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EXISTENCE AND CONTINUOUS DEPENDENCE OF SOLUTIONS OF SOME FUNCTIONAL-DIFFERENTIAL EQUATIONS

by

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1. Introduction.

Let us consider the following differential equation

$$y'(x) = f(x,y(y(x))),\ x \in [a,b].$$

Relatively to this equation we consider the Cauchy's problems

$$(1)\qquad \begin{cases} y'(x) = f(x,y(y(x))), & x \in [a,b] \\ y(x_o) = y_o \end{cases}$$

where $x_o, y_o \in [a,b]$ and $f \in C([a,b]\times[a,b])$;

$$(2)\qquad \begin{cases} y'(x) = f(y(y(x))), & x \in [a,b] \\ y(x) = t(x), & x \in [a-h,a] \end{cases}$$

where $h \in \mathbb{R}_+^*$, $t \in C([a-h,a],[a-h,b])$ and $f \in C([a,b],[a-h,b])$.

This kind of problems was studied by Coman, Pavel, Rus, Rus [1], Oberg [2]. In this paper we shall study this problems using fixed point theorems and the Bielecki's metric we shall give existence, existence and uniqueness and data dependence theorems.

We use the following notations:

$$c_{x_o} = \max\{x_o-a, b-x_o\};$$

$$c_{y_o} = \max\{y_o-a, b-y_o\};$$

$M \geq 0$ such that $M \geq \max\{|f(s,u)|: (s,u) \in D_f\}$;

$\|y\|_C = \max_{x \in [a,b]} |y(x)|, \ \forall\, y \in C[a,b]$;

$\tau > 0,\ x_o \in [a,b]$,

$\|y\|_B = \max_{x \in [a,b]} |y(x)| e^{-\tau(x-x_o)}, \ \forall\, y \in C[a,b]$;

$d_C(y,z) = \|y-z\|_C$;

$d_B(y,z) = \|y-z\|_B$;

$L > 0,\ \mathcal{C}_L = \{y \in C([a,b],[a,b]) \mid |y(x_1)-y(x_2)| \leq L|x_1-x_2|,\ \forall\, x_1,x_2 \in [a,b]\}$

$\lambda > 0,\ \mathcal{C}_{L,\lambda} = \{y \in \mathcal{C}_L([a,b],[a,b]) \mid y(x) \leq \lambda x,\ \forall\, x \in [a,b]\}$

$h > 0,\ \mathcal{C}_{L,h,\lambda} = \{y \in \mathcal{C}_L([a-h,b],[a-h,b]) \mid y(x) \leq \lambda x,\ \forall\, x \in [a,b]\}$.

2. **Lemma 1.** Let $\|\cdot\| \in \{\|\cdot\|_C, \|\cdot\|_B\}$. Then $\mathcal{C}_L$, $\mathcal{C}_{L,\lambda}$ and $\mathcal{C}_{L,\lambda,h}$ are convex and compact in the Banach space $(C[a,b],\|\cdot\|)$, respectively $(C[a-h,b,\|\cdot\|)$.

Proof. We shall prove that $\mathcal{C}_L$ is convex and compact in the Banach space $C[a,b]$. For the other subsets the proof is similar.

Let $y,z \in \mathcal{C}_L$, $t \in (0,1)$.

Let $x \in [a,b]$, $(ty+(1-t)z)(x) = ty(x)+(1-t)z(x) \in [a,b]$.

Then $ty+(1-t)z \in C([a,b],[a,b])$.

$|(ty+(1-t)z)(x_1)-(ty+(1-t)z)(x_2)| = |t(y(x_1)-y(x_2))+(1-t)(z(x_1)-z(x_2))| \leq tL|x_1-x_2|+(1-t)L|x_1-x_2| = L|x_1-x_2|$.

Then $ty+(1-t)z \in \mathcal{C}_L$, so $\mathcal{C}_L$ is convex.

It's easy to prove that $\mathcal{C}_L$ is equicontinuous and bounded. Using Arzela-Ascoli theorem we deduce that $\mathcal{C}_L$ is relatively compact. But $\mathcal{C}_L$ is closed, then $\mathcal{C}_L$ is compact.

3. **Theorem 1.** Assume that the following conditions are satisfied:

(i) $f \in C([a,b]\times[a,b])$; $\exists\, T > 0$: $|f(s,u)-f(s,v)| \leq T|u-v|$

(ii) $M \leq L$

(iii) $\begin{cases} \text{a) } M \leq \dfrac{c_{y_o}}{c_{x_o}} \text{ or} \\ \text{b) } x_o = a,\ M \leq \dfrac{b-y_o}{b-a},\ f(s,u) \geq 0\ \forall\, s,u \in [a,b] \text{ or} \\ \text{c) } x_o = b,\ M \leq \dfrac{y_o-a}{b-a},\ f(s,u) \geq 0\ \forall\, s,u \in [a,b] \end{cases}$

Then there exists at least one solution $y^* \in \mathcal{C}_L$ of CP(1).

Proof. We set

$$A: \mathcal{C}_L \to C[a,b]$$

$$(Ay)(x) = y_o + \int_{x_o}^{x} f(s,y(y(s)))ds,\ \forall\, x \in [a,b],\ \forall\, y \in \mathcal{C}_L$$

To prove that there exists a solution y^* of CP(1) is equivalent to prove that the mapping A has a fixed point $y^* \in \mathcal{C}_L$. For this we intend to apply the Schauder's theorem. We have that $\mathcal{C}_L$ is a convex and compact subset of the Banach space $(C[a,b],\|\cdot\|)$. So, we need to verify if $\mathcal{C}_L$ is an invariant set for A, and A is a continuous mapping.

For $y \in \mathcal{C}_L$ and $x \in [a,b]$ we have

$$(Ay)(x) \leq y_o + \left|\int_{x_o}^{x} f(s,y(y(s)))ds\right| \leq y_o+M|x-x_o| \leq y_o+Mc_{x_o} \leq y_o+c_{y_o} \leq b$$

$$(Ay)(x) \geq y_o - \left|\int_{x_o}^{x} f(s,y(y(s)))ds\right| \geq y_o-M|x-x_o| \geq y_o-Mc_{x_o} \geq y_o-c_{y_o} \geq a$$

where we used the condition (iii)a). It's easy to prove the same thing using the condition (iii)b) or (iii)c).

Let $x_1,x_2 \in [a,b]$

$$|(Ay)(x_1)-(Ay)(x_2)| = \left|\int_{x_o}^{x} f(s,y(y(s)))ds\right| \leq M|x_1-x_2| \leq L|x_1-x_2|$$

So $Ay \in \mathcal{C}_L$, $\forall\, y \in \mathcal{C}_L$.

Let $y,z \in \mathcal{C}_L$, $x \in [a,b]$

$$|(Ay)(x)-(Az)(x)| \leq \left|\int_{x_o}^{x} |f(s,y(y(s)))ds-f(s,z(z(s)))|ds\right| \leq$$

$$\le \left|\int_{x_o}^{x} T|y(y(s))-z(z(s))|ds\right| \le T\left|\int_{x_o}^{x} [y(y(s))-y(z)s))|+\right.$$

$$\left.+|y(z(s))-z(z(s))||ds\right| \le T\left|\int_{x_o}^{x} [L|y(s)-z(s)|+|y(z(s))-z(z(s))||ds\right| \le$$

$$\le T(L+1)\|y-z\|_C|x-x_o| \le Tc_{x_o}(L+1)\|y-z\|_C$$

$$\|Ay-Az\|_C \le Tc_{x_o}(L+1)\|y-z\|_C$$

A is Lipschitz, so is a continuous mapping.

Theorem 2. Assume that the conditions of Theorem 1 are satisfied and, in addition, we have:

(iv) $Tc_{x_o}(L+1) < 1$.

Then there exists an unique solution $y^* \in \mathscr{C}_L$ of CP(1).

Proof. The mapping A: $\mathscr{C}_L \to \mathscr{C}_L$ is a contraction and $(\mathscr{C}_L, d_C)$ is a complete metric space. Applying contraction principle we have thatA has an unique fixed point, which is the unique solution of CP(1).

Relativelv to this theorem we shall give a data dependence theorem.

One consider the Cauchy's problem:

$$(\tilde{1}) \qquad \begin{cases} y'(x) = g(x,y(y(x))), \ x \in [a,b] \\ y(\tilde{x}_o) = z_o \end{cases}$$

where $\tilde{x}_o$, $z_o \in [a,b]$, $g \in C([a,b]\times[a,b])$.

Let $M_g \ge 0$ be such that $M_g \ge \max_{(s,u)\in[a,b]^2} |g(s,u)|$.

Theorem 3. Assume that the following conditions are satisfied:

(i) CP(1) satisfies the conditions of Theorem 2 (so CP(1) has an unique solution, $y^* \in \mathscr{C}_L$);

(ii) CP(2) satisfies the conditions of Theorem 1 (so CP($\tilde{1}$) has a solution, $z^* \in \mathscr{C}_L$);

(iii) $\exists\ \varepsilon > 0$ such that $|f(s,u)-g(s,u)| \le \varepsilon\ \forall\ s,u \in [a,b]$.

Then

$$d_C(y^*,z^*) \le \frac{|y_o-z_o|+\varepsilon c_{x_o}+M_g|\tilde{x}_o-x_o|}{1-Tc_{x_o}(L+1)}$$

Proof. We intend to apply the data dependence theorem for the complete metric space $(\mathscr{C}_L, d_C)$ and the mappings

$$A,B: \mathscr{C}_L \to \mathscr{C}_L$$

$$(Ay)(x) = y_o + \int_{x_o}^{x} f(s,y(y(s)))ds$$

$$(By)(x) = z_o + \int_{x_o}^{x} g(s,y(y(s)))ds, \ \forall\ x \in [a,b], \ \forall\ y \in \mathscr{C}_L.$$

Let $y \in \mathscr{C}_L$, $x \in [a,b]$

$$|(Ay)(x)-(By)(x)| = \left|y_o+\int_{x_o}^{x} f(s,y(y(s)))ds-z_o-\int_{x_o}^{x} g(s,y(y(s)))ds\right| \le$$

$$\le |y_o-z_o|+\left|\int_{x_o}^{x} |f(s,y(y(s)))-g(s,y(y(s)))|ds\right|+\left|\int_{x_o}^{x} g(s,y(y(s)))ds\right| \le$$

$$\le |y_o-z_o|+\varepsilon|x-x_o|+M_g(\tilde{x}_o-x_o) \le |y_o-z_o|+\varepsilon c_{x_o}+M_g|\tilde{x}_o-x_o|$$

A: $\mathscr{C}_L \to \mathscr{C}_L$ is a contraction mapping with the constant $Tc_{x_o}(L+1)$ and $F_A = \{y^*\}$, $z^* \in F_B$.

Then $d_C(y^*,z^*) \le \dfrac{|y_o-z_o|+\varepsilon c_{x_o}+M_g|\tilde{x}_o-x_o|}{1-Tc_{x_o}(L+1)}$.

4. Theorem 4. Assume that the following conditions are satisfied:

(i) $y_o \le \lambda x_o$;

(ii) $\exists\ T > 0$: $|f(s,u)-f(s,v)| \le T|u-v|$, $\forall\ s,u,v \in [a,b]$;

(iii) $M \le \min\{\lambda, L\}$;

(iv) $\begin{cases} \text{a) } M \le \dfrac{c_{y_o}}{c_{x_o}} \text{ or} \\ \text{b) } x_o = a,\ M \le \dfrac{b-y_o}{b-a},\ f(s,u) \ge 0\ \forall\ s,u \in [a,b] \text{ or} \\ \text{c) } x_o = b,\ M \le \dfrac{y_o-a}{b-a},\ f(s,u) \ge 0\ \forall\ s,u \in [a,b] \end{cases}$

(v) $M(x_o-a) \ge y_o-\lambda a$.

Then there exists at least one solution $y^{o} \in \mathcal{C}_{L,\lambda}$ of CP(1).

Proof. The set $C_{L,\lambda}$ is a convex and compact subset of the Banach space $C[a,b]$ endowed with Bielecki's metric.

The proof is similar with the proof of Theorem 1.

The conditions (ii), (iii), (iv) assure that $A(\mathcal{C}_{L,\lambda}) \subset \mathcal{C}_L$.

Let $y \in \mathcal{C}_{L,\lambda}$, $x \in [a,b]$

$(Ay)(x) \le y_o+M(x-x_o) = y_o+M(x-a)-M(x_o-a) \le y_o+\lambda(x-a)-(y_o-\lambda a) = \lambda x$,

where we used (iii) and (v).

So, $\mathcal{C}_{L,\lambda}$ is an invariant set for A.

Also, for $y,z \in \mathcal{C}_{L,\lambda}$, $x \in [a,b]$ we get

$$|(Ay)(x)-(Az)(x)| \le T\left|\int_{x_o}^{x} L|y(s)-z(s)|ds+\int_{x_o}^{x} |y(z(s))-z(z(s))|ds\right| \le$$

$$\le T\left[\left|\int_{x_o}^{x} L\|y-z\|_B e^{\tau(s-x_o)} ds\right|+\left|\int_{x_o}^{x} \|y-z\|_B e^{\tau(z(s)-x_o)} ds\right|\right]\le$$

$$\le T\left[L\left|\int_{x_o}^{x} e^{\tau(s-x_o)} ds\right|+\left|\int_{x_o}^{x} e^{\tau(z\lambda s-x_o)} ds\right|\right]\|y-z\|_B =$$

$$= T\left[\frac{L}{\tau}|e^{\tau(x-x_o)}-1| + \frac{1}{\tau\lambda}|e^{\tau(\lambda x-x_o)}-e^{\tau(\lambda x_o-x_o)}|\right]\|y-z\|_B.$$

Then:

$$|(Ay)(x)-(Az)(x)|e^{-\tau(x-x_o)} \le T\left[\frac{L}{\tau}|1-e^{\tau(x-x_o)}|+\frac{1}{\tau\lambda}|e^{\tau(\lambda-1)x}-e^{\tau(\lambda x_o-x)}|\right]\cdot$$

$$\cdot\|y-z\|_B = \frac{T}{\tau}\left[L|1-e^{\tau(x-x_o)}|+ \frac{1}{\lambda} e^{\tau(\lambda-1)x}|1-e^{-\tau\lambda(x-x_o)}|\right]\|y-z\|_B =$$

$$= L_A(x)\|y-z\|_B$$

L_A: $[a,b] \longrightarrow \mathbb{R}$ is a continuous function. Then $\exists\ L_A > 0$ such that

$$L_A \ge \max_{x\in[a,b]} L_A(x).$$

Then, we get

$$\|Ay-Az\|_B \le L_A\|y-z\|_B\ \forall\ y,z \in \mathcal{C}_{L,\lambda}$$

The operator A is Lipschitz, so is continuous.

The hypothesis of Schauder's theorem are fullfiled, then A has at least one fixed point $y^{*} \in \mathcal{C}_{L,\lambda}$ which is a solution of CP(1).

Remark. For $a,b > 0$ and $\lambda \ge b/a$ the Theorems 1 and 4 coincide. In this case $\mathcal{C}_{L,\lambda} = \mathcal{C}_L$ and the conditions (i), (v) and $M \le \lambda$ from Theorem 4 are verified.

Theorem 5. Assume that the conditions of Theorem 4 are satisfied and, in addition, we have:

$$\text{vi) } \max\left\{(\lambda-1)b,\ (\lambda-1)a,\ \frac{x_o-a}{\ln 2},\ \frac{\lambda(x_o-a)}{\ln 2},\ 0\right\} T\left[L + \frac{e}{\lambda}\right] < 1.$$

Then there exists an unique solution $y^{*} \in \mathcal{C}_{L,\lambda}$ of CP(1).

Proof. $(\mathcal{C}_{L,\lambda}, d_B)$ is a complete metric space and $A:\mathcal{C}_{L,\lambda} \longrightarrow \mathcal{C}_{L,\lambda}$ is Lipschitz with constant L_A.

We intend to apply the contraction principle.

We shall prove that the condition (vi) assure A is a contraction.

$$\forall\ x \in [a,b],\ L_A(x) \le \frac{TL}{\tau} \max\left\{e^{-\tau(a-x_o)}-1,\ e^{-\tau(b-x_o)}\right\} +$$

$$+ \frac{T}{\tau\lambda} e^{\tau \max\{(\lambda-1)a,(\lambda-1)b,0\}} \max\left\{e^{-\tau\lambda(a-x_o)}-1,\ 1-e^{-\tau\lambda(b-x_o)}\right\};$$

$$\text{(vi)} \Rightarrow \quad \frac{\max\{1,\lambda\}(x_o-a)}{\ln 2} < \frac{1}{T\left[L + \frac{e}{\lambda}\right]}$$

and

$$\max\{(\lambda-1)a,(\lambda-1)b,0\} < \frac{1}{T\left[L + \frac{e}{\lambda}\right]}$$

We choose τ such that:

$$\frac{\max\{1,\lambda\}(x_o-a)}{\ln 2} \le \frac{1}{\tau} < \frac{1}{T\left[L + \frac{e}{\lambda}\right]}$$

and

$$\max\{(\lambda-1)a, (\lambda-1)b, 0\} \le \frac{1}{\tau}$$

If $x_o = a$ it's easy to notice that

(3) $$L_A(x) \le \frac{TL}{\tau} + \frac{T}{\tau\lambda} e^{\tau \max\{(\lambda-1)a, (\lambda-1)b, 0\}}$$

If $x_o \ne a$ we get

$$\tau \le \frac{\ln 2}{\frac{(x_o-a)}{\max\{1,\lambda\}}} \Rightarrow e^{-\tau(a-x_o)} - 1 < 1$$

and

$$e^{-\tau\lambda(a-x_o)} - 1 < 1.$$

Then we obtain again the relation (3).

$$\tau \max\{(\lambda-1)a, (\lambda-1)b, 0\} \le 1 \Rightarrow L_A(x) \le \frac{TL}{\tau} + \frac{Te}{\tau\lambda} = \frac{T}{\tau}\left[L + \frac{e}{\lambda}\right] < 1.$$

So, A: $(\mathscr{C}_{L,\lambda}, d_B) \longrightarrow (\mathscr{C}_{L,\lambda}, d_B)$ is a contraction with $L_A = \frac{T}{\tau}\left[L + \frac{e}{\lambda}\right]$.

Theorem 6. Assume that:

(i) CP(1) satisfies the conditions of Theorem 5 [so CP(1) has an unique solution $y^* \in \mathscr{C}_{L,\lambda}$ and there exists $\tau > 0$ such that

$$\max\left\{(\lambda-1)a,\ (\lambda-1)b,\ \frac{x_o-a}{\ln 2},\ \frac{\lambda(x_o-a)}{\ln 2},\ 0\right\} \le \frac{1}{\tau} < \frac{1}{T\left[L + \frac{e}{\lambda}\right]}\Bigg]$$

(ii) CP($\tilde{1}$) satisfies the conditions of Theorem 4 (so CP($\tilde{1}$) has a solution $z^* \in \mathscr{C}_{L,\lambda}$)

(iii) $\exists\ \varepsilon > 0$ such that $|f(s,u)-g(s,u)| \le \varepsilon$, $\forall$ s,u $\in$ [a,b].

Then

$$d_C(y^*,z^*) \le \frac{|y_o-z_o| + \varepsilon C x_o + Mg|\tilde{x}_o-x_o|}{1 - \frac{T}{\tau}\left[L + \frac{e}{\lambda}\right]}$$

Proof. Let A,B: $\mathscr{C}_{L,\lambda} \longrightarrow \mathscr{C}_{L,\lambda}$ be defined like in Theorem 3.

For $y \in \mathscr{C}_{L,\lambda}$, $x \in [a,b]$ we get

$$|(Ay)(x)-(By)(x)|e^{-\tau(x-x_o)} \le \left[|y_o-z_o| + \varepsilon C x_o + Mg|x_o-\tilde{x}_o|\right]e^{-\tau(x-x_o)}$$

$$d_B(Ay,By) \le \left[|y_o-z_o| + \varepsilon C x_o + Mg|x_o-\tilde{x}_o|\right]e^{-\tau(x-x_o)}$$

A: $(\mathscr{C}_{L,\lambda}, d_B) \longrightarrow (\mathscr{C}_{L,\lambda}, d_B)$ is a contraction mapping with the constant

$$L_A = \frac{T}{\tau}\left[L + \frac{e}{\lambda}\right] < 1 \text{ and } F_A = \{y^*\}.$$

$$z^* \in F_B.$$

Applying the data dependence theorem we get

$$d_B(y^*,z^*) \le \frac{|y_o-z_o| + \varepsilon C x_o + Mg|\tilde{x}_o-x_o|}{1 - \frac{T}{\tau}\left[L + \frac{e}{\lambda}\right]} e^{-\tau(a-x_o)}$$

We have that $\|y\|_C \le \|y\|_B \cdot e^{\tau(b-x_o)}$, $\forall$ y $\in$ C[a,b] who leads us to the relation by the theorem.

5. Theorem 7. Assume that:

(i) $t(a) \le \lambda a$;

(ii) $\exists\ T > 0$: $|f(s,u)-f(x,v)| \le T|u-v|$, $\forall$ s $\in$ [a,b], $\forall$ u,v $\in$ [a-h,b];

(iii) $f(s,u) \ge 0$, $\forall$ s $\in$ [a,b], $\forall$ u $\in$ [a-h,b];

(iv) $M \le \min\left\{\lambda, L, \frac{b-t(a)}{b-a}\right\}$;

(v) $|t(x_1)-t(x_2)| \le L|x_1-x_2|$, $\forall\ x_1,x_2 \in$ [a-h,b].

Then there exists at least one solution $y^* \in \mathscr{C}_{L,\lambda,h}$ of CP(2).

Proof. Let A: $\mathscr{C}_{L,\lambda,h} \longrightarrow C[a-h,b]$

$$(Ay)(x) = \begin{cases} t(a), & x \in [a-h,a] \\ t(a) + \int_a^x f(s,y(y(s)))ds, & x \in [a,b],\ \forall\ y \in \mathscr{C}_{L,\lambda,h} \end{cases}$$

We have that $\mathscr{C}_{L,\lambda,h}$ is a convex and compact subset of the Banach space $(C[a-h,b], \|\cdot\|_B)$.

It's easy to prove that $\mathscr{C}_{L,\lambda,h}$ is an invariant set for A.

Let $y,z \in \mathscr{C}_{L,\lambda,h}$,

for $x \in [a-h,a]$, $|(Ay)(x)-(Az)(x)| = 0$

for $x \in [a,b]$, $|(Ay)(x)-(Az)(x)| \le$

$$\le \left[\frac{TL}{\tau}\left(e^{\tau(x-a)} - \right) + \frac{T}{\tau\lambda}\left(e^{\tau(\lambda x-a)} - e^{\tau(\lambda-1)a}\right)\right]\|y-z\|_B \le$$

$$\le \left[\frac{TL}{\tau}e^{\tau(x-a)} + \frac{T}{\tau\lambda}e^{\tau(\lambda x-a)}\right]\|y-z\|_B$$

$$|(Ay)(x)-(Az)(x)|e^{-\tau(x-a)} \le \left[\frac{TL}{\tau} + \frac{T}{\tau\lambda}e^{\tau(\lambda-1)}\right]\|y-z\|_B \le$$

$$\le \left[\frac{TL}{\tau} + \frac{T}{\tau\lambda}e^{\tau\max\{(\lambda-1)a,(\lambda-1)b,0\}}\right]\|y-z\|_B, \ \forall\, x \in [a-h,b]$$

Then $\|Ay-Az\|_B \le \frac{T}{\tau}\left[L + \frac{1}{\lambda}e^{\tau\max\{(\lambda-1)a,(\lambda-1)b,0\}}\right]\|y-z\|_B$,

$$\forall\, y,z \in \mathscr{C}_{L,\lambda,h}$$

This means that A is Lipschitz, so is a continuous mapping. Applying Schauder's theorem, the mapping A has at least one fixed point $y^* \in \mathscr{C}_{L,\lambda,h}$, which is a solution of CP(2).

Theorem 8. Assume that the conditions of Theorem 7 are satisfied and, in addition, we have:

(vi) $\max\{(\lambda-1)b,(\lambda-1)a,0\}\ T\left[L + \frac{e}{\lambda}\right] < 1$.

Then there exists an unique solution $y^* \in \mathscr{C}_{L,\lambda,h}$ of CP(2).

Proof. $(\mathscr{C}_{L,\lambda,h}, d_B)$ is a complete metric space.

A: $(\mathscr{C}_{L,\lambda,h}, d_B) \longrightarrow (\mathscr{C}_{L,\lambda,h}, d_B)$ is Lipschitz.

(vi) $\Rightarrow$ $\max\{(\lambda-1)a,(\lambda-1)b,0\} < \dfrac{1}{T\left[L + \frac{e}{\lambda}\right]}$

$$\Rightarrow \exists\, \tau > 0:\ \max\{(\lambda-1)a,(\lambda-1)b,0\} \le \frac{1}{\tau} < \frac{1}{T\left[L + \frac{e}{\lambda}\right]}$$

Then $d_B(Ay,Az) \le \frac{T}{\tau}\left[L + \frac{e}{\lambda}\right]\|y-z\|_B$ and $\frac{T}{\tau}\left[L + \frac{e}{\lambda}\right] < 1$. Hence A is a contraction mapping.

Applying the contraction principle, the mapping A has an unique fixed point $y^* \in \mathscr{C}_{L,\lambda,h}$, which is the unique solution of CP(2).

Let us consider the Cauchy's problem:

$$(\tilde{2})\qquad \begin{cases} y'(x) = g(x,y(y(x))), & x \in [a,b] \\ y(x) = \tilde{t}(x), & x \in [a-h,b] \end{cases}$$

where $g \in C([a,b]\times[a-h,b])$, $\tilde{t} \in C([a-h,a],[a-h,b])$.

Theorem 9. Assume that:

(i) CP(2) satisfies the conditions of Theorem 8 (so CP(2) has an unique solution $y^* \in \mathscr{C}_{L,\lambda,h}$ and

$$\exists\, \tau > 0:\ \max\{(\lambda-1)b,(\lambda-1)a,0\} \le \frac{1}{\tau} < \frac{1}{T}\left[L + \frac{e}{\lambda}\right]\Big);$$

(ii) CP($\tilde{2}$) satisfies the conditions of Theorem 7 (so CP($\tilde{2}$) has a solution $z^* \in \mathscr{C}_{L,\lambda,h}$);

(iii) $\exists\, \varepsilon > 0$ such that $|f(s,u)-g(s,u)| \le \varepsilon$, $\forall\, s \in [a,b]$, $\forall\, u \in [a-h,b]$;

(iv) $\exists\, \eta > 0$ such that $|t(x)-\tilde{t}(x)| \le \eta$, $\forall\, x \in [a-h,b]$.

Then

$$d_C(y^*,z^*) \le \frac{\max\{\eta, |t(a)-\tilde{t}(a)|+\varepsilon(b-a)\}}{1 - \frac{T}{\tau}\left[L + \frac{e}{\lambda}\right]}e^{\tau(b-a)}$$

6. We shall now give an example in order to illustrate the above theorems.

Let us consider the following Cauchy's problem:

$$(3)\qquad \begin{cases} y'(x) = \mu y(y(x)), & x \in [0,1] \\ y(0) = 0 \end{cases}$$

where $\mu > 0$.

We have $f: [0,1]\times[0,1] \longrightarrow \mathbb{R}$, $f(s,u) = \mu u$. $f \in C[0,1]\times[0,1]$. We can consider $M = \mu$ and $T = \mu$.

Let $L,\lambda > 0$.

Applying, one by one, the Theorems 1,2,4 and 5, we get:

(1) $\Rightarrow$ If $\mu \le L$, $\mu \le 1$ then CP(3) has at least one solution $y^* \in \mathscr{C}_L$.

(2) $\Rightarrow$ If $\mu \le L$, $\mu \le 1$, $\mu(L+1) < 1$ then CP(3) has an unique solution $y^* \in \mathscr{C}_L$.

(4) $\Rightarrow$ If $\mu \le L$, $\mu \le 1$, $\mu \le \lambda$ then CP(3) has at least one solution $y^* \in \mathscr{C}_{L,\lambda}$.

(5) $\Rightarrow$ If $\mu \le L$, $\mu \le 1$, $\mu \le \lambda$, $\max\{(\lambda-1),0\}\cdot\mu\left[L + \frac{e}{\lambda}\right] < 1$ then CP(3) has an unique solution $y^* \in \mathscr{C}_{L,\lambda}$.

Consequently, if $\mu \le L$ and $\mu \le 1$ we get:

I CP(3) has an unique solution in $\mathscr{C}_{L,1}$ and, moreover, the solution is subject to the condition:

$$y(x) \le \mu x, \ \forall\, x \in [0,1].$$

II If $\mu(L+1) < 1$ then CP(3) has an unique solution $y^* \in \mathscr{C}_L$ and, moreover, $y^*(x) \le \mu x$, $\forall\, x \in [0,1]$.

III If $\lambda > 1$ and $(\lambda-1)\mu\left[L + \frac{e}{\lambda}\right] < 1$ then CP(3) has an unique solution in $\mathscr{C}_{L,\lambda}$.

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