Defining numbers in some of the Harary graphs

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In a given graph G = (V, E), a set of vertices S with an assignment of colors to them is said to be a defining set of the vertex coloring of G if there exists a unique extension of the colors of S to a c ≥ χ(G) coloring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number. In this note, we study the chromatic number, the defining number and the strong defining number in some of the Harary graphs.

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1. Introduction

A k-coloring of a graph G is a labelling f : V(G) → T, where |T| = k and it is proper if the adjacent vertices have different labels. A graph is k-colorable if it has a proper k-coloring. The chromatic number χ(G) is the least k such that G is k-colorable. Let χ(G) ≤ k ≤ |V(G)|. A set S of the vertices of G with an assignment of colors to them is called a defining set of the vertex coloring of G if there exists a unique extension of S to a k-coloring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set and its cardinality is the defining number, denoted by d(G, k). Let G be a graph with n vertices. A defining set S (|S| = s), with an assignment of colors in graph G, is called a strong defining set of the vertex coloring of G if there exists an ordering set {v1, v2, . . . , vn−1} of the vertices of (G − S) such that in the induced list of colors in each of the subgraphs (G − S), (G − S ∪ {v1}), (G − S ∪ {v1, v2}) . . . and (G − S ∪ {v1, v2, . . . , vn−2−1}) there exists at least one vertex whose list of colors is of cardinality 1 (see [1,2]). The strong defining number, sd(G, k), of G is the cardinality of its smallest strong defining set (for more see [3,4,2,5,6]).

Given m ≤ n, place n vertices around a circle, equally spaced. If m is even, form Hm,n by making each vertex adjacent to the nearest m/2 vertices in each direction around the circle. If m is odd and n is even, form Hm,n by making each vertex adjacent to the nearest (m − 1)/2 vertices in each direction and to the diametrically opposite vertex. In each case, Hm,n is m-regular. When m and n are both odd, index the vertices with the integers modulo n. Construct Hm,n from Hm−1,n by adding the edges i ↔ i + (n − 1)/2 for 0 ≤ i ≤ (n − 1)/2 (see [7]).

We would like to study the chromatic number, the defining number and the strong defining number of some Harary graphs.

For example: H2,2n = Cn and Hm,m+1 = Km+1. And so χ(H2,2n ) = 2, χ(H2,2n+1 ) = 3, χ(Hm,m+1) = m + 1. Also, d(H2,2n+1, χ) = n, d(H2,2n, χ) = 1, d(Hm,m+1, χ) = m.

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2. The chromatic numbers

In this section, the chromatic numbers of some Harary graphs are studied.

Lemma 1. i. Let \( H = H_{2m,n} \) or \( H_{2m+1,n} \), and \( m \geq 2 \). Then,
\[
\chi(H) \geq \begin{cases} 
  m + 2, & \text{if } m + 1 \mid n \\
  m + 1, & \text{if } m + 1 \nmid n 
\end{cases}
\]
ii. \( \chi(H_{3,2n}) \geq \chi(C_{n+1}) \) \((n \geq 1)\).
iii. \( \chi(H_{3,2n+1}) \geq 3 \).

Proof. i. Every \( m \) + 1 consecutive vertices such as \( i, i + 1, \ldots, i + m \pmod{n} \) in \( H \) induce a clique subgraph \( K_{m+1} \); hence \( \chi(H) \geq \chi(K_{m+1}) = m + 1 \). In these graphs the coloring function with \( m + 1 \) colors is a congruence function, to modulo \( m + 1 \); therefore if \( m + 1 \mid n \), then \( \chi(H) \geq m + 2 \).
ii. Let \( V(H_{3,2n}) = \{1, 2, 3, \ldots, 2n\} \). The set of vertices \( \{1, 2, \ldots, n + 1\} \) induces a cycle \( C_{n+1} \), so \( \chi(H_{3,2n}) \geq \chi(C_{n+1}) \), \( n \geq 1 \).
iii. Let \( V(H_{3,2n+1}) = \{1, 2, 3, \ldots, 2n + 1\} \). The set of vertices \( \{1, n + 1, 2n + 1\} \) induces \( C_3 \); therefore \( \chi(H_{3,2n+1}) \geq 3 \). \( \square \)

Proposition 2. For every \( n = (m + 1)k + r \), and \( k \geq r \), we have
\[
\chi(H_{2m,n}) = \begin{cases} 
  m + 1, & \text{if } r = 0 \\
  m + 2, & \text{otherwise.}
\end{cases}
\]

Proof. Since \( k \geq r \) we may have \( n = (m + 1)(k + r) + r(m + 2) \). If \( m + 1 \nmid n \), then the coloring function \( f \), to modulo \( m + 1 \), i.e.,
\[
f(i) = j, \quad i \equiv j \pmod{m + 1}, \quad (1 \leq i \leq n),
\]
is a proper coloring. So \( \chi = m + 1 \), by Lemma 1. Now, we suppose that \( m + 1 \nmid n \). The coloring function \( f \) with criterion
\[
f(i) = j, \quad i \equiv j \pmod{m + 1}, \quad 1 \leq i \leq (k - r)(m + 1),
f(i) = j, \quad i - (k - r)(m + 1) \equiv j \pmod{m + 2}, \quad (k - r)(m + 1) + 1 \leq i \leq n,
\]
and Lemma 1 imply \( \chi = m + 2 \). \( \square \)

We know that if \( n \geq m(m+1) \), then
\[
\chi(H_{2m,n}) = \begin{cases} 
  m + 1, & \text{if } m + 1 \mid n \\
  m + 2, & \text{otherwise}
\end{cases}
\]
(see [7]). \( \chi(H_{2m,n}) \) is now determined for some \( n < m(m+1) \).

Proposition 3. If \( n = m(m + 1) - t \) for \( 2 \leq t \leq m \), then \( \chi(H_{2m,n}) = m + 2 \).

Proof. Since \( m + 1 \nmid n \) therefore \( \chi \geq m + 2 \). Now, the proper coloring function \( f \) with criterion
\[
f(i) = j, \quad i \equiv j \pmod{m + 1}, \quad 1 \leq i \leq (t - 2)(m + 1),
f(i) = j, \quad i - (t - 2)(m + 1) \equiv j \pmod{m + 2}, \quad (t - 2)(m + 1) + 1 \leq i \leq n,
\]
implies \( \chi(H_{2m,n}) = m + 2 \). \( \square \)

If \( m \geq 3 \) and \( n = 3m + 2 \), then \( n < m(m+1) \). Since \( m + 1 \nmid n \), so \( \chi \geq m + 2 \). For example \( \chi(H_{8,14}) = 7 = 4 + 3 = \lceil \frac{14}{2} \rceil \) and \( \chi(H_{10,17}) = 9 = 5 + 4\lceil \frac{17}{2} \rceil \) since each color cannot be used for at most two vertices. We in general have:

Theorem 4. \( \chi(H_{2m,3m+2}) = \lceil \frac{3m+2}{2} \rceil \) for \( m \geq 3 \).

Proof. We first show that each color is used at most twice. Let the color 1 be used more than twice. Without loss of generality assume the vertices \( v_1, v_k, v_l \) take the color 1. Hence \( k \geq m + 2 \) and \( l \geq 2m + 3 \). It is easy to see that the vertex \( v_i \) is adjacent to \( v_1 \). That is a contradiction and so \( \chi(H_{2m,3m+2}) \geq \lceil \frac{3m+2}{2} \rceil \).

On the other hand if \( f \) is a coloring function with \( \lceil \frac{3m+2}{2} \rceil \) colors then \( f \) is a proper coloring of \( H_{2m,3m+2} \). It is sufficient if we say \( f(i) = f(i + \lceil \frac{3m+2}{2} \rceil) = i \) for even \( n \), and \( 1 \leq i \leq \lceil \frac{3m+2}{2} \rceil \), and \( f(i) = f(i + \lceil \frac{3m+2}{2} \rceil) = i \), \( f(\lceil \frac{3m+2}{2} \rceil) = \lceil \frac{3m+2}{2} \rceil \) for odd \( n \) and \( 1 \leq i \leq \lceil \frac{3m+2}{2} \rceil \). Hence \( \chi(H_{2m,3m+2}) \leq \lceil \frac{3m+2}{2} \rceil \). Thus \( \chi(H_{2m,3m+2}) = \lceil \frac{3m+2}{2} \rceil \). \( \square \)

Proposition 5. For every \( n \geq m + 1 \), we have
\[
\chi(H_{2m+1,n+2}) = \begin{cases} 
  m + 1, & \text{if } 2n = (m + 1)t \text{ and } t \text{ is odd} \\
  m + 2, & \text{if } 2n = (m + 2)t \text{, } t \text{ is odd and } m + 1 \nmid t
\end{cases}
\]
Proof. Suppose that $2n = (m + 1)t$ and $t$ is odd. The coloring function $f$, to modulo $m + 1$, is a proper coloring. Because $n \not\equiv 0 \pmod{m + 1}$ implies $f(i + n) \neq f(i)$ for $1 \leq i \leq n$. Therefore $\chi(H_{2m+1,2n}) = m + 1$.

Now, we suppose that $2n = (m + 2)t$, $t$ is odd and $m + 1 \not\mid t$. The coloring function $f$, to modulo $m + 2$, is a proper coloring. Because $n \not\equiv 0 \pmod{m + 2}$ implies $f(i + n) \neq f(i)$ for $1 \leq i \leq n$. Since $m + 1 \not\mid 2n$, so $\chi = m + 2$ by Lemma 1. \hfill $\square$

As an immediately result, we have:

**Corollary 6.** If $2n = (m + 1)(m + 2)t$ such that both of $t$ and $m$ are odd, then $\chi(H_{2m+1,2n}) = m + 1$.

**Proposition 7.** If $n = (m + 1)k + r$ where $1 \leq r \leq m$, then

$$\chi(H_{2m+1,2n}) = m + 1 \iff m \text{ is odd and } r = (m + 1)/2.$$  

Proof. Let $f$ be a coloring function with $m + 1$ colors for $H_{2m+1,2n}$. By Lemma 1, $\chi(H_{2m+1,2n}) \geq m + 1$. But $\chi(H_{2m+1,2n}) = m + 1$ if and only if $n \not\equiv 0 \pmod{m + 1}$ and $f(2n) = m + 1$. Since $n \not\equiv 0 \pmod{m + 1}$ and $f(2n) = f(2r)$, so $\chi(H_{2m+1,2n}) = m + 1$ if and only if $2r = m + 1$. \hfill $\square$

Obviously, $\chi(H_{2m+1,2n}) \geq m + 2$ if $m$ is even or if $m$ is odd and $r \neq (m + 1)/2$, by Proposition 7.

**Proposition 8.** For every $n \geq 2$, we have $\chi(H_{3,2n+1}) = 3$.

Proof. Let $V = \{1, 2, 3, \ldots, 2n + 1\}$; we know $\chi \geq 3$, by Lemma 1. We now consider three cases as follows.

Case i. $2n + 1 = 4k + 5, k \geq 0, 3 \not\mid n$.

The coloring function $f$ with criterion

$$f(i) = \delta_i, \quad \text{where } \delta_i \equiv \begin{cases} i \pmod{3}, & 1 \leq i \leq n \\ i + 1 \pmod{3}, & n + 1 \leq i \leq 2n \end{cases} \quad \text{and} \quad 1 \leq \delta_i \leq 3,

$$

$$f(2n + 1) = 3,$$

is a proper coloring with three colors, $f(n + 1) = 2$ isn't equal to $f(2n + 1)$ and $f(1)$, and $3 \mid n$ imply $f(i + n) = f(i + 1) \neq f(i)$, $1 \leq i \leq n$.

Case ii. $2n + 1 = 4k + 5, k \geq 0, 3 \mid n$.

Let $n = 3t + r, 1 \leq r \leq 2$. The coloring function $f$, to modulo 3, is a proper coloring, because $f(i + n) = f(i + r) \neq f(i)$ for $1 \leq i \leq n$. Also, $f(n + 1) = r + 1 \neq f(2r + 1) = f(2n + 1)$.

Case iii. $2n + 1 = 4k + 7, k \geq 0$.

The coloring function $f$ with criterion

$$f(i) = j, \quad i \equiv j \pmod{2}, 1 \leq i \leq 2n, 1 \leq j \leq 2, f(2n + 1) = 3$$

is a proper coloring with three colors, because $f(i + n) = f(i + 1) \neq f(i)$ for $1 \leq i \leq n$ and $f(n + 1) = 2 \neq f(2n + 1)$. So, we conclude that $\chi(H_{3,2n+1}) = 3$, for $n \geq 2$. \hfill $\square$

**Proposition 9.** If $m + 1 \not\mid n$, then

$$\chi(H_{2m+1,2n+1}) = \begin{cases} m + 2, & m \text{ is even, } n \equiv \frac{m}{2} \pmod{m + 2}, \quad n \not\equiv \frac{m}{2} \pmod{m + 1} \\ m + 2, & m \text{ is odd, } n \equiv \frac{m + 1}{2} \pmod{m + 2} \\ m + 1, & m \text{ is even, } n \equiv \frac{m^2}{2} \pmod{m + 1}. \end{cases}$$

Proof. In the first two cases, $m + 1 \not\mid 2n + 1$ so $\chi(H_{2m+1,2n+1}) \geq m + 2$. Now, it is trivial that the coloring function, to modulo $m + 2$, is a proper coloring. So, $\chi = m + 2$. In the third case, the coloring function, to modulo $m + 1$, is proper and so $\chi = m + 1$. \hfill $\square$

3. The strong defining numbers of some Harary graphs

The strong defining numbers and the bounds of defining number of some Harary graphs are subjects that we are really interested in studying now.

**Proposition 10.** If $n \equiv 1 \pmod{m + 1}$, then

$$d(H_{2m,n}, \chi) \leq m + [(n - m)/(m + 1)].$$
Proof. Let \( n = (m + 1)k + 1, k \geq 2 \), and let \( f \) be a coloring function. We know \( \chi = m + 2 \). The set
\[
S = \{1, 2, \ldots, m\} \cup \{(m + 1) + m|1 \leq i \leq k - 1\}
\]
with the following coloring function:
\[
f(i) = i, \quad 1 \leq i \leq m, \quad f(i + 1) = \begin{cases} m + 1, & \text{if } i \text{ is odd} \\ m + 2, & \text{if } i \text{ is even} \end{cases}, \quad 1 \leq i \leq k - 1
\]
is a defining set for \( H_{2m,n} \), because, from the first \( m + 1 \) elements of \( S \), we get the result
\[
f(m + 1) = m + 2, \quad f(m + i) = i - 1, \quad 2 \leq i \leq m.
\]

Now,
\[
f((2m + 2) - i) = m + 1 - i, \quad 2 \leq i \leq m, \quad f(2m + 1) = m + 1 \quad \text{and} \quad f(3m + 2) = m + 2,
\]

imply
\[
f(2m + 2) = m, \quad f(2m + 2 + i) = i, \quad \text{for } 1 \leq i \leq m - 1.
\]

We continue this process. Since \( n = m + (m + 1)(k - 1)/2 \), so two vertices \( n - 1 \) and \( n \) have not been colored, yet. Therefore
\[
f(n) = \begin{cases} m + 2, & \text{if } k \text{ is even} \\ m + 1, & \text{if } k \text{ is odd} \end{cases}, \quad f(n - 1) = m. \quad \square
\]

Remark. We note that in each coloring function of \( H_{2m,n} \) with \( m + 2 \) colors there exists at least a vertex \( i \) such that \( m + 1 \) colors appear in a subset \( T \) of the neighborhood set of \( i \):
\[
N(i) = \{i - m, \ldots, i - 2, i - 1, i + 1, i + 2, \ldots, i + m\}, \quad \text{(mod } n\text{)}.
\]

Let \( S \) be a defining set for \( H_{2m,n} \). One can see that if \( i + j \) or \( i - j \), for some \( j \), is the nearest vertex to \( i \) such that it belongs to \( T - S \), then after the coloring of the vertex \( i \), this vertex can be colored. Suppose that we continue this procedure \( t \) times, by using the first \( m + 1 \) vertices of \( S \). Therefore, \( |T| = t + m + 1 \). Since \( m + 2 \leq t + m + 1 \leq 2m + 1 \), if \( t = m \) (i.e. \( T = N(i) \)), then \( S \) is a strong defining set with the minimum cardinality and hence \( |i - m, i + m| \subseteq S \). Also, we note that after the coloring of the above \( 2m + 1 \) vertices, with a permutation on the color set of these vertices, we can suppose that \( m \) vertices \( i - m, i - m - 1, \ldots, i - 1 \) belong to \( S \) and they take the colors of \( 1, 2, \ldots, m \), respectively and also the vertex \( i + m \) has one of the colors \( m + 1, m + 2 \).

**Theorem 11.** i. If \( n \equiv 1 \pmod{m + 1} \), then
\[
sd(H_{2m,n}, \chi) = m + [(n - m)/(m + 1)].
\]

ii. If \( n \equiv 1 \pmod{m + 1} \) and \( 2m + 1 \leq n \leq 3m + 1 \), then
\[
d(H_{2m,n}, \chi) = m + [(n - m)/(m + 1)].
\]

**Proof.** i. The remark and Proposition 10 imply this.

ii. For every graph \( G \), \( d(G, \chi) \geq \chi - 1 \). If \( 2m + 1 \leq n \leq 3m + 1 \), then \( \chi - 1 = m + 1 = m + [(n - m)/(m + 1)] \); now use Proposition 10 (ii). \( \square \)

By the remark and Proposition 10, we have:

**Lemma 12.** If \( n \neq 3m + 2 \), then
\[
sd(H_{2m,n}, m + 2) \geq m + [(n - m)/(m + 1)].
\]

**Conjecture.** If \( n = 3m + 2 \), then
\[
sd(H_{2m,n}, \chi) = \begin{cases} 2m, & \text{for even } m \\ 2m + 1, & \text{for odd } m. \end{cases}
\]

**Proposition 13.** For every positive integer \( m \) if \( m + 1 \mid n \), then:

i. \( d(H_{2m,n}, \chi) = m \),

ii. \( d(H_{2m,n}, \chi + 1) \leq m + [(n - m)/(m + 1)] \),

iii. \( sd(H_{2m,n}, \chi + 1) = m + [(n - m)/(m + 1)] \).
Proof. Proposition 2 says that $\chi(H_{2m,n}) = m + 1$. Let $n = (m + 1)k$, $k \geq 2$ and let $S$ be a defining set with a defined coloring function $f$ on $H_{2m,n}$.

i. The set $S = \{i | f(i) = i, 1 \leq i \leq m\}$ has minimum cardinality.

ii. The set $S$ given in the Proposition 10 is a defining set and its proof is similar. In this case, the only non-colored vertex is $n$ and we say

$$f(n) = \begin{cases} 
  m + 1, & \text{if } k \text{ is odd} \\
  m + 2, & \text{if } k \text{ is even}.
\end{cases}$$

iii. The set

$$S = \{1, 2, \ldots, m\} \cup \{m + i(m + 1) | 1 \leq i \leq k - 1\}$$

with coloring function

$$f(i) = i, \quad 1 \leq i \leq m, \quad f(m + i(m + 1)) = \begin{cases} 
  m + 1, & \text{if } i \text{ is odd} \\
  m + 2, & \text{if } i \text{ is even}.
\end{cases} \quad 1 \leq i \leq k - 1$$

is a strong defining set for $H_{2m,n}$ with $\chi + 1 = m + 2$ colors. So

$$sd(H_{2m,n}, \chi + 1) = m + \lceil(n - m)/(m + 1)\rceil. \quad \square$$

Proposition 14. Let $n = (m + 1)k + r$, $2 \leq r \leq m$ and $n \neq 3m + 2$. For a strong defining set $S$ of $H_{2m,n}$ if the first $m$-tuple $1, 2, \ldots, m$ of vertices belong to $S$, then there exist at least two vertices $\alpha, \beta \in S - \{1, 2, \ldots, m\}$ such that $|\beta - \alpha| \leq m$.

Proof. Let $\{i | f(i) = i, 1 \leq i \leq m\} \subseteq S$. If for every $\alpha, \beta \in S - \{1, 2, \ldots, m\}$ we have $|\beta - \alpha| = m + 1$, then the vertices between every two consecutive vertices of $S$ take the colors $a, 1, 2, \ldots, m - 1, b$ or $b, 1, 2, \ldots, m - 1, a$ according to the ordering of vertices, where $a, b \in \{m + 1, m + 2\}$. And so the second vertex from the last $(m + 1)$-tuple takes color 1, which is impossible, because this vertex is adjacent to vertex 1 with color 1. Thus the proposition is proved. \quad \square

Proposition 15. If $n \equiv 2 \pmod{3}$, $n \geq 8$, then $d(H_{4,n}, \chi) \leq 2 + (n - 2)/3$.

Proof. Let $n = 3k + 2, k \geq 2$. We have $\chi(H_{4,n}) = 4$ by Proposition 2. Let $k$ be odd. The set

$$S = \{1, 2, 4\} \cup \{3t + 4 | 1 \leq t \leq k - 1\}$$

with the coloring function

$$f(i) = i, \quad i = 1, 2, 4; \quad f(3t + 4) = \begin{cases} 
  1, & \text{if } t \text{ is odd} \\
  2, & \text{if } t \text{ is even}, \quad 1 \leq t \leq k - 1
\end{cases}$$

is a defining set. If $k$ is even, then the set

$$S = \{1, 2, n - 1\} \cup \{3t + 2 | 1 \leq t \leq k - 1\}$$

with the coloring function

$$f(i) = i, \quad i = 1, 2; \quad f(3t + 2) = \begin{cases} 
  3, & \text{if } t \text{ is odd} \\
  4, & \text{if } t \text{ is even}, \quad 1 \leq t \leq k - 1; f(n - 1) = 2$$

is a defining set. \quad \square

Theorem 16. If $n \equiv 2 \pmod{3}$ and $n \geq 8$, then $sd(H_{4,n}, \chi) = 2 + (n - 2)/3$.

Proof. It is obvious by Proposition 15 and the remark. \quad \square

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