{s_{j}} \in \mathbb{R}^{s_{j}}, \quad p_{j}^{s_{j}} \in \mathcal{P}_{j}^{s_{j}}, \quad \sum_{m \in \mathcal{M}} h_{m}^{s_{j}^{p_{m}}} \leq Q \}.
\]

The function \( F_j(p_{j}^{s_{j}}, r_{j}^{s_{j}}) \) is defined as

\[
F_j(p_{j}^{s_{j}}, r_{j}^{s_{j}}) = \left[ \frac{F_{j}}{F_{j}} \right] = \left[ \begin{array}{c} \frac{\partial u_j}{\partial p_{j}} \\ \frac{\partial u_j}{\partial r_{j}} \end{array} \right].
\]

Now we have the following theorem.

**Theorem 1.** The game \( \mathcal{G} \) (see (19) and (21)) is equivalent to the variational inequality problem\(^2\) \( VI(\mathcal{X}, F), \) where the set \( \mathcal{X} \) is defined in (22), \( F = [F_{1}^{s_{1}}, \ldots, F_{M}^{s_{M}}]^{T}, \) and \( F_{j} \) is defined in (23).

**Proof:** From the KKT conditions of the game in (19) and of \( VI(\mathcal{X}, F), \) the proof follows from the fact that the respective two sets of KKT conditions are equivalent.

**Theorem 1** states that instead of the game (19), we can analyze the equivalent variational inequality problem. Therefore, instead of GNE’s uniqueness, we pursue the conditions under which the solution to the corresponding variational inequality problem is unique. The following theorem deals with the existence and uniqueness of GNE of the game (19).

**Theorem 2.** The variational inequality \( VI(\mathcal{X}, F) \) admits at least one solution. Moreover, the solution to \( VI(\mathcal{X}, F) \) is unique when for each SU \( j, \) conditions (24) and (25) at the top of next page are satisfied, where \( I_{j}^{p_{\min}} = \frac{S_{j}^{p_{\min}}}{I_{j}^{p_{\max}}}, \) \( I_{j}^{p_{\max}} = I_{j}^{s_{j}}, \) \( p_{j}^{0} \) is a vector consisting of the maximum transmit power levels of PUs, and \( p_{j}^{s_{j}} \) is a vector consisting of the maximum transmit power levels of SUs.

**Proof:** See Appendix A.

Although when the conditions of Theorem 2 are satisfied, the GNE is guaranteed to be unique, they are very restrictive and in many situations are impractical. In fact, the above conditions are satisfied when the strategy space of each user is tight and the interference of SUs on each other is very low. This situation can happen when SUs are far apart and interference gains are very low, as in an ad-hoc network in which transmitters and receivers are far apart. In other cases, this is very restrictive and may not be practical. In what follows, we obtain another set of sufficient conditions for GNE’s uniqueness, which is easier to satisfy in practice.

### C. The Effect of Pricing

Note that the condition in Theorem 2 is a sufficient condition, meaning that when it is satisfied, Theorem 2 can be used. This also means that there may exist some other conditions under which Theorem 2 can be used. Such conditions may be more practical and easy to satisfy. Besides, these conditions may give the utility parameters a key role, such as defining a target SINR. Here, we obtain such conditions, and show that they can be very useful in satisfying users’ certain requirements.

\(^1\)The generalized Nash equilibrium problem (GNEP), or the generalized game is denoted by \( G^{\text{GNEP}}(\mathcal{M}, S_{j}, \{u_{j}\}) \) where \( \mathcal{M} \) is the set of players, \( S_{j}(s_{j}) \) is the strategy set of player \( j \) that depends on strategies of other players, i.e., on \( s_{-j} \), and \( u_{j} \) is the utility function of player \( j \). The outcome of a generalized game, if exists, is called the generalized Nash equilibrium (GNE). A non-cooperative game is denoted by \( G^{\text{GNE}}(\mathcal{M}, S_{j}, \{u_{j}\}) \), and its outcome, if exists, is called the Nash equilibrium (NE). The difference between these two games is that in the latter, the set of strategy space of each player \( j \), i.e., \( S_{j} \), does not depend on other players’ chosen strategies, i.e., \( s_{-j} \). In general, GNEPs are difficult to analyze.

\(^2\)For a given set \( \mathcal{X} \subset \mathbb{R}^{n} \) and mapping \( F : \mathcal{X} \to \mathbb{R}^{n}, \) the variational inequality problem, denoted by \( VI(\mathcal{X}, F), \) is to find \( x^{*} \in \mathcal{X} \) such that \( (x-x^{*})^{T} F(x^{*}) \geq 0 \) for all \( x \in \mathcal{X} \) (Definition 1.1.1 in [29]).
where the utility function \( \hat{G} \) as in Theorem 3 below, this constraint is readily satisfied, in which case, there is no need to consider it. The analysis of the game \( \hat{G} \) is much simpler than that of \( G \) and is considered in [17], where it is shown that if the optimal solution to (26) lies within the strategy space of SUs, the transmit power \( p^S_j \) and the data rate \( r^S_j \) of SU \( j \) can be obtained from

\[
\begin{align*}
\left\{ \begin{array}{ll}
p^S_j = \sqrt{\frac{\frac{2}{2\alpha_1} I^S_{j\max} + \frac{1}{2\alpha_2} \lambda I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2}}{I^S_{j\max} + \frac{1}{\alpha_2} I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2} \\
r^S_j = \sqrt{\frac{\frac{2}{2\alpha_1} I^S_{j\max} + \frac{1}{2\alpha_2} \lambda I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2}}{I^S_{j\max} + \frac{1}{\alpha_2} I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2}
\end{array} \right.
\end{align*}
\]

To reach the NE of the game, the following iterative algorithm is proposed

\[
(p^S_j(k+1), r^S_j(k+1)) = (T^S_{p^S_j}(p^S_j(k), r^S_j(k)), T^S_{r^S_j}(p^S_j(k), r^S_j(k))),
\]

where the update functions for transmit power levels and data rates are

\[
\begin{align*}
\left\{ \begin{array}{ll}
T^S_{p^S_j}(p^S_j, r^S_j) = \sqrt{\frac{\frac{2}{2\alpha_1} I^S_{j\max} + \frac{1}{2\alpha_2} \lambda I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2}}{I^S_{j\max} + \frac{1}{\alpha_2} I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2} \\
T^S_{r^S_j}(p^S_j, r^S_j) = \sqrt{\frac{\frac{2}{2\alpha_1} I^S_{j\max} + \frac{1}{2\alpha_2} \lambda I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2}}{I^S_{j\max} + \frac{1}{\alpha_2} I^S_{j\min} (2\alpha_2 I^S_{j\max} + 2\alpha_1 I^S_{j\min})^2}
\end{array} \right.
\end{align*}
\]

Moreover, it is shown in [17] that the game \( \hat{G} \) always admits a unique NE, at which, the distributed algorithm (28) converges. The following theorem provides the condition under which, the game \( G \) can be replaced by \( \hat{G} \).

**Theorem 3.** Let the pricing factor \( \lambda \) be such that

\[
\left( \frac{\sum_{m \in \mathcal{M}} h_{m,R_m}^S}{\sum_{m \in \mathcal{M}} h_{m,R_m}^S} \right)^2 \leq \frac{\alpha_1}{\alpha_2} \lambda.
\]

In this case, the game \( G \) reduces to the game \( \hat{G} \).

**Proof:** See Appendix B.

Theorem 3 states that for a set of channel realizations, one can choose the pricing factor \( \lambda \) according to (30), which guarantees that the interference constraint (14) is strictly satisfied. Hence, instead of the game \( G \), one can use the game \( \hat{G} \). Note that the choice of \( \lambda \) according to (30) depends on the channel realization, meaning that the value of \( \lambda \) must change when channel gains change. Now, each SU \( j \) updates its transmit power level and data rate at each iteration, say iteration \( k + 1 \), via the following iterative algorithm

\[
(p^S_j(k+1), r^S_j(k+1)) = \arg \max_{(p^S_j, r^S_j)} u_j(p^S_j, r^S_j),
\]

From Theorem 3 and the results in [17], the following theorem can be proved.

**Theorem 4.** Let the pricing factor \( \lambda \) be as in Theorem 3. The game admits a unique GNE, at which the distributed algorithm (31) converges. Besides, when the convergence point of the distributed algorithm is within the strategy space of users, the transmit power and the data rate of SUs are such that for each SU \( j \) we have

\[
\frac{1}{r^S_j} \frac{p^S_j}{I^S_j} = \frac{\alpha_2}{\alpha_1}.
\]

This means that the SINR of SU \( j \) at the GNE is

\[
\gamma^S_j = \frac{W p^S_j}{r^S_j I^S_j} = \frac{\alpha_2}{\alpha_1} W.
\]

**Proof:** See Appendix C.
V. CONVERGENCE ANALYSIS OF THE PROPOSED FRAMEWORK

In Lemma 1 and Theorem 4, we showed the convergence of PUs’ power control algorithm for arbitrary fixed transmit power levels of SUs, and the convergence of SUs’ joint rate and power control algorithm for arbitrary fixed transmit power levels of PUs. However, for variable transmit power, the PUs’ algorithm and/or the SUs’ algorithm may not converge at all. To analyze such cases, we first consider the case where the PUs’ transmit power levels as well as the SUs’ transmit power levels and data rates are not constrained. This assumption, in addition, means that we are ignoring the SUs’ interference constraint on the primary network. In practice, this may not be valid, as in general, processing gains are limited, meaning that the corresponding data rate is upper bounded. Very small data rates are not of interest, meaning that data rates should be lower bounded as well. However, for the sake of mathematical presentation, we consider this case. Generally, since the PUs and the SUs power update functions are standard interference functions (see [28] for the primary network, and Theorem 4 for the secondary network), we have the following theorem on convergence of PUs’ and SUs’ resource allocation algorithms.

**Theorem 5.** Consider the concatenated power update function

\[
(p^P(k + 1), p^S(k + 1)) = T^C(p^P(k), p^S(k), r^S(k))
\]

(34)

where \(T^C(p^P, p^S, r^S) = (T^{pp}(p^P, p^S, r^S), T^{sp}(p^P, p^S, r^S))\), \(T^{pp}\) is defined in (11), and \(T^{sp}\) is defined in (29). When \(T^C\) is feasible, i.e., when it has a fixed point, the fixed point is unique and \(T^C\) converges to this unique fixed point. This means that when both PUs’ and SUs’ resource allocation algorithms are applied in parallel, they converge to their respective unique points.

**Proof:** The proof comes from the fact that \(T^{pp}\) and \(T^{sp}\) are standard functions, and from Theorems 1 and 2 in [28].

The function \(T^C\) may not be feasible. In such cases, when both PUs and SUs simultaneously apply their respective resource allocation schemes, they may not converge to any point, as there may not exist an equilibrium point. However, in what follows, we show that only feasibility of PUs is sufficient for convergence of resource allocation schemes of both PUs and SUs.

**Theorem 6.** When the primary network is feasible, i.e., when the condition of Lemma 1 is satisfied, the concatenated power update function \(T^C\) defined in (34) has a fixed point.

**Proof:** From Lemma 1, we know that for arbitrary fixed transmit power levels of SUs, the PUs’ power update function has a fixed point given by (12). Note that each PU’s transmit power at the fixed point is a function of SUs’ transmit power levels, i.e., \(p^P(p^S) = (I - D^P H^P)^{-1} D^P v^P(p^S)\). Therefore, the transmit power of each PU is a linear function of SUs’ transmit power levels. One can substitute each PU’s transmit power into the SUs’ power update function, i.e., into \(T^{sp}\), and follow the same steps as in Theorem 4 in [17] to show that the SUs’ power update function has a fixed point.

From Theorem 6, the feasibility of primary network resource allocation problem is sufficient for the convergence of both the primary and the secondary networks’ resource allocation algorithms. Now we impose a lower bound on each SU \(j\) data rate, i.e., \(\inf \{R^S_{j}\} > 0\), do not impose an upper bound on SUs’ data rate, and keep PUs’ and SUs’ transmit power levels unbounded. Theorem 7 below is on convergence of PUs’ and SUs’ resource allocation schemes under such conditions.

**Theorem 7.** When each SU \(j\) data rate strategy space \(R^S_{j}\) is lower bounded to a positive value, i.e., \(\inf \{R^S_{j}\} > 0\), the joint data rate and transmit power algorithm of SUs always converges to a unique point. Besides, when the condition in Lemma 1 is satisfied, the power control algorithm of PUs also converges to a unique point.

**Proof:** See Appendix D.

Theorem 7 states that when the data rate strategy space of each SU is lower bounded to a positive value, regardless of the feasibility of PUs resource allocation algorithm, the SUs’ resource allocation problem always converges. However, upper bounding the SUs’ data rate strategy space has no effect on convergence of the proposed framework, and Theorem 5 is applicable in this case. Theorem 8 below is on convergence of PUs’ and SUs’ resource allocation schemes for other cases.

**Theorem 8.** a) When each SU \(i\) power strategy space \(P^P_{i}\) is upper bounded, the SUs’ joint data rate and transmit power allocation algorithm always converges to a unique point. Besides, when the condition in Lemma 1 is satisfied, the PUs’ power control algorithm also converges to a unique point.

b) When each PU \(i\) power strategy space \(P^P_{i}\) is upper bounded, the PUs’ and the SUs’ resource allocation algorithms converge to their respective unique points.

Assuming that users are randomly spread in the coverage area of the network, in the resource allocation problem for primary users, the spectral radius constraint in Lemma 1 is more likely violated when the number of PUs is increased. When this violation happens, PUs increase their transmit power levels to infinity if the transmit power strategy space is unbounded, or transmit at their maximum power level if the transmit power strategy space is upper bounded.

On the other hand, when the PUs’ transmit power levels are fixed and the SUs’ transmit power strategy is unbounded, if the number of SUs is increased, they will increase their transmit power levels and reduce their data rates to reach their target SINR at GNE. However, when the number of SUs is increased and their strategy space is bounded, they will increase their transmit power levels until the upper bound is reached, and reduce their data rate until the lower bound is attained, but will not achieve their target SINRs.

VI. PRACTICAL CONSIDERATIONS

In the proposed scheme, the primary network and the secondary network simultaneously apply their respective resource allocation algorithms. The proposed scheme rely on the channel state information (CSI) of the network. This means that CSIs are assumed to be available. We also assume that during the time that each algorithm converges, CSIs do not change. When CSIs change due to changes in the network, some parameters in the proposed scheme should be changed accordingly as described below.

There are four parameters in the proposed scheme, namely \(\alpha_1, \alpha_2, \lambda,\) and \(Q\); each with a different impact on transmit
power levels and data rates of users. In Theorem 4, the impacts of $\alpha_1$ and $\alpha_2$ are shown. As stated in (33), at the NE of the game $\mathcal{G}'$, depending on the values of $\alpha_1$ and $\alpha_2$, each user will achieve a specific SINR. This means that by setting the values of $\alpha_1$ and $\alpha_2$, the target SINR for each user is set. In general, the values of $\alpha_1$ and $\alpha_2$ can be set by the base stations and/or by the service provider based on the corresponding service and the user’s required quality of service, meaning that setting these values are not part of solving the optimization problem. As such, depending on channel conditions, each user chooses its transmit power and data rate to achieve its SINR.

The value of pricing factor $\lambda$ affects each SU’s transmit power and data rate. Increasing $\lambda$ would decrease the transmit power and the data rate. Therefore, the pricing factor could be used to control each SU’s transmit power such that the interference constraint (14) is satisfied. One example of such a setting has already been mentioned in Theorem 3. However, setting the pricing factor $\lambda$ such that the condition (30) is satisfied needs cooperation between the primary base station and the secondary base station. Another approach is to start from an initial value for $\lambda$ and monitor the interference constraint (14). If this constraint is not satisfied, the pricing factor should be increased by $\Delta \lambda$. In this way, SUs decrease their transmit power levels. This procedure continues until the interference constraint is satisfied. Note that this approach does not need cooperation between the primary base station and the secondary base station.

Lastly, the parameter $Q$ determines to what extent SUs’ interference on the primary network is permissible. The value of $Q$ is chosen such that $p^P \geq (1 - D^H H^P)^{-1} D^P v^P(Q)$ is satisfied. In this way, the convergence point for resource allocation to the primary network lies within the power strategy space of PUs. When the PUs’ channel conditions are bad, the value of $Q$ is small, and when the PUs’ channel conditions are good, since PUs can tolerate more interference, the value of $Q$ is high. Note that setting the value of $Q$ does not need additional massage passing as $Q$ captures the aggregate impact of SUs.

The primary network solves the problem (6), for which users update their transmit power levels according to (10), meaning that each PU updates its transmit power level in a distributed manner by using power update function (11). In (11), one can observe that for obtaining the transmit power level at each iteration, each PU needs to know the value of the normalized interference plus noise $I^P_i$, which is the ratio of interference for the user at its base station $\sum_{n \in N} h^P_{in} p^P_n + \sum_{m \in M} h^P_{jm} p^P_m + N^P_0$ to its direct channel gain $h^P_j$. The direct channel gain $h^P_j$ can be estimated by the PU, or can be estimated by the primary base station and sent to the PU via a feedback channel. On the other hand, the primary base station broadcasts to PUs the value of the total power it receives, i.e., $\sum_{n \in N} h^P_{in} p^P_n + \sum_{m \in M} h^P_{jm} p^P_m + N^P_0$. As such, knowing the value of $\sum_{n \in N} h^P_{in} p^P_n + \sum_{m \in M} h^P_{jm} p^P_m + N^P_0$, the direct channel gain $h^P_j$, and the transmit power level in the previous iteration, each PU can obtain the value of interference it receives, i.e., $\sum_{n \notin i, n \in N} h^P_{in} p^P_n + \sum_{m \in M} h^P_{jm} p^P_m + N^P_0$.

The secondary network applies the game $\mathcal{G}'$, in which each SU $j$ solves the optimization problem (26) with utility $u_j$ given in (13). To update its transmit power level and data rate, each SU applies the iterative algorithm (31). This requires each SU $j$ to know the value of its normalized interference plus noise $I^S_j$ at each iteration, which is the ratio of interference for this user at its base station $\sum_{n \in N} h^S_{jn} p^S_n + \sum_{m \notin j, m \in M} h^S_{jm} p^S_m + N^S_0$ to its direct channel gain $h^S_j$. Similar to the case for the primary network, each SU $j$ can obtain the value of $I^S_j$.

### VII. Simulation Results

We now evaluate the proposed framework via simulations by considering the uplink of a cognitive radio network consisting of a single primary cell and a single secondary network. Noise power is assumed to be the same for receivers in the secondary and the primary networks, and its value is $N^S_0 = N^P_0 = N_0 = 10^{-10}$ Watts. Interference limit is set to 10 times the noise power, i.e., $Q = 10 \times N_0 = 10^{-9}$ Watts. The channel gain from SU $j$ to its base station is $h^S_j = \frac{d^{-3}}{t^3}$, where $d_j$ is the distance between SU $j$ and its base station, $t$ is the path loss exponent, and $\xi$ models power variations due to shadowing. We arbitrarily set $t = 4$ and $\xi = 0.097$. Similarly, we define the channel gain from PU $i$ to its base station, i.e., $h^P_i$. In addition, we assume that the channel gain from SU $j$ to the primary base station $(h^P_j)$ is randomly selected from $(0, h^S_j)$, and the gain from PU $i$ to the secondary base station $(h^P_S)$ is randomly selected from $(0, h^P_i)$. The bandwidth is $W = 10^6$ Hz, and the bounds on the strategy spaces are $p_{j,\text{min}} = 10^{-6}$ Watts, $p_{i,\text{max}} = 2$ Watts, $r_j,\text{min} = 0.1$ bps, and $r_j,\text{max} = 96000$ bps. We also set the target rate of each PU $i$ to $R_i = 10000$ bps.

First, we simulate the case where the convergence point of the algorithm is on the boundaries of the strategy space of users. There are 3 SUs, each located at a distance of $d_j = 120$ meters from its base station, and 2 PUs, each located at a distance of $d_i = 120$ meters from its base station. We set $\alpha_1 = 10^6$ for all SUs, and $\alpha_2 = 10, 15, 20$ for SUs 1 to 3, respectively. Each SU’s data rate is arbitrarily required to be above 10000 bps, and each PU’s target SINR is set to 20 (13 dB). The results are shown in Fig. 1. Since the convergence point of the PUs’ algorithm is within the transmit power strategy spaces of PUs, all PUs achieve their target SINRs. The same is true about SU 1. But, as SUs 2 and 3 achieve their lower bound on the data rate strategy spaces, they do not achieve their target SINRs.

Next, we study how interference affects allocated resources. There are 3 PUs and 3 SUs in the network located at the distance of 100 meters from their respective base stations. Each PU’s data rate is fixed at 12 kbps. We set $\alpha_1 = 10^6$ and $\alpha_2 = [10, 15, 20]$ such that target SINRs of SUs 1 to 3 are 10, 15, and 20, respectively ($=[10, 11.76, 13]$ dB, respectively). Target SINRs of PUs 1 to 3 are set to 15, 15, and 20, respectively ($=[11.76, 11.76, 13]$ dB, respectively). The maximum transmit power for PUs is 0.4 Watts and for SUs is 0.45 Watts. Suppose that all SUs and PUs 1 and 2 are active. For $\lambda = 10^{-4}$, we simultaneously apply PUs’ and SUs’ resource allocation schemes. Figs. 2 and 3 show transmit power levels of active users, and data rates and SINRs of...
SUs, respectively. Now, suppose PU 3 enters the network at iteration 20. This increases interference in the network, causes all users to increase their transmit power levels, and forces SUs to reduce their data rates. Figs. 4 and 5 show transmit power levels of users, and data rates and SINRs of SUs, respectively, after PU 3 enters the network. As can be seen in Fig. 4, PU 3 transmits at its maximum power limit. Now, the primary network should change the interference limit to bring the transmit power of PU 3 within its power strategy space. The value of $Q$ is chosen such that $\bar{p}_P \geq (I - D_PH_P)^{-1}D_Pv_P(Q)$ is satisfied. The new value for $Q$ is $8 \times 10^{-10}$, for which, from Theorem 3, the new value of $\lambda$ is set to $5.31 \times 10^{-4}$. These changes are applied at iteration 40, and the results are shown in Fig. 6 for transmit power levels of users and in Fig. 7 for data rates and SINRs of SUs. When pricing is increased, each SU decreases its transmit power and data rate. Consequently, SUs interference on PUs is reduced, and PUs transmit at less power to achieve their target SINRs. Also note that all users’ transmit power levels are within their strategy spaces, and SUs’ data rates are within their data rate strategy spaces as well. In addition, both PUs and SUs achieve their target SINRs.

Finally, via extensive simulations, we studied the convergence of the secondary network when its strategy space was unbounded. In doing so, for 3 to 30 SUs, we changed their SINR from 10 to 30 and their distance to their base station from 50 to 150 meters. Each simulation was ended when the difference between the target SINR and the achieved SINR at each iteration was less than $10^{-3}$. We observed that for all such cases, 15-18 iterations were needed for convergence.

VIII. Conclusions

We proposed a distributed framework for resource allocation in cognitive radio networks in which PUs and SUs apply their respective resource allocation schemes. We obtained the sufficient conditions for convergence of the proposed schemes, and via simulations, demonstrated the performances of our proposed schemes and validity of our analysis.

APPENDIX A

Proof of Theorem 2

In $VI(\mathcal{X}, F)$, the set $\mathcal{X}$ defined in (22) is non-empty, convex, and compact; and the function $F = [F_1 \cdots F_M]^T$, where $F_i$ is defined in (23), is continuous. From Corollary 2.2.5 in [29], $VI(\mathcal{X}, F)$ always has one solution. Therefore, the equivalent game always has at least one GNE.

From Theorem 2.3.3-b in [29], when $F$ is continuous and strongly monotone on the convex and closed set $\mathcal{X}$, the solution to $VI(\mathcal{X}, F)$ is unique. The strong monotonicity of $F$ means that there exists a constant $c > 0$ such that for every
\(\mathbf{x}, \mathbf{x}' \in \mathcal{X},\)
\[(\mathbf{x} - \mathbf{x}')^T (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}')) \geq c||\mathbf{x} - \mathbf{x}'||^2,\]  \hspace{1cm} \text{(A.1)}

where \(||\cdot||\) is the Euclidean norm.

To prove that the solution to \(VI(\mathcal{X}, F)\) is unique, one must prove the strong monotonicity of the function \(F\). To this end, for each user \(j\), we first obtain the followings.

\[
\begin{align*}
\mathcal{F}_j(p^S_j, r^S_j, \mathbf{p}^S_j) - \mathcal{F}_j(p'^S_j, r^S_j, \mathbf{p}^S_j) &= \frac{\alpha_1^2(p^S_j - p'^S_j) + \alpha_1 \alpha_2 I^S_j (r^S_j - r'^S_j)}{(\alpha_1 p^S_j + \alpha_2 I^S_j r^S_j) (\alpha_1 p'^S_j + \alpha_2 I^S_j r'^S_j)} + \lambda \frac{\alpha_1}{\alpha_2} I^S_j (p^S_j - p'^S_j), \\
\mathcal{F}_j(p^S_j, r^S_j, \mathbf{p}^S_j) - \mathcal{F}_j(p^S_j, r'^S_j, \mathbf{p}^S_j) &= \frac{\alpha_1 \alpha_2 I^S_j (p^S_j - p'^S_j) + \alpha_2^2 (I^S_j)^2 (r^S_j - r'^S_j)}{(\alpha_1 p^S_j + \alpha_2 I^S_j r^S_j) (\alpha_1 p^S_j + \alpha_2 I^S_j r'^S_j)} + \lambda \frac{\alpha_2}{\alpha_1} I^S_j (p^S_j - p'^S_j), \\
\mathcal{F}_j(p^S_j, r^S_j, \mathbf{p}^S_j) - \mathcal{F}_j(p^S_j, r^S_j, \mathbf{p}^S_{-j}) &= \frac{\alpha_1^2 p^S_j - \alpha_2^2 (I^S_j)^2 (r^S_j - r'^S_j)}{(\alpha_1 p^S_j + \alpha_2 I^S_j r^S_j) (\alpha_1 p^S_j + \alpha_2 I^S_j r'^S_j)} + \lambda \frac{\alpha_1}{\alpha_2} I^S_j (p^S_j - p'^S_j). \end{align*}
\]  \hspace{1cm} \text{(A.2) to (A.5)}

From (A.2) and (A.3), we obtain (A.6), (A.7), and (A.8) shown in the next page, where in (A.8), we assumed that the value of \(\lambda\) is large enough so that taking the maximum value in the denominator is allowed. Similarly, from (A.4) and (A.5), we obtain (A.9) in the next page. Adding these two inequalities, we obtain (A.10) shown in the next page. Besides, we need to obtain (A.11) shown in the page following the next page, which is computed in the same way as (A.10).

Now, from (A.10) and (A.11), it is obvious that when the conditions (24) and (25) are satisfied, there exists a \(c_j > 0\) for which we have
\[
\begin{align*}
\left[\left(\frac{(p^S_j - p'^S_j)}{(r^S_j - r'^S_j)}\right) + (\mathbf{F}_j(p^S_j, r^S_j, \mathbf{p}^S_j) - \mathbf{F}_j(p'^S_j, r^S_j, \mathbf{p}^S_j))\right] &= \frac{\left(\frac{(p^S_j - p'^S_j)}{(r^S_j - r'^S_j)}\right)^2 + (\frac{(p^S_j - p'^S_j)}{(r^S_j - r'^S_j)})^2}{c_j}. \\
\text{(A.12)}
\end{align*}
\]

Therefore, the function \(\mathbf{F}\) is strongly monotone with strong monotonicity constant \(c = \min_j c_j\).

**APPENDIX B**

**PROOF OF THEOREM 3**

In the utility function (20), the term \(\lambda^S h^{SP} p^S_j\) works as pricing, meaning that when \(\lambda^S\) is increased, the optimal transmit power for SU \(j\) is reduced. Thus, the transmit power of SU \(j\) for any value of \(\lambda^S\), i.e., \(p^S_j(\lambda^S)\), is less than or equal to the transmit power of that user when \(\lambda^S = 0\). For \(\lambda^S = 0\), the optimal transmit power can be computed from (27). In such cases, the maximum transmit power of SU \(j\) is obtained when all other SUs transmit at their maximum power levels, i.e.,
\[
p^S_{j,\text{max}} = \sqrt{\frac{1}{\alpha_1} \frac{\sum_m h_{m}^{SP} I_{m,\text{max}}^{S, \text{max}}}{\lambda^S}}. \hspace{1cm} \text{(B.1)}
\]

Substituting (B.1) into (14), we obtain
\[
\sum_m h_{m}^{SP} \frac{1}{\alpha_1} \frac{I_{m,\text{max}}^{S, \text{max}}}{\lambda^S} \leq Q. \hspace{1cm} \text{(B.2)}
\]

From (B.2), it is evident that when the pricing \(\lambda^S\) satisfies the condition (30), the inequality (14) is strictly satisfied. This means that for a set of channel gain realizations, when the pricing \(\lambda^S\) is greater than a given threshold, the SUs’ total interference on the primary network is less than \(Q\). In such
Lemma 2. Existence of the NE follows from the way SU’ strategy
\[ (p_j^S - p_j^S') \left( F_j^p(p_j^S, r_j^S, p_j^S) - F_j^p(p_j^S, r_j^S, p_j^S) \right) + (r_j^S - r_j') \left( F_j^I(p_j^S, r_j^S, p_j^S) - F_j^I(p_j^S, r_j^S, p_j^S) \right) \]
\[ \geq \frac{\alpha_1^2 (p_j^S - p_j^S')^2 - 2 \alpha_1 \alpha_2 I_{j,S}^S (p_j^S - p_j^S') (r_j^S - r_j')} {\alpha_1 p_j^S + \alpha_2 I_{j,S}^S} + \lambda \frac{1}{\alpha_2} I_{j,S}^S (r_j^S - r_j')^2 \]  
(A.6)
\[ \geq \frac{\alpha_1^2 (p_j^S - p_j^S')^2 - \alpha_1 \alpha_2 I_{j,S}^S (p_j^S - p_j^S')^2 + \alpha_1^2 (I_{j,S}^S)^2 (r_j^S - r_j')^2} {\alpha_1 p_j^S + \alpha_2 I_{j,S}^S} + \lambda \frac{1}{\alpha_2} I_{j,S}^S (r_j^S - r_j')^2 \]  
(A.7)
\[ \geq \frac{\alpha_1 \alpha_2 \left( I_{j,S}^S, \left(p_j^S + \alpha_2 I_{j,S}^S \right) \right) \left( \alpha_1 I_{j,S}^S, \alpha_2 \right)^2 \left( I_{j,S}^S, \alpha_2 \right)^2 \right)} {\alpha_1 p_j^S + \alpha_2 I_{j,S}^S} \left( \alpha_1 I_{j,S}^S, \alpha_2 \right)^2 \left( I_{j,S}^S, \alpha_2 \right)^2 \right)} {\alpha_1 p_j^S + \alpha_2 I_{j,S}^S} \left( \alpha_1 I_{j,S}^S, \alpha_2 \right)^2 \left( I_{j,S}^S, \alpha_2 \right)^2 \right) \]  
(A.8)
\[ \geq \frac{\alpha_1 \alpha_2 \left( I_{j,S}^S, \left(p_j^S + \alpha_2 I_{j,S}^S \right) \right) \left( \alpha_1 I_{j,S}^S, \alpha_2 \right)^2 \left( I_{j,S}^S, \alpha_2 \right)^2 \right)} {\alpha_1 p_j^S + \alpha_2 I_{j,S}^S} \left( \alpha_1 I_{j,S}^S, \alpha_2 \right)^2 \left( I_{j,S}^S, \alpha_2 \right)^2 \right) \]  
(A.9)
\[ \geq \frac{\alpha_1 \alpha_2 \left( I_{j,S}^S, \left(p_j^S + \alpha_2 I_{j,S}^S \right) \right) \left( \alpha_1 I_{j,S}^S, \alpha_2 \right)^2 \left( I_{j,S}^S, \alpha_2 \right)^2 \right)} {\alpha_1 p_j^S + \alpha_2 I_{j,S}^S} \left( \alpha_1 I_{j,S}^S, \alpha_2 \right)^2 \left( I_{j,S}^S, \alpha_2 \right)^2 \right) \]  
(A.10)

a case, from the complementarity condition (21), one can see that \( \lambda^S \) must be zero. Thus, the game (19) can be replaced by (26), and hence the proof.

APPENDIX C

PROOF OF THEOREM 4

Existence of the NE follows from the way SU’ strategy spaces and their utility functions are formed. We focus on uniqueness of the NE and convergence of the distributed algorithm. In the following lemma, we obtain the transmit power and the data rate update functions.

Lemma 2: The transmit power and data rate update functions of each SU \( j \) are
\[ p_j^S(k + 1) = \min \left\{ p_j^{S,max}, \max \left\{ p_j^{S,min}, [T_j^S]_j(p(k)) \right\} \right\}, \quad (C.1) \]
\[ [T_j^S]_j(p) = \min \left\{ F_j^S(p), \max \left\{ F_j^1(p), F_j^2(p) \right\} \right\}, \quad (C.5) \]  

where \([T_j^S]_j(p)\) and \([T_j^S]_j(p)\) defined in (C.3) and (C.4) shown in the next page, are the best response functions for the transmit power and the data rate, respectively.

Since only the transmit power levels of users are coupled, convergence and uniqueness of the NE depends only on the power update function, which is considered in Theorem 9.

Theorem 9. The transmit power’s best response function (C.3) is equivalent to
\[ (p_j^s - p_j^S) \left( F_j^S(p_j^s, r_j^S, p_j^S) - F_j^S(p_j^s, r_j^S, p_j^S) \right) + (r_j^S - r_j^S) \left( F_j^S(p_j^s, r_j^S, p_j^S) - F_j^S(p_j^s, r_j^S, p_j^S) \right) \leq \frac{\alpha_1 \alpha_2}{(\alpha_1 p_j^{S,min} + \alpha_2 I_j^{S,min})^2} \left( \frac{\alpha_1}{\alpha_2} + I_j^{S,max} + \frac{1}{\alpha_2} I_j^{S,max} \right) \left( \frac{\alpha_1 p_j^{S,max} + \alpha_2 I_j^{S,max} r_j^{S,max}}{\alpha_2} \right)^2 (p_j^s - p_j^S)^2 + \frac{\alpha_1 \alpha_2}{(\alpha_1 p_j^{S,min} + \alpha_2 I_j^{S,min})^2} \left( \frac{\alpha_2}{\alpha_1} (I_j^{S,max})^2 + I_j^{S,max} + \frac{1}{\alpha_1} I_j^{S,max} \left( \frac{\alpha_1 p_j^{S,max} + \alpha_2 I_j^{S,max} r_j^{S,max}}{\alpha_2} \right)^2 \right) (r_j^S - r_j^S)^2. \]  

(A.11)

\[
\begin{align*}
[T^{Sp}]_j(p) = \begin{cases} 
-\alpha_2 \lambda r_j^{S,max} I_j^S(p) + \frac{1}{2} \frac{\lambda r_j^{S,max} I_j^S(p)}{2 \alpha_1} & \text{if } I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2, \\
\frac{1}{2} \frac{\lambda r_j^{S,max} I_j^S(p)}{2 \alpha_1} & \text{if } \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2 \leq I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2, \\
\alpha_2 \lambda r_j^{S,max} I_j^S(p) + \frac{1}{2} \frac{\lambda r_j^{S,max} I_j^S(p)}{2 \alpha_1} & \text{if } I_j^S(p) \geq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2.
\end{cases}
\end{align*}
\]

(C.3)

\[
[T^{Sr}]_j(p) = \begin{cases} 
\frac{1}{2} \frac{\lambda r_j^{S,max} I_j^S(p)}{2 \alpha_1} & \text{if } I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2, \\
\frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2 & \text{if } \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2 \leq I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2, \\
\frac{1}{2} \frac{\lambda r_j^{S,max} I_j^S(p)}{2 \alpha_1} & \text{if } I_j^S(p) \geq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2.
\end{cases}
\]

(C.4)

Besides, the function \( T^{Sp}(p) \) is a standard function and has a unique fixed point.

**Proof**: To prove the equivalence, we need the following lemmas.

**Lemma 3.** The function \( f(x) = -ax + \sqrt{(ax)^2 + b} \) is strictly decreasing in \( x \).

**Lemma 4.** The function \( f(x) = -ax + \sqrt{(ax)^2 + b} - c \) is strictly increasing in \( x \).

**Lemma 5.** The equation \(-ax + \sqrt{(ax)^2 + b} = 0\) with \( c \leq b \) has only two solutions, \( x = 0 \) and \( x = (b - c^2)/(4ac) \).

Using Lemma 3, one can see that for each value of \( p \), we have \( F_j^1(p) < F_j^S(p) \). For the values of \( p \) such that \( I_j^S(p) \) is near zero, we have \( F_j^2(p) < F_j^S(p) \). Therefore, by using Lemma 5, for values of \( p \) that \( 0 \leq I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2 \), we have \( F_j^2(p) < F_j^1(p) \), and for \( \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2 \leq I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2 \), we have \( F_j^2(p) > F_j^1(p) \). Similarly, for values of \( p \) that \( 0 \leq I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2 \), we have \( F_j^2(p) < F_j^3(p) \), and for \( \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2 \leq I_j^S(p) \leq \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2 \), we have \( F_j^2(p) > F_j^3(p) \). Note that \( \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,min})^2 < \frac{2 \alpha_1}{\alpha_2} \lambda (p_j^{S,max})^2 \). Therefore, the function in (C.3) is equivalent to (C.5).

Since all functions \( F_1(p) \), \( F_2(p) \), and \( F_3(p) \) are standard, from Theorems 7 and 8 in [28], the function \( T^{Sp}(p) \) is standard. Note that there exists a value \( z < \infty \) such that for all \( p \geq 0 \), we have \([T^{Sp}]_j(p) \leq z\). Therefore, one can use the Brouwer’s fixed point theorem [30] to prove that the function \( T^{Sp}(p) \) has a fixed point. Since \( T^{Sp}(p) \) is a standard function, this fixed point is unique.

From Theorem 9, the game (26) has a unique NE to which the distributed algorithms (C.1) and (C.2) converge. In addition, when the convergence point of the algorithm is within its strategy space, one can easily use (27) to obtain (32). This completes the proof.

**APPENDIX D**

**PROOF OF THEOREM 7**

When the SUs’ data rates are lower bounded, the SUs’ power update function (C.3) is upper bounded by function \( F_j^3 \) in (C.6). For \( p \rightarrow \infty \), where \( p \) is a vector consisting of transmit power levels of all users in the network, we have \( F_j^3(p) \rightarrow \infty \). Now, for function \( F_j^3 \), we have (D.1) and (D.2) in the next page, which means that the function \( F_j^3 \) is bounded. One can use the Brouwer’s fixed point theorem [30] to show that \( T^{Sp}(p_j^r, p_j^r, r_j^r) \) always has a fixed point, which from Theorem 1 in [28] is unique and, from Theorem 2 in [28], the distributed resource allocation scheme for SUs converges to this fixed point.

When the condition in Lemma 1 is satisfied, the PUs’ power control algorithm converges to a unique point, and hence, both the PUs’ and the SUs’ resource allocation schemes converge.

**REFERENCES**


\[
\lim_{p \to \infty} F_j^S(p) = \lim_{I_j \to \infty} F_j^S(I_j^S) = \frac{-\alpha_2 \lambda I_j^S + \sqrt{(\alpha_2 \lambda I_j^S)^2 + 4 \alpha_1 \alpha_2 \lambda I_j^S}}{2 \alpha_1 \lambda} = \frac{4 \alpha_1 \alpha_2 \lambda}{(2 \alpha_1 \lambda)(2 \alpha_2 \lambda I_j^S) + \sqrt{(\alpha_2 \lambda I_j^S)^2 + 4 \alpha_1 \alpha_2 \lambda I_j^S}} = \frac{1}{\lambda I_j^S}
\]

\[\text{(D.1)}\]

\[\text{(D.2)}\]