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Advances and applications of chance-constrained approaches to systems optimisation under uncertainty
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A chance-constrained optimisation (CCOPT) model has a dual goal: guaranteeing performance as well as system reliability under uncertainty. The beginning of CCOPT methods dates back in the 1950s. Recently, CCOPT approaches are gaining momentum as modern engineering and finance applications are forced to consider reliability and risk metrics at the design and planning stages. Although theoretical development and practical applications have been made, many open problems remain to be addressed in this area. This article attempts to provide a brief survey of major application areas, structure properties, challenges and solution approaches to CCOPT. In particular, we present our research results achieved in the past few years.

Keywords: chance constraints; nonlinear optimisation; dynamic optimisation; back-projection; analytic approximation; sparse-grid integration; quasi-Monte Carlo method

1. Introduction

Engineering and financial applications have to deal with uncertainties. Frequently, it is required to guarantee an optimal and reliable performance in the presence of uncertainties. The optimisation of some performance function under uncertainty is most preferably accomplished through stochastic optimisation (Kall and Wallace 1995; Prékopa 1993, 1995, 2001). The conventional way of solving optimisation problems with uncertainties is by using expected (nominal) values for the uncertain parameters. Nevertheless, the solutions obtained through such simplifications may not be robust and reliable. Therefore, stochastic optimisation methods are essential to consider the full effect of stochastic parameters.

In practical applications, constraint violations are unavoidable due to unexpected and extreme events, measurement errors, etc. In addition, a requirement for deterministic satisfaction of constraints may incur high operating costs. In practice, to cope with inevitable uncertainties, some degree of violations of constraints are tolerated. Therefore, it is reasonable to demand constraint satisfaction with some level of probability. This calls for the use of chance constraints. Chance constraints can be used to specify a guaranteed level of fulfilment of product specifications, a guaranteed level of availability of products or outputs, safety conditions, a degree of fault tolerance, risk-aversion, etc. Therefore, optimal and reliable performance of a given system can be gained by methods of chance constrained optimisation (CCOPT) by demanding satisfaction of constraints with some pre-given reliability (probability) levels.

A CCOPT model was first introduced in the 1950s by Charnes, Cooper, and Symmonds (1958) and Charnes and Cooper (1959) in connection with financial planning problems. Since then, many contributions have been made in the theoretical analysis and computational methods of CCOPT; among these are the outstanding contributions from Prékopa (1973a,b, 1993, 1995, 2001).

Depending on the process under consideration the chance-constrained model can be a steady-state or a dynamic optimisation problem. Since most processes are dynamic in nature, we consider the following form of dynamic CCOPT problems:

\[
\text{(DynCCOPT)} \min_u \left[ \mathbb{E}[J_1(\tilde{x}, x, u, \xi)] + \gamma \text{Var}[J_2(\tilde{x}, x, u, \xi)] \right],
\]

\[\text{s.t. } g(\tilde{x}(t), x(t), u(t), \xi, t) = 0, \quad x(t_0) = x_0, \quad (2)\]

\[
\Pr \left\{ x_{i_{\min}} \leq x_i(t) \leq x_{i_{\max}} \right\} \geq \alpha_i, \quad i \in I, \quad (3)
\]

\[
u \in \mathcal{U}, \quad \xi \in \Omega, \quad (4)
\]
where $x$ represents state variables, $u \in \mathcal{U} = \{v \mid u_{\text{min}} \leq v(t) \leq u_{\text{max}}, \, v(t) \in \mathbb{R}^n \forall t \in [t_0, t_f]\}$ are controls, $\xi \in \mathcal{X} \subseteq \mathbb{R}^p$ are uncertain variables and $I \subset \{1, 2, \ldots, n\}$. The functionals in the objective $J_1, J_2$: $\mathcal{C}([t_0, t_f], \mathbb{R}^p) \times \mathcal{C}([t_0, t_f], \mathbb{R}^n) \times \mathcal{U} \times \mathcal{X} \to \mathbb{R}$ are performance measures and $g$: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^m$ defines the nonlinear dynamic model equations. The operators $\Pr(\cdot), \mathbb{E}(\cdot)$ and $\text{Var}(\cdot)$ represent probability, expected value and variance, respectively, and $\gamma \geq 0$ is a weighting factor. The uncertain variables $\xi$ can be either time-dependent or time-invariant. In general, it is assumed that the uncertain variables $\xi$ have a known distribution. The state variables $x$ are related to the uncertain variables $\xi$ through the model equations. Hence, they are uncertain. In general, the control decisions $u$ are taken to be deterministic.

The chance constraints $\Pr\{x_i^{\min} \leq x_i(t) \leq x_i^{\max}\} \geq \alpha_i, \, i \in I$, represent single or separate chance constraints. Each of these constraints specifies the reliability (probability) level for holding the state trajectory $x_i(t)$ between the bounds $x_i^{\min}$ and $x_i^{\max}$ by a user specified value $\alpha_i \in [\frac{1}{2}, 1]$, separately for each state-constraint. On the other hand, $\Pr\{x_i^{\min} \leq x_i(t) \leq x_i^{\max}, i \in I\} \geq \alpha$ states that all the specified state constraints to hold with the same (jointly given) reliability level $\alpha \in [\frac{1}{2}, 1]$. Note also the equivalence that

$$\Pr\{x_i^{\min} \leq x_i(t) \leq x_i^{\max}\} \geq \alpha$$

$$\equiv \Pr\{x_i(t) < x_i^{\min} \text{ or } x_i(t) > x_i^{\max}\} \leq 1 - \alpha.$$

That is, the risk for the state trajectories to go out of the bound $[x_i^{\min}, x_i^{\max}]$ should not be more than $1 - \alpha_i$. This risk becomes smaller if higher reliability levels ($\alpha$ near 1) are demanded. In this article, we are exclusively interested in single chance constraints and give only a few complementary remarks on joint chance constraints.

The rest of this article is organised as follows. Section 2 surveys briefly major application areas of CCOPT methods. The chance-constrained model predictive scheme is presented in Section 3 as a highly competitive strategy in dealing with nonlinear stochastic dynamic optimisation problems from various areas of application. Section 4 summarises well-known properties of chance constraints as well as major challenges, specifically, associated with nonlinear CCOPT models. Major state-of-the-art and some widely used solution approaches to steady-state CCOPT problems are compactly presented in Section 5. In this section we present a novel analytic approximation strategy for chance constraints (without detailed theoretical analysis) which will be used later in the case-studies. Brief commentaries

2. Some applications areas of CCOPT

Recently, an ever increasing demand for system reliability, fault-tolerance and risk management has motivated a strong interest in CCOPT. Steady improvements of computational capability of computers and the availability of efficient computational algorithms for large-scale deterministic optimisation problems are among the additional driving forces. As a result CCOPT models are widely used in several fields of applied sciences. The list of application areas for CCOPT models is quite long. The literature can be roughly categorised as below considering only some major and accessible contributions. Therefore, the following categories should not be taken as an exhaustive review.

Water resources management: One of the classical applications of chance-constrained models is in water reservoir management. The use of water reservoirs has been long recognised to rectify water shortages and avoid disruption in water supply. However, water reservoirs are highly influenced by seasonal inflow variations as well as demand uncertainties. Therefore, scarcity of water call for efficient management, optimal control and reliable distribution by taking uncertainties into account. This involves the efficient management and control of water reservoirs in water distribution networks to guarantee the reliable satisfaction of forecasted demand (Dupachova, Giavornski, Kos, and Szantai 1991; Prékopa 1995; Sreenivasan and Vedula 1996; Ouarda and Labadie 2001; Labadie 2004; Dhar and Datta 2006; Andrieu, Henrion, and Römisch 2010; Van Ackooij, Henrion, Möller, and Zorgati 2010).

Optimisation of power system operation and planning: Power generation, transmission and supply systems are prone to system inherent uncertainties and external influences. Internal uncertainties such as measurement errors, forecast inaccuracies, outage of system elements, load uncertainties and power losses, hamper reliable demand satisfaction affecting profitability. In addition, recent interest in renewable energy generation, using wind and solar energy, necessitated efficient energy storage and regenerative power input
management strategies, since wind speed and solar intensity are naturally random (Ibanez, Rosell, and Beckman 2003; Karki, Hu, and Billinton 2006; Usalo 2009; Celik, Makkawi, and Muneer 2010).

The work by Zhang and Li (2010, 2011) considers load uncertainties for optimal power dispatch problems with chance constraints on generator capacities, line active power flows, line currents and bus voltage magnitudes. Related studies of Chan et al. (2006), Mazadi, Rosehart, Malik, and Aguado (2009), Yang and Wen (2005) and Yu, Chung, Wong, and Zhang (2009) consider CCOPT for transmission system expansion to cope with future power demand. Unit commitment problems under chance constraints are also used to address temporal uncertainties in energy markets (Anders 1981; Ozturk, Mazumdar, and Norman 2004).

**Process engineering:** Feed variations, measurement errors, inaccuracies in model parameters propagate through the process and cause random disturbances on the output of the process. As a result product quality specifications, safety conditions and process constraints can be violated. For this reason stochastic optimisation approaches can play a key role to gain robustness of process engineering applications (e.g. Sahinidis 2004). The work in Diwekar (2003) and Shastri and Diwekar (2006) consider CCOPT problems, using nonlinear optimisation algorithms coupled with sampling techniques (Diwekar and Kalagnanam 1997; Kalagnanam and Diwekar 1997; Kim and Diwekar 2002). These studies along with other related investigations by Bernardo, Pistikopolous, and Saraiva (1999, 2001) and Rooney and Biegler (1999, 2001) are mainly related to planning and design problems using steady-state process models.


**Financial risk management and portfolio optimisation:** In order to cope with market risks caused by random changes in commodity prices, interest rates, exchange rates, etc., financial institutions use risk management strategies like value-at-risk (VaR) and conditional value-at-risk (cVaR). VaR is used to measure the amount that may be lost in a portfolio over a given period of time; while cVaR measures conditional expected losses for the loss exceeding VaR. VaR constraints are found to provide lower estimates for cVaR constraints (Rockafellar and Uryasev 2000, 2002). The presence of a VaR constraint in an optimisation problem leads to a non-convex CCOPT problem. Tractability of such problems can be improved through convex approximation strategies as suggested by Nemirovski and Shapiro (2006a), Wozabal (2010) and Wozabal, Hochreiter, and Pflug (2010), etc.

**Risk metrics** like VaR and cVaR or chance constraint methods can be interchangeably used in portfolio selection and optimisation (Ermoliev, Ermoliev, Macdonald, and Norkin 2000; Henrion 2006; Wang, Chen, and Zhang 2007; Bonami and Lejeune 2009; Chen, Sim, Sun, and Teo 2010, etc.) to help make decisions of risk diversification as well as for efficient hedging strategies against risk. Chance-constrained methods in production planning, budget management and control, etc., provide strategies to cope with probable shortfalls and to guarantee optimal system performance (Charnes and Cooper 1959; Parlar 1984; Gurgur and Luxhoj 2003; Lejeune and Ruszczyński 2007; Verderame and Floudas 2010).

**Reliability-based design optimisation:** In engineering design uncertainties in material properties, modelling errors, measurement errors, variability in external conditions such as temperature, loading, etc, seriously affect the performance characteristics of the design. These design parameters are usually considered as random and are assumed to have certain probability distributions. The field of reliability-based design optimisation (RBDO) studies engineering design problems. The major target in RBDO is to determine design strategies that yield higher reliability at lower design costs; e.g. to minimise material costs with constraints on the probability of failure. A greater reliability level is obtained by demanding a smaller probability of failure. Here, chance constraints can be interpreted as measures of safety.

Application areas of RBDO are mainly structural designs such as wind turbine design and aircraft structure designs. Problems in these areas are usually large-scale with multiple uncertain variables (Royset and Polak 2004; Royset, Kiureghian, and Polak 2006).

**Control and guidance of unmanned robots:** Control of unmanned aerial and ground vehicle (UAV and
UGV, respectively) systems have aroused profound interest in probabilistic methods (Thrun, Burgard, and Fox 2005). Inaccuracies in model dynamics, sensor errors, environment uncertainties, etc., have severe impact on the motion and targeting of unmanned vehicles. Field applications of unmanned vehicles bring additional exposures to rough terrains, stationary and randomly moving obstacles, etc. Therefore, optimal robot path tracking and reliable collision avoidance strategies are highly demanding for safer robot navigation and trajectory-planning problems (Blackmore and Ono 2009; Blackmore, Ono, Bektassov, and Williams 2010; Hess and Sattel 2011; Lyons, Calliessy, and Hanebec 2005).

In general, the nature and characteristics of processes input uncertainties, random errors and disturbances vary with the type of application being considered. Independent random variables are simpler to handle. Strongly correlated input uncertainties are more difficult to deal with. When input uncertainties posses joint (Gaussian) normal distributions, de-correlation of the random variables can be achieved by standardisation using variables transformation. Nevertheless, recent research has shown that distribution of process input uncertainties can be non-Gaussian; e.g. wind speed frequency in wind power generation (Celik et al. 2010), clearness index in solar energy (Ibanez et al. 2003), etc., are found to be non-Gaussian. These and other findings show that there are input uncertainties that can be fairly described with multimodal and multi-parametric distributions (see also Kim and Diwekar 2002; Diwekar 2003; Chen, Julian, and Elaine 2006; Vasquez, Whiting, and Meerschaert 2010).

In most applications, input uncertainties are assumed to be continuously distributed. There are relatively a few studies on CCOPT models with discrete input random variables (Prékopa, Vizvári, and Badics 1998; Dentecheva, Prékopa, and Ruszczyński 2000; Ruszczyński 2002; Luedtke, Ahmed, and Nemhauser 2010).

In a complex dynamic process with uncertain disturbances it is difficult to predict future outcomes exactly. While steady-state processes always possess time-independent uncertain variables, in dynamic processes there may be both time-independent and time-dependent uncertain variables.

The solution of dynamic optimisation problems under chance constraints presents enormous computational difficulties, especially when time-dependent uncertain variables are considered. This difficulty becomes more complicated when the objective function is stochastic. The dynamics of most processes are commonly described by a large set of differential (algebraic) equations. In order to optimally control such processes along reference trajectories, the model predictive control (MPC) scheme provides an efficient control mechanism (Allgöwer, Findeisen, and Nagy 2004; Findeisen, Imsland, Allgöwer, and Foss 2003; Mayne, Rawlings, Rao, and Scokaert 2000). In the MPC scheme, the dynamic model equations are discretised in time intervals. Time-dependent input uncertainties also need to be discretised on the considered time intervals. Then optimisation problems, with large number of equality constraints, are solved on a finite moving horizon. On each prediction horizon, the solution of the corresponding optimisation problem yields the optimal control profiles. However, due to input uncertainties future process outputs cannot be predicted exactly. There can be a big variation between the predicted and realistic output reference trajectories. In deterministic MPC approaches, uncertain input variables are either considered as small amplitude noises that can be corrected by the controller or are taken as being constants with nominal values. In this way, due to input uncertainties future process outputs cannot be predicted exactly and constraints on output variables can be violated (Scokaert and Rawlings 1999; Scokaert et al. 1999; Grimm, Messian, Tuna, and Teel 2004; Tatjewski 2007). Several studies (Tatjewski 2007; Cannon, Kouvaritakis, and Wu 2009b, etc.) use soft-constraints on output variables in order to accommodate for possible infeasibility, but such considerations do not exclusively allow the user to specify reliability levels. Therefore, chance-constrained MPC techniques provide a framework to find optimal control strategies to guide the process along predicted trajectories with higher reliability of holding product specifications and process constraints.

The studies of Li and coworkers (Li et al. 2000, 2002a,b; Li, Hui, Li, and Li 2004b) consider linear chance-constrained MPC partly with time-independent uncertain input variables (see also Schwarm and Nikolaiou 1999; Cannon, Ng, and Kouvaritakis 2009a; Cannon et al. 2009b) for chance-constrained MPC. Uncertain time-discretised process input variables are usually assumed to have a Gaussian (normal) distribution. Since most practical processes have non-linear model equations, the investigations of Li and associates (Henrion et al. 2001; Wendt et al. 2002; Flemming et al. 2007; Xie, Li, and Wozny 2007; Li et al. 2008; Klöppel et al. 2011) apply chance-constrained nonlinear MPC (NMPC) schemes.
The model predictive formulation of a chance-constrained dynamic optimisation problem takes the form

\[(\text{CCNMPC}) \quad \min_u \left\{ E[f_1(x_k, u_k, \xi)] + \gamma \text{Var}[f_2(x_k, u_k, \xi)] \right\} \]

\[\text{s.t. } x_{k+1} = g(x_k, u_k, \xi), \quad x(0) = x_0, \]

\[\Pr\{x_{\text{min}} \leq x_{i,j} \leq x_{\text{max}}\} \geq \alpha_i, \quad i \in I, \]

\[u_k \in \mathcal{U}, \quad \xi \in \Omega, \quad k = 0, 1, \ldots, N - 1,\]

where the variable set is similar to the one given for the problem DynCCOPT in Equations (1)–(5), except that we have here a discrete dynamical system. Consequently, after appropriate aggregation of variables, on each prediction horizon a CCOPT problem of the form

\[(\text{CCOPT}) \quad \min_u \left\{ E[f_1(x, u, \xi)] + \gamma \text{Var}[f_2(x, u, \xi)] \right\} \]

\[\text{s.t. } G(x, u, \xi) = 0, \]

\[\Pr\{x_{\text{min}} \leq x_i(u, \xi) \leq x_{\text{max}}\} \geq \alpha_i, \quad i \in I, \]

\[u \in \mathcal{U}, \quad \xi \in \Omega,\]

needs to be solved.

The numerical tractability of CCOPT depends on its structural properties. In general, joint CCOPT problems are known to pose more difficulties than those with single chance constraints. As indicated in Equation (12) the state variables \(x(u, \xi)\) are related to the controls \(u\) and the uncertainties \(\xi\) through the model equation \(G(x, u, \xi) = 0\). Therefore, by projecting the problem CCOPT on the solution set of the nonlinear system of equations \(G(x, u, \xi) = 0\), the problem above can be written as

\[(\text{CCOPT}) \quad \min_u \left\{ E[f_1(x, u, \xi)] + \gamma \text{Var}[f_2(x, u, \xi)] \right\} \]

\[\text{s.t. } \Pr\{c(u, \xi) \leq 0\} \geq \alpha_i, \quad i \in I, \]

\[u \in \mathcal{U},\]

for brevity of notations and using the same name of CCOPT without distinguishing it from the former one. Define the probability functions \(h_i(u) := \Pr\{c(u, \xi) \leq 0\}, \quad i \in I\). When attempting to solve problems of the form CCOPT, it is helpful to know if each \(h_i(u)\) has a direct analytic representations as a function of \(u\); if each \(h_i(u)\) is convex and differentiable. The convexity property is a central topic in optimisation, since it guarantees global optimality of local optimal solutions. Local optimal solution can be obtained by using gradient-based optimisation algorithms. In fact, almost all efficient large-scale optimisation solvers are gradient-based. Therefore, the knowledge of differentiability of \(h\) also plays a central role in the solution of CCOPT models.

4. Properties and challenges of CCOPT problems

A knowledge of the structural properties of a CCOPT problem facilitates the selection of solution algorithms as well as the interpretation of results obtained through the chosen algorithm. Properties like convexity, differentiability, feasibility, etc., are most important properties of an optimisation problem. In general, for a given probability level \(\alpha\), an \textit{a priori} verification of feasibility of chance constraints can be a difficult task. Therefore, the optimisation solver for CCOPT may need to be capable of working with infeasible start points.

4.1. Deterministic representations and convexity properties of chance constraints

Some special instances of chance constraints can have exact analytic (deterministic) representations explicitly as functions of \(u\) when the uncertain variables \(\xi\) are independently normally distributed. Simplifications of these types are frequently used in several applications (see, Schwarm and Nikolaou 1999; Bonami and Lejeune 2009). Except for such special cases, analytic representations are difficult to obtain even for linear forms like:

- chance constraints with additive uncertainties: \(h(u) = \Pr\{A_0u + B_0\xi + d \leq 0\} \geq \alpha\) (such constraints are prevalent in chance-constrained linear MPC problems; e.g. Oldewurtel, Jones, and Morari 2008; Blackmore and Ono 2009; Cannon et al. 2009a, b, etc.);
- chance constraints with linear matrix inequalities: \(h(u) = \Pr\{A_0u + \sum_{k=1}^{p} \xi_k A_k[u] \leq 0\}\) (e.g. Kothare, Balakrishnan, and Morari 1996; Ben-Tal and Nemirovski 1998; Nemirovski and Shapiro 2006b);
- chance constraints with multiplicative uncertainty: \(h(u) = \Pr\{A(\xi)u + d \leq 0\}\), where \(A(\xi)\) is a random matrix (e.g. Prékopa, Yoda, and Subasi 2011).

In practice, tractability of these constraints is attained through approximation strategies for independent random variables with normal distribution or for bounded random variables. Little is known about
analytic representations in the presence of general non-Gaussian input uncertainties.

Convexity properties of chance constraints are dependent not only on the convexity of the defining functions, but also on the type of distribution function of the uncertain input variables \( \xi \). In the 1970s, Prékopa introduced the notion of logarithmic-concave (log-concave) probability measures (Prékopa 1973b). This has been the basis for the study of convexity properties of chance constraints (Prékopa 1993, 1995). Recently Prékopa et al. (2011) have provided generalisations of some earlier convexity results (see also Henrion and Strugarek 2008).

4.2. Nonlinear chance constraints

Nonlinear chance constraints are basically hard to deal with. Frequently, such constraints do not have direct deterministic representations and convexity properties are also rarely available. Sometimes, for very large and complex systems (as in power flow (PF) problems considered by Zhang and Li (2010, 2011) and time critical and fast changing processes, linearisation of the considered by Zhang and Li (2010, 2011) and time

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Difficulties.

As in Equation (12) pose major computational difficulties.

Now, by further defining \( F_1(u) := E[f_1(x(u, \xi), u, \xi)] \) and \( F_2(u) := \text{Var}[f_2(x(u, \xi), u, \xi)] \), we can write the CCOPT problem as a nonlinear programming problem (NLP):

\[
\text{(NLP)} \quad \min_u \left\{ F_1(u) + \gamma F_2(u) \right\} \\
\text{s.t. } h_i(u) - \alpha_i \geq 0, \quad i \in I, \\
\quad u \in \mathcal{U}. 
\]

The solution of NLP can be done numerically by using either a gradient-based or a gradient-free optimisation algorithm or a combination of both. Hence, it should be possible to compute objective function values \( F_1(u), F_2(u) \) and values of the constraints \( h_i(u), i \in I \). Gradient-based algorithms further require the gradients \( \nabla F_1(u), \nabla F_2(u) \) and \( \nabla h_i(u), i \in I \), for each given value of \( u \in \mathcal{U} \) when these derivatives exist.

In fact, the major challenge in the numerical solution of nonlinear CCOPT problems is the computation of probability values of chance constraints and their gradients. These fundamental difficulties commonly arise when the uncertain (input) variables and model (output) variables have nonlinear relations. In the case of known differentiability of \( h_i(u), i \in I \), the derivative formulae of chance constraints from Uryasev (1995) or of Marti (1996) are more of theoretical nature and are quite complicated to be used for practical computations. For a set of linear chance constraints the derivative formulae of Van Ackooij et al. (2010) and Prékopa et al. (2011) are practical. Differentiability of the expected values \( F_1(u) \) and \( F_2(u) \) can also be guaranteed by a compactness assumption on the set \( \Omega \) of the uncertain variables \( \xi \) (see, e.g. Qi, Shapiro, and Ling 2005).

Frequently, the computation of values and derivatives of chance constraints and objective functions is attained through the evaluation of multidimensional probability integrals. Since optimisation is done with respect to the control variable \( u \), these values need to be computed at each step of the optimisation algorithm. Therefore, the efficiency of the solution strategy is highly dependent on how fast and accurate these integrals are evaluated (see Section 6).

4.3. Optimality vs. feasibility

Another very important issue in CCOPT is the question of optimality of the problem versus the demand for higher reliability. This can be studied by investigating the parametric problem

\[
\varphi(\alpha) = \min_u (F_1(u) + \gamma F_2(u)),
\]

s.t. \( u \in \mathcal{M}(\alpha) = \{u \in \mathcal{U} \mid h_i(u) \geq \alpha_i, i \in I\}, \]

where \( \alpha = (\alpha_1, \ldots, \alpha_n) \in (0, 1)^n \) and \( n_1 = |I| \). Note that if \( \alpha^{(1)} \leq \alpha^{(2)} \) lexicographically, then \( \mathcal{M}(\alpha^{(1)}) \supseteq \mathcal{M}(\alpha^{(2)}) \) and \( \varphi(\alpha^{(1)}) \leq \varphi(\alpha^{(2)}) \). This implies that increasing the reliability level increases the minimum value of the objective function, since a continuous increase in the reliability level \( \alpha \) results in a shrink on the feasible set. In many cases, this leads to a maximum reliability level \( \alpha^* \in (0, 1) \) beyond which it is not possible to hold the chance constraints. Hence, there is a maximum \( \alpha^* \) for which the problem remains feasible and the feasible set \( \mathcal{M}(\alpha) \) becomes empty for \( \alpha > \alpha^* \). Therefore, the choice of \( \alpha \) is a trade-off (compromise) between expected cost reduction and the level of reliability. The determination of the maximum attainable reliability level \( \alpha^* \), without causing infeasibility, is by itself an optimisation problem. Li and group (Li et al. 2002a; Geletu, Hoffmann, Klöppel, and Li 2011) investigated the feasibility issues of chance-constrained linear and nonlinear optimisation problems to determine the maximum confidence level \( \alpha \).

5. Approaches to CCOPT

The literature on the numerical solution of nonlinear CCOPT models suffers from a deficit, despite the theoretical guarantee of convexity for various
distributions (Prékopa 2001; Prékopa et al. 2011; Henrion and Strugarek 2008). Basically, considering only non-heuristic methods, the following computational approaches to nonlinear CCOPT problems can be identified.

5.1. Sampling approaches

The robust optimisation techniques (Ben-Tal and Nemirovski 1998; Ben-Tal, El Ghaoui, and Nemirovski 2009) consider the worst-case problem (Nemirovski 2009) and the worst-case problem (Henrion and Strugarek 2008) for nonlinear CCOPT problems can only non-heuristic methods, the following computational approaches to nonlinear CCOPT problems can be identified. While convexity structures can be preserved and the resulting optimisation problem can be simple to implement and solve, a very large number of scenarios are required only to warrant approximate feasibility of the obtained solutions to the original problem. In this way, higher reliability of satisfaction of constraints can be computationally expensive. Scenario generation methods (Califore and Campi 2005, 2010; Henrion et al. 2001) are among such approaches. Despite the fact that these approaches are applicable irrespective of the type of distribution function of the uncertain variables (Gaussian or non-Gaussian), the requirement of feasibility of constraints for almost all possible realisations of the uncertain variables leads to a conservative approach and a deterioration in the reliability of satisfaction of constraints for generated samples to the original problem. In this way, higher reliability of satisfaction of constraints can be computationally expensive. Scenario generation methods (Henrion, Küchler, and Römisch 2009; Campi and Garetti 2011) show the potential to provide some improvements of computational burdens.

On the other hand sample average approximation (SAA) techniques (Kookos 2003; Royston and Polak 2004; Wang and Ahmed 2008; Pagnoncelli, Ahmed, and Shapiro 2009, 2011, etc.) are based on the generation of uniformly distributed deterministic (QMC) samples with low-discrepancy properties. The SAA approach replaces a chance constraint \Pr\{c(u, \xi) \leq 0\} by a relative-frequency count of satisfaction of this constraint for generated samples \{\xi^1, \ldots, \xi^N\} \subset \Omega; i.e.

\[
C_N(u) = \frac{1}{N} \sum_{k=1}^{N} \mathbb{I}_{(-\infty,0]}(c(u, \xi^k)) \geq \alpha,
\]

where

\[
\mathbb{I}_{(-\infty,0]}(\xi) = \begin{cases} 1, & \text{if } \xi \leq 0 \\ 0, & \text{if } \xi > 0 \end{cases}.
\]

Despite the fact that the SAA method avoids the need to compute multidimensional integrals, there are potential difficulties associated with it: it requires a very large sample size to yield estimates for the probability constraints; solution obtained through the SAA approach may fail to be feasible to the original problem, and the discrete approximations of the chance constraints lead to a combinatorial optimisation problem.

5.2. The back-mapping approach

The back-mapping approach transforms chance constraints on output variables into the space of uncertain input variables by using monotonic relations (Wendt et al. 2002). Consider now a chance constraint

\[
\Pr\{c(u, \xi) \leq 0\} \geq \alpha,
\]

where the distribution of the random variable \(z = c(u, \xi)\) is difficult to directly compute from that of \(\xi\). To facilitate the computation of this chance constraint using the back-mapping approach, among \((\xi_1, \xi_2, \ldots, \xi_p)\) find a \(\xi_j\) that has a strict monotonic relation with \(z\); so that \(z = \psi_{u,i}(\xi_j)\). Then

\[
\xi_j \uparrow z \text{ implies } \Pr\{c(u, \xi) \leq 0\} = \Pr\{\xi_j \leq \psi_{u,i}^{-1}(0)\} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi(\xi)\,d\xi.
\]

(23)

\[
\xi_j \downarrow z \text{ implies } \Pr\{c(u, \xi) \leq 0\} = \Pr\{\xi_j \geq \psi_{u,i}^{-1}(0)\} = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi(\xi)\,d\xi.
\]

(24)

In Equations (23) and (24) integrals are given in terms of the density function \(\phi(\xi)\).

The required monotonic relation, between a chance-constrained output (state) variable and one uncertain input variable, can be determined from experience, through a simulation step or by using a mathematical analysis technique. For instance, Flemming et al. (2007), Li et al. (2002a, 2008) and Wendt et al. (2002) identify monotonic relations by a simulation step carried on model equations of process engineering optimisation problems (similar ideas are also used in Arelano-Garcia and Wozny 2009; Arelano-Garcia, Martini, Wendt, Li, and Wozny 2003; Arelano-Garcia et al. 2006, 2007). Moreover, the mathematical analysis of monotonic relations can be done through the so-called global implicit function
corresponds to an expected value in the space with density function

\[ 
\phi(\xi) = \phi_1(\xi_1) \cdot \phi_2(\xi_2) \cdots \phi_p(\xi_p) = \prod_{i=1}^{p} \phi_i(\xi_i), 
\]

the integral in Equation (23) can be written as

\[
\Pr\{G(u, \xi) \leq 0\} = \Pr\{\xi \leq \varphi^{-1}_u(0)\} 
= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} \left[ \int_{-\infty}^{\varphi^{-1}_u(0)} \phi(\xi_j)d\xi_j \right] \phi(\xi)d\xi, \quad (25)
\]

where \( \tilde{\xi} \) is as defined above and \( \phi(\tilde{\xi}) = \prod_{i=1}^{p} \phi_i(\xi_i) \).

Define

\[
M(u, \tilde{\xi}) := \left[ \int_{-\infty}^{\varphi^{-1}_u(0)} \phi(\xi_j)d\xi_j \right]
\]

so that

\[
\Pr\{G(u, \xi) \leq 0\} = \Pr\{\xi \leq \varphi^{-1}_u(0)\} 
= \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} M(u, \tilde{\xi}) \phi(\tilde{\xi})d\tilde{\xi} = E[M(u, \tilde{\xi})], \quad (26)
\]

This indicates that the value of the chance constraint corresponds to an expected value in the space \( \mathbb{R}^p \).

The back-mapping approach is usable if monotonic relations are easy to find and provides a direct representation of chance constraints in terms of the input uncertain variables with known distributions. As a result, the chance constraints can be computed by multidimensional integration on the constrained input space. While models from process engineering applications frequently display monotonic relations, owing to the nature of this specific application area, it is unrealistic to expect the existence of monotonic relations in all other applications. In fact, the existence of a monotonic relation might be difficult to verify and such relations may not in general exist. Under such circumstance, the analytic approximation of chance constraints can serve as a more general alternative to the back-projection approach.

5.3. Analytic approximation strategies

Analytic approximation strategies can be relaxation of constraints to accommodate for possible violation of constraints due to uncertainties or replacing the chance constraint with an approximation.

Some relaxation strategies replace chance constraints with bounding confidence regions that are easy to derive for normally distributed uncertainties. These considerations are mainly confined to linear process model equations. The majority of these approaches replace chance constraints with quadratic constraints (Cannon et al. 2009a,b). Frequently, bounding constraints are used to accommodate for maximum likely violation of constraints on output variables and are usually directly used without any mention of chance constraints (Rooney and Biegler 1999, 2001; Heine, Kawohl, and King 2006). Such state estimation or approximation methods use the estimated mean and covariance matrix of the state variables to set up confidence regions. These strategies are also widely used in CCOPT of unmanned vehicle navigation systems (Blackmore and Ono 2009; Blackmore et al. 2010). Nevertheless, when disturbance variables are non-Gaussian, their mean and covariance may not have sufficient information to tell about the nature of the state variables. In addition, these approaches have the danger of either over or under estimation of constraints, leading to conservative or unreliable approximations. Therefore, a numerical solution scheme that facilitates easier evaluation of chance constraints at the same time guaranteeing feasibility of obtained solutions is currently quite demanding for the treatment of general nonlinear CCOPT problems.

The concept of analytic approximations of chance constraints is based on probability inequalities. Pinter (1989) was the first to propose such approximation schemes. Recently Nemirovski and Shapiro (2006a) suggested a convex approximation of chance constraints using a parametrised function. Recall now the equivalent representation of chance constraints as

\[
\Pr\{c(u, \xi) \leq 0\} \geq \alpha \equiv \Pr\{c(u, \xi) > 0\} \leq 1 - \alpha.
\]

Define the function

\[
h(u, \xi) := \begin{cases} 
0, & \text{if } c(u, \xi) < 0, \\
1, & \text{if } c(u, \xi) \geq 0. 
\end{cases}
\]

Then it is easy to show that \( \Pr\{c(u, \xi) > 0\} = E[h(u, \xi)] \) so that

\[
\Pr\{c(u, \xi) \leq 0\} \geq \alpha \equiv \Pr\{c(u, \xi) > 0\} \leq 1 - \alpha \equiv E[h(u, \xi)] \leq 1 - \alpha.
\]

The constraint \( E[h(u, \xi)] \leq 1 - \alpha \) provides an exact representation of the chance constraint. Hence, the problem CCOPT can be equivalently written as an expected-value constrained optimisation problem

\[
(\text{ENLP}) \min_u E[f(u, \xi)]
\]

(27)
But, the function $h$ is discontinuous and is not amenable for numerical computations. Consequently, convenient approximation strategies could facilitate the solution of the problem ENLP. The analytic approximations approach replaces the discontinuous function $E[h(u, \xi)]$ by a continuous (possibly smooth) parametric function $\psi(\tau, u)$, with a scalar parameter $\tau > 0$, satisfying

\begin{equation}
E[h(u, \xi)] \leq \psi(\tau, u), \quad \text{for all } \tau > 0.
\end{equation}

A good choice for $\psi$ is if it satisfies

\[
\inf_{\tau > 0} (\psi(\tau, u) - E[h(u, \xi)]) = 0.
\]

Basically, $\psi(\tau, u) := E[\Psi(\tau, c(u, \xi))]$ for some given function $\Psi: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$. In the literature, there are some suggestions for $\Psi$ in order to set up analytic approximations:

- $\Psi(c(u, \xi), \tau) = \exp(\tau c(u, \xi))$, $\tau > 0$ (Pinter 1989);
- $\Psi(c(u, \xi), \tau) = \exp(\tau^{-1} c(u, \xi))$, $\tau > 0$ (Nemirovski and Shapiro 2006a);
- $\Psi(c(u, \xi), \tau) = \tau + \frac{1}{\tau - 1} [c(u, \xi) - \tau, 0]_+$, $\tau > 0$ (Rockafellar and Uryasev 2000), where $[s, \tau, 0]_+ = \max(s - \tau, 0)$.

Therefore, we solve the following optimisation problem:

\[
(NLP)_\tau \quad \min_u E[f(u, \xi)]
\]

\[
\text{s.t. } \inf_{\tau > 0} \psi(\tau, u) \leq 1 - \alpha.
\]

Obviously, from the relation (29), any optimal solution of the problem $(NLP)_\tau$ is a feasible point of the CCOPT problem.

The approximation techniques of Nemirovski and Shapiro (2006a) uses the function $\psi(\tau, u) = \tau E[\Psi(\tau, c(u, \xi))] = \tau E[\exp(\tau^{-1} c(u, \xi))]$ which is convex if $c(u, \xi)$ is convex w.r.t. $u$. Hence, the constraint $\inf_{\tau > 0} \psi(\tau, u) \leq 1 - \alpha$ provides a convex approximation for the chance constraint $E[h(u, \xi)] \leq 1 - \alpha$. In this case a global solution of $(NLP)_\tau$ can be guaranteed, but a detailed numerical study remains still to be done on the quality of this approximation strategy. The strategy of Rockafellar and Uryasev (2000) uses a continuous piecewise-differentiable function. Recently, this has been used to approximate VaR and cVaR constraints through difference-convex optimisation problems (Wozabal 2010; Wozabal et al. 2010).

We suggest here a novel idea for the analytic approximation of chance constraints, so that the inf-constraint $\inf_{\tau > 0} \psi(\tau, u) \leq 1 - \alpha$ is easier to handle. We propose to use a function $\psi(\tau, u)$ that is monotonically decreasing w.r.t. the parameter $\tau$ and still satisfying the relation (29). Then it follows

\[
\inf_{\tau > 0} \psi(\tau, u) = \lim_{\tau \downarrow 0^+} \psi(\tau, u).
\]

Under this assumption it is possible to consider the solution of the problem $(NLP)_\tau$ for a decreasing sequence $\{\tau_k\}_{k \in \mathbb{N}}$ of parameters, where $\tau_k \downarrow 0^+$. In particular, for a sufficiently small and fixed value $\tau_k$, the solution of the problem

\[
(NLP)_\tau \quad \min_u E[f(u, \xi)]
\]

\[
\text{s.t. } \psi(u, \tau_k) \leq 1 - \alpha
\]

provides a good approximation to the solution of CCOPT. The function

\[
\psi(u, \tau) = E[\Psi(\tau, G(u, \xi))] = E \left[ \frac{1 + m_1 \tau}{1 + m_2 \tau \exp(-\frac{1}{4} G(u, \xi))} \right]
\]

(30)

with $m_2 < m_1$ and $0 < \tau < 1$, is monotonically decreasing w.r.t. $\tau$, also satisfies the relation (29).

Figure 1 shows a comparison of the function $\psi(\tau, s) = \exp(\tau s)$ used in the analytic approximation of Nemirovski and Shapiro (2006a) and the function $\psi(\tau, s) = \frac{1 + m_2 \tau}{1 + \tau \exp(-\frac{1}{4} G(u, \xi))}$ which is related to the approximating function in Equation (30), for $-\infty < s < +\infty$. It is easy to see that the latter function provides a smooth and closer approximation to the function

\[
\mathbb{I}_{[0, +\infty)}(s) = \begin{cases} 0, & \text{if } s < 0, \\ 1, & \text{if } s \geq 0. \end{cases}
\]

In fact, despite its preservation of convexity of chance constraints, the approximation proposed by Nemirovski and Shapiro (2006a) has shown poor performance in our numerical experimentations.

![Figure 1. Comparison of the functions.](image-url)
It can be seen that, as \( t \) decreases, the approximated profile approaches near to the step function \( \mathbb{1}_{[0, +\infty)} \). A detailed theoretical analysis of this new approach is beyond the scope as well as the size of this article and thus will be presented elsewhere.

6. Numerical methods for computing multidimensional integrals

As indicated in Section 4, the computation of values and gradients of constraints and objective functions require the evaluation of multidimensional integrals of the form

\[
E_u[f] = \int_\Omega \cdots \int f(u, \xi) \phi(\xi) \, d\xi.
\]

Since optimisation of CCOPT is done w.r.t. the variable \( u \), it is compulsory to evaluate a set of integrals at each step of the optimisation algorithm. This, coupled with the size of the dimension of the probability integrals, renders numerical solution of CCOPT problems resource intensive and computationally expensive. Therefore, the overall optimisation strategy for CCOPT is highly dependent on numerical methods for the computation of multidimensional probability integrals. Hence, it is imperative to look for efficient methods for multidimensional integrals with low number of function evaluation and higher degree of accuracy.

Since the integration in Equation (31) is done only w.r.t. \( \xi \), we drop the parametrisations with \( u \) and consider the integral \( E[f] \) by a weighted sum as

\[
E[f] = \int_\Omega \cdots \int f(\xi) \phi(\xi) \, d\xi \approx \sum_{i=1}^N w_i f(\xi(i)).
\]

Major computation methods of multidimensional integrals can be classified as: deterministic cubature rules and sampling-based methods such as MC and QMC methods. All these methods approximate the integral \( I[f] \) by a weighted sum as

\[
I[f] = \int_\Omega \cdots \int f(\xi) \phi(\xi) \, d\xi \approx \sum_{i=1}^N w_i f(\xi(i)).
\]

The difference in multidimensional integration methods lies in the strategy of selection (construction) of integration weights \( w_i \), \( i = 1, \ldots, N \) and integration nodes \( \xi(i) \in \Omega \subset \mathbb{R}^p \), \( i = 1, \ldots, N \).

6.1. Cubature rules for multidimensional probability integrals

Deterministic cubature rules are usually constructed from one-dimensional quadrature rules based on the weight (probability density) function \( \phi \) and integration domain \( \Omega \). In general, the following are central assumption in numerical methods of multidimensional integrals. The domain of integration \( \Omega \) is the product of \( p \) identical sets

\[
\Omega = \prod_{k=1}^p \Omega_k, \quad \text{where} \quad \Omega_1 = \Omega_2 = \cdots = \Omega_p
\]

and each \( \Omega_k \subset \mathbb{R} \) is either a finite or an infinite interval. And, the joint probability density function \( \phi(\xi) \) is positive and can be written as

\[
\phi(\xi) = \phi(\xi_1, \xi_2, \ldots, \xi_p) = \prod_{i=1}^p \phi_i(\xi_i),
\]

where each \( \phi_i : \mathbb{R} \rightarrow [0, \infty) \) is a positive one-dimensional probability density function. This assumptions holds true if the variables \( \xi_1, \ldots, \xi_p \) are independent random variables.

When the random variables \( \xi_1, \ldots, \xi_p \) are correlated it might be necessary to de-correlate them. For instance, normally distributed random variables with a regular covariance matrix can be easily de-correlated (see, e.g. Geletu et al. 2011). Unless in such simple cases, de-correlation of general random variables is not always possible. In fact, even if only partial de-correlation is possible, this can be of advantage.

6.1.1. Full-grid integration rules

Based on the assumptions above, let for each \( k \in \{1, 2, \ldots, p\} \), \( X_k = \{\xi^{(k)}_1, \xi^{(k)}_2, \ldots, \xi^{(k)}_{N_k}\} \subset \Omega_k \) is a set of quadrature node and \( w^{(k)}_1, w^{(k)}_2, \ldots, w^{(k)}_{N_k} \) are corresponding weights for the computation of the one-dimensional integral \( \int_{\Omega_k} f(\xi) \phi_i(\xi) \, d\xi \) on the set \( \Omega_k \) with the weight function \( \phi_i \); so that \( \sum_{i=1}^{N_k} w^{(k)}_i f(\xi^{(k)}_i) = Q^{(k)}_I[f] \). Among most efficient techniques for the generation of quadrature nodes and weights are Gauss-quadrature rules (Gautschi 2004), extensions of Gauss-quadrature rules (Kronrod 1965; Patterson 1968) and Clenshaw–Curtis quadrature rules (Trefethen 2008).

A full-grid cubature rule for the approximation of the multidimensional integral \( I[f] \) is

\[
Q[I] = \left( Q^{(1)}_I \otimes Q^{(2)}_I \otimes \cdots \otimes Q^{(p)}_I \right)[f]
\]

\[
= \sum_{k_1=1}^{N_1} \sum_{k_2=1}^{N_2} \cdots \sum_{k_p=1}^{N_p} \left( w^{(1)}_{k_1} w^{(2)}_{k_2} \cdots w^{(p)}_{k_p} \right) f(\xi^{(1)}_{k_1}, \xi^{(2)}_{k_2}, \ldots, \xi^{(p)}_{k_p}).
\]

This is also known as a full-grid tensor product of one-dimensional quadrature rules, \( Q^{(k)}_I, k = 1, \ldots, p \), or product rule. The number of nodes in the full-grid rule \( Q[I] \) is given by \( N_1 \times N_2 \times \cdots \times N_p \). If each
one-dimensional quadrature rule uses equal number of nodes; i.e. \( N_1 = N_2 = \cdots = N_p = N \), then the number of nodes \( Q(N) \) will be \( N^p \). This leads to a large number of function evaluations in the numerical solution of stochastic optimisation problems. Möller (1976), Mysovskikh (1968) and Tchakaloff (1957) indicated redundancy in the full-grid rule, so that only a subset of the \( N^p \) nodes are enough to obtain a good approximation of multidimensional integrals.

The largest \( p \)-dimensional polynomial of degree \( d \) for which the equality \( Q(\mathcal{P}_d^p) = \mathcal{I}(\mathcal{P}_d^p) \) holds true, where \( \mathcal{P}_d^p \) is a degree \( d \) polynomial in the \( p \)-variables \( \xi_1, \xi_2, \ldots, \xi_p \) determines the quality of the cubature rule. In this case, the value \( d \) is called the polynomial exactness or degree of accuracy of the cubature rule \( Q(\cdot) \). Accordingly Möller (1976), Mysovskikh (1968) and Tchakaloff (1957) have verified that only \( \binom{p+d}{d}_p \) integration nodes are sufficient for a degree of accuracy equal to \( d \). However, these authors did not indicate how to construct a cubature rule with this property (see Cools 2002; Davis and Rabinowitz 2007 for details). In 1963, Smolyak gave a scheme for the construction of cubature rules with number of nodes less than or equal to \( \frac{(p+d)}{p} \). These rules are widely known as sparse-grid integration techniques or Smolyak’s tensor-product integration rules.

### 6.1.2. Sparse-grid integration techniques

Besides the previous assumptions, we further claim that the random variables \( \xi_1, \xi_2, \ldots, \xi_p \) to be independently identically distributed so that \( \phi_1 \equiv \cdots \equiv \phi_p \). Thus, we consider the same quadrature rule for each one-dimensional \( \Omega_k \times \phi_k \) pair. Hence, the index \( k \) can be dropped and a sequence of sets \( X(1), X(2), \ldots \) of one-dimensional quadrature nodes, with the property \( X(i) \subseteq X^{(i+1)}, \quad i = 1, 2, \ldots, \) is considered. Correspondingly, there is a sequence of quadrature rules \( Q_1, Q_2, \ldots \) satisfying the property \( Q_i(f) \leq Q_i^{(i+1)}(f) \), \( i = 1, 2, \ldots \). Such a sequence of quadrature rules is called a nested sequence or embedded cubature rules.

Now define a multi-index \( i = (i_1, i_2, \ldots, i_p) \in \mathbb{N}^p \) and its corresponding norm \( \|i\| = i_1 + i_2 + \cdots + i_p \).

**Definition 6.1** (The sparse-grid cubature technique; Smolyak 1963; Wasilkowski and Woźniakowski 1995; Gerstner and Griebel 1998): The Smolyak or tensor-product sparse-grid cubature rule, with accuracy \( d \), for approximation of a \( p \)-dimensional integral \((d \geq p)\) is given by

\[
\mathcal{A}(d, p)[f] = \sum_{d-p+1 \leq \|i\| \leq d} (-1)^{d-\|i\|} \left( \begin{array}{c} p - 1 \\ d - \|i\| \end{array} \right) \\
\times (Q(i) \otimes \cdots \otimes Q(i))[f],
\]

where \( i = (i_1, i_2, \ldots, i_p) \) and

\[
(Q(i) \otimes \cdots \otimes Q(i))[f] = \sum_{k_1=1}^{N(i_1)} \ldots \sum_{k_p=1}^{N(i_p)} \left( \prod_{i=1}^{p} w_{k_i}^{(i)} \right) f(x_{k_1}^{(i_1)}, \ldots, x_{k_p}^{(i_p)}).
\]

As an example to compute a two-dimensional integral \( \mathcal{A}(7, 2)[f] = \int_a^b \int_a^b f((\xi_1, \xi_2) \phi_1(\xi_1) \phi_2(\xi_2))d\xi_1 d\xi_2 \) with a degree of accuracy \( d = 7 \), we have

\[
\mathcal{A}(7, 2)[f] = \sum_{7-2+1 \leq \|i\| \leq 7} (-1)^{7-\|i\|} \left( \begin{array}{c} 2 \\ 7 - \|i\| \end{array} \right) \\
\times (Q(i) \otimes Q(i))[f].
\]

This example shows that the sparse-grid rule uses a linear combination of tensor products of quadrature rules with lower and higher degrees of accuracy. The number of grid points in \( \mathcal{A}(7, 2)[f] \) is 11. Raising the degree of accuracy from 7 to 9 brings only 4 additional integration nodes. (See Heiss and Winschel 2006 for a more accessible discussion).

The approximation of \( \mathcal{E}[f] \) by using the sparse-grid cubature rule \( \mathcal{A}(d, p)[f] \) has the following properties:

- For functions \( f \) that are polynomial with respect to \( \xi \), \( \mathcal{E}[f] \) can be computed exactly i.e. \( \mathcal{E}[f] = \mathcal{A}(d, p)[f] \) (Novak and Ritter 1999; Barthelmann, Novak, and Ritter 2000; Petras 2003; Heiss and Winschel 2006).
- The number of integration nodes in the formula (32) is equal to the cardinality of the set

\[
\mathcal{X}(d, p) := \bigcup_{d-p+1 \leq \|i\| \leq d} (X(i_1) \times X(i_2) \times \cdots \times X(i_p))
\]

which, for a fixed level of precision \( d \) and embedded one-dimensional quadrature rules, is given by (Novak and Ritter 1999)

\[
|\mathcal{X}(d, p)| = \frac{(2p)^d}{d!}.
\]

Thus, the number of integration nodes in (32) grows only polynomially with respect to the dimension \( p \). Therefore, sparse-grid rules require a few integration nodes reducing function evaluations by enormous amount. In addition to this, when the underlying quadrature rules are embedded, then the sparse-grid rule is also embedded, i.e. \( \mathcal{X}(d, p) \subseteq \mathcal{X}(d+1, p) \).
Table 1. Error estimations for sparse grid, MC and QMC methods.

<table>
<thead>
<tr>
<th>Sparse grid</th>
<th>MC</th>
<th>QMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Error estimation</td>
<td>$O(N^{-(\log N)^{(p-1)(r+1)}})$</td>
<td>$O(N^{-2})$</td>
</tr>
</tbody>
</table>

- The error for approximating $E[f]$ (see Table 1) by $A(d, p)[f]$ is given by

$$|E[f] - A(d, p)[f]| = O(N^{-r}(\log N)^{(p-1)(r+1)})$$  \(35\)

if $f$ is a function of smoothness of degree $r$; i.e. $f \in C^r$ on a bounded integration domain $\Omega$; i.e. $\Omega = [a, b]^d$ and $N := |A(d, p)|$ (Wasilkowski and Woźniakowski 1995; Gerstner and Griebel 1998; Bungartz and Griebel 2004). For unbounded integration domains, e.g. $\Omega = \mathbb{R}^p$, it may be necessary to perform a transformation on to a bounded domain $\Omega = [a, b]^p$ (Chen and Mehrotra 2010).

In the pure sparse-grid integration technique above, the construction of integration nodes and weights is done independent of the function $f$ to be integrated. This has the advantage that integration nodes and weights have to be computed only once and can be used for various types of integrands $f$. Unfortunately, the error estimation (35) of the sparse-grid method depends on the smoothness of the integrand $f$. Hence, integrands of lower order of smoothness may require a large number $N$ of integration nodes for a good sparse-grid approximation of $E[f]$. Bungartz and Dünstorfer (2003) suggest an adaptive sparse-grid method so that smoothness properties of the integrand $f$ are reflected in the $A(d, p)[f]$. Nevertheless, an adaptive sparse-grid integration may impose intensive computations in the solution of CCOPT problems like chance-constrained nonlinear model predictive (CCNMPC).

Sometimes, cubature rules can be constructed by exploiting symmetric properties of the weight function $\phi$ and the integration domain $\Omega$; e.g. $\phi$ normal density function and $\Omega = \mathbb{R}^d$ are symmetric. As a result there is a large class of the so-called fully symmetric integration rules. Lu and Darmofal (2004), Genz and Keister (1996) are two recent works in this direction. Theoretically, fully symmetric cubature rules require quite a few number of integration nodes and can be taken as being a subclass of sparse-grid integration techniques (Hinrichs and Novak 2007). In practice, among all fully-symmetric integration rules, only the one from Genz and Keister (1996) has a good reputation of performance in the computation of multivariate normal integrals.

We have first proposed the use of the sparse-grid integration method for computing probability values in solution of CCOPT problems (Geletu et al. 2011). We have also demonstrated the efficiency of sparse-grid technique, in comparison to full-grid integration, and solved several process optimisation problems with uncertainties (Geletu et al. 2011; Klöppel et al. 2011).

6.2. MC and QMC integration

MC and QMC methods use the equal weight sum

$$Q(N, P)[f] = \frac{1}{N} \sum_{k=1}^{N} f(\xi_k)$$

to approximate the integral $E[f]$. Basically, the MC and QMC methods are used for higher dimensional integrals on the bounded integration domain $\Omega = [0, 1]^p$. Therefore, integrals on arbitrary domains need to be transformed onto $\Omega = [0, 1]^p$. Hence, the integration nodes $\xi_k$, $k = 1, \ldots, N$, in MC method are commonly generated from $\Omega = [0, 1]^p$ using pseudo-random generators, while QMC uses deterministic integration nodes with low discrepancy properties (see Caflisch 1998 and Niederreiter 1992 for details).

Besides the simplicity of construction, the extensive and practical use of the MC and QMC methods stems from some important advantages. The MC and QMC methods can be used irrespective of the type of probability distribution of the random variables $\xi = (\xi_1, \ldots, \xi_p)$ as long as the random variables are independent. They are applicable for integrands of low order of smoothness, even when they have a finite number of discontinuities. Basically, they are methods of choice for integrals of very high dimensions (see, e.g. Sloan and Woźniakowski 1998; Deák 2010).

Practical error estimation for the approximation of $E[f]$ by $Q(N, P)[f]$ is independent of the dimension $p$ of the integral (e.g. Sloan and Woźniakowski (1998) have experimentally verified that the actual error estimation for integration using QMC is independent of $p$.) In contrast, the sparse-grid method has error estimation given in Equation (35) which is dependent on the dimension $p$ and the smoothness order of the integrand $f$. Comparative numerical experimentations on sparse-grid and QMC are to be found from Schürer (2003). Needless to say that good error estimations are
achieved only with a very large \( N \); consequently, with intensive computations.

In summary, these three methods follow different philosophies. MC method attempts to reduce variations in computed results of \( Q(N, P)[f] \) from sample to sample using large sample size \( N \) for error reduction; QMC method tries to cover the integration domain \( \Omega \) with uniformly distributed integration nodes; while sparse-grid techniques use interpolatory properties of polynomials at a few appropriately selected points from \( \Omega \) with corresponding weights.

7. Case-studies

This section presents two cases-studies to show the relevance of the CCOPT approach in practical engineering applications. The first case-study considers the MPC scheme for a batch reactor. The discussion considers two types of objective functions: a deterministic and a stochastic objective functions. Results show that, in the presence of input uncertainties, it is possible to maximise the predicted average product concentration without exceeding allowed reactor temperature with a higher reliability. This is shown to be achievable by subduing the fluctuations in the cooling water inflow.

The second case-study considers the behaviour of electrical networks in the presence of uncertainties from wind power generators. The discussion shows possible violation of deterministic demand constraints in the face of non-Gaussian uncertainties.

7.1. Batch reactor

In this case study a batch reactor with a series reaction of the form

\[
A \rightarrow B \rightarrow C
\]

is considered, where species B is the desired product, but the reaction of B into C cannot be totally avoided. A similar process was considered as a case-study by Shah and Madhavan (2004). There, only uncertainties in one of the pre-exponential Arrhenius factors were considered. In contrast, we consider uncertainties in both pre-exponential Arrhenius factors and activation energies. The principal configuration of a batch reactor can be seen in Figure 2.

The process in the reactor is described by a system of nonlinear differential equations

\[
\dot{x}_1 = -k_1 x_1 x_2, \tag{36}
\]

\[
\dot{x}_2 = k_1 x_1 x_2 - k_2 x_2, \tag{37}
\]

with initial conditions

\[
x_1(0) = 1, \quad x_2(0) = 0, \quad T(0) = T_M(0) = T_J(0) = 320 \text{ K},
\]

where \( x_1 \) and \( x_2 \) are the concentration of species A and B, \( T, T_M \) and \( T_J \) are the temperatures of the reaction mass, wall and jacket, respectively, \( k_1, k_2 \) describe the reaction rates, \( E_1, E_2 \) the activation energies, \( k_{10}, k_{20} \) the pre-exponential Arrhenius factors and \( F_J \) is the amount of cooling water per hour. The physical parameters of the model are described in Table 2. The behaviour of the temperature \( T \) for the deterministic system for different constant levels of \( F_J \) is shown in Figure 3. We assume that the variables \( E_1, E_2, k_{10} \) and \( k_{20} \) are time-independent and underlie a joint normal distribution with the parameters given in Table 3. This is motivated by the fact, that these four quantities are determined experimentally and therefore underlie measurement errors. Using an Euler
can be formulated as chance-constrained dynamic optimisation problem discretisation of the dynamic system (36)–(42), a

\[
\begin{align*}
\text{s.t. } k_1(t) &= k_{10} \exp \left( -\frac{E_1}{RT(t-1)} \right), \\
k_2(t) &= k_{20} \exp \left( -\frac{E_2}{RT(t-1)} \right), \\
x_1(t) &= x_1(t-1) - \Delta t(k_1(t)x_1(t-1) - k_2(t)x_2(t-1)), \\
x_2(t) &= x_2(t-1) + \Delta t(k_1(t)x_1(t-1) - k_2(t)x_2(t-1)), \\
T(t) &= T(t-1) \\
&\quad + \Delta t \left( \frac{(\Delta H_{r1}k_1(t)x_1(t-1) + \Delta H_{r2}k_2(t)x_2(t-1))V_p}{V_p \rho_p c_p} - hA_f(T(t-1) - T_M(t-1)) \right), \\
T_M(t) &= T_M(t-1) \\
&\quad + \Delta t \left( \frac{hA_f(T(t-1) - 2T_M(t-1) - Z_j(t-1))}{V_M \rho_M c_M} \right), \\
T_j(t) &= T_j(t-1) \\
&\quad + \Delta t \left( \frac{F_j(t)\rho_j c_J(T_j(t-1) - T_j(t-1))}{V_f \rho_j c_J} \right), \\
\Pr \{ T(t) \leq 328 \text{K} \} &\geq 0.8, \\
t &\in \{ \hat{t}, \ldots, \hat{t} + h - 1 \}, \\
x_1(0) &= 1, \quad x_2(0) = 0, \quad T(0) = T_M(0) = T_j(0) = 320 \text{ K},
\end{align*}
\]

where \( h \) is the length of the prediction horizon, \( \hat{t} \) is the actual time interval, \( \Delta t = 2 \text{ min} \) is the time interval used for the discretisation and the other quantities are defined as above. Firstly, a deterministic objective function in the form

\[
\begin{align*}
f(F_J, x_1, x_2, T, T_M, T_j, \hat{t}, h) &= \Delta F_J^2 \Delta F_J + \sum_{t=\hat{t}}^{\hat{t}+h-1} F_J^2(t) 
\end{align*}
\]
is considered, where the term

$$
\Delta F_j = (F_j(\hat{t}) - F_j(\hat{t} - 1), \ldots, F_j(\hat{t} + h - 1) - F_j(\hat{t} + h - 2))^T 
$$

describes the fluctuation in the cooling water inflow. We use an MPC scheme for the optimal control of the process. The objective of the optimisation is to reduce the fluctuations in the cooling water stream (described by the first term in the objective function) and at the same time to minimise the amount of cooling water used (described by second term), whereas the temperature of the reaction mass should not exceed 328 K. It is visible from Figure 3 that reaction mass temperatures above 328 K typically only occur within the first 60 min of the process. Therefore, this time period was chosen for optimisation together with a prediction horizon $h = 8$. The results for one specific realisation of the uncertain variables and using different approaches to the calculation of the probabilities (convex approximation, analytic approximation, back-mapping) are shown in Figure 4. For the back-mapping approach the positive monotonic relationship $k_{20} T$ was used. This relation is clear from Equations (47) and (48): A higher value of $k_{20}$ leads to an increased reaction $B \rightarrow C$, which produces reaction heat and therefore increases reaction mass temperature as described in (48). As shown in Figure 4, the convex approximation approach uses a higher amount of cooling water, due to the underestimation of the probabilities of holding the constraints, leading to increased costs. Furthermore it should be noted that no feasible solution could be found in the first five time horizons when using this approach. On the other hand, the cooling water stream profiles for the analytic approximation and back-mapping approach are similar and lead to similar costs (amounting to 5181.2 for the analytic approximation, 5551.1 for the back-mapping, and 15,345.3 for the convex approximation approaches, respectively). The differences in the profiles can be mainly attributed to the behaviour of the NLP solver used (fmincon from Matlab\textsuperscript{TM}).

In order to further show the viability of the presented approach, an optimisation problem with the stochastic objective function

$$
f(F_j, x_1, x_2, T, T_M, T_J, \hat{t}, h) 
= \omega_1 \left[ \sum_{i=1}^{\hat{t}+h-1} (-E(x_2(t)) + \text{Var}(x_2(t))) \right] + \cdots 
+ \omega_2 \left[ F_j^2 \Delta F_j + \sum_{i=1}^{\hat{t}+h-1} F_j^2(t) \right] 
$$

is solved under the same set of constraints (44)–(53). Here, $\omega_1$ and $\omega_2$ are weighting factors. This objective function is a weighted sum: between the function in (54) and the expected concentration of product $B$ (i.e. $E[x_2(t)]$) while decreasing its variations ($\text{Var}(x_2(t))$). For this problem only the analytic approximation approach was considered. The results of the optimisation can be found in Figure 5. Figure 5(a) presents the difference between the concentration of $x_2$ for the stochastic objective (55) and the deterministic objective (54). It can be seen that the difference is always non-negative, showing a success in the optimisation. The fact that the margin of difference is not very large can be attributed to the model used. Figure 5(b) shows that there exist qualitative and quantitative differences in the cooling water stream profiles.

Since the presented problems have only four uncertain input variables (the corresponding sparse-grid cubature rule uses only 441 grid points), the
computation of probabilities, expectations and variances can be carried out without significant computational load. The typical computation time for the solution of both problems for an optimisation horizon of 30 is less than 1 min using Matlab\textsuperscript{TM} optimisation toolbox.

7.2. PF under uncertainty

In this case study, the behaviour of an electrical network under uncertain, non-Gaussian distributed influences from wind generators is analysed. As a test case we choose the 9-bus system present in Matpower (Zimmerman, Murillo-Sánchez, and Thomas 2011), which is shown in Figure 6. Additionally, it is assumed, that the generators at buses 2 and 3 are wind generators. According to Fabbri, San Román, Abbad, and Quezada (2005), the generation of a wind generator can be described by a Beta distribution and the distribution parameters can be calculated from weather forecasts. In this application, we assume that the generation is independent (i.e. the wind generators are located in distinct areas) and the distribution parameters are $\alpha = 5$ and $\beta = 3$ for both generators.

We are now interested in the calculation of constraint satisfaction probabilities

$$\Pr(V_i \geq V_{\text{min}}),$$

where $V_i$ is the voltage magnitude at bus $i$ and is determined using the following model equations

$$P_i - P_{gi} + P_{di} = 0, \quad (56)$$

$$Q_i - Q_{gi} + Q_{di} = 0, \quad (57)$$

$$P_i = G_i(e_i e - f_i f) + B_i(f_i e - e_i f), \quad (58)$$

$$Q_i = G_i(f_i e - e_i f) + B_i(e_i e - f_i f), \quad (59)$$

$$e_i + if_i = V_i \exp(i\theta_i), \quad i = 1, \ldots, n. \quad (60)$$

Here $n$ is the number of buses in the system, $e_i$ and $f_i$ are the real and imaginary components of the complex voltage at bus $i$, $\theta_i$ is the voltage angle at bus $i$, $P_{gi}$ and $Q_{gi}$ are active (real) and reactive (imaginary) powers of the generator at bus $i$, $P_{di}$ and $Q_{di}$ are active and
reactive loads at bus $i$, $P_i$ and $Q_i$ are the active and reactive power injections into bus $i$, $1 \leq s \leq n$ determines the slack (or reference bus), $1 \leq j_1, \ldots, j_k \leq n$ determines buses with deterministic generators attached, $u = (e_i, f_i, P_{gj_1}, \ldots, P_{gj_k})$ are control variables, $x = (P_{gs}, Q_{gs}, f_{j_1}, \ldots, f_{j_k}, Q_{j_1}, \ldots, Q_{j_k}, e_1, \ldots, f_1, \ldots)$ are the state variables (for buses with generators (except the slack bus) real voltage and real power generation are controls, imaginary component of voltage and generation are states, for buses without generator real and imaginary component of the voltage are states) and $\xi = (P_2, P_3)$ are the uncertain input variables. Equations (56)-(59) are the so-called PF equations which describe the distribution of power throughout the network.

In the following we consider bus 5. The discussion remains the same for the rest of the buses. In order to determine the system behaviour the probabilities

$$\Pr[V_i \geq V_{\min}]$$

are calculated for different values of $V_{\min}$ and the given distribution parameters of the Beta distribution. For this calculation the analytic approximation approach in conjunction with different integration algorithms was used. To solve the underlying PF problems Matpower (Zimmerman et al. 2011) was used. We employed full- and sparse-grid integration rules based on Clenshaw–Curtis rules (Trefethen 2008), which are adapted to the Beta weight function, and a QMC scheme based on the Sobol sequence. The sparse-grids were generated using the Sandia sparse library from Burkardt (2011). As can be seen in Figure 8, both full grid technique and QMC method give good results even with a low number of grid points. On the contrary, the results of sparse-grid integration are inaccurate, even though a higher number of grid points were used. This can be explained by the distribution of grid points in the sparse-grid rule, which is shown in Figure 7. The sparse-grid has only few points in the relevant area near the expectation of the given
distribution (which is \((0.625, 0.625)\)) in comparison to the other integration approaches. Sparse grids with higher number of grid points can approximate the probabilities better.

The result shown in Figure 8 depicts the probability values of \(\Pr\{ V_5 > V_{\min} \}\) for different lower bounds \(V_{\min}\). These results are obtained through the analytic approximation method (Section 5.3, Equation (30) with \(m_1 = 2, m_2 = 1\)) for \(\Pr\{ V_5 \geq V_{\min} \}\) and computing the values of the function

\[
1 - E \left[ \frac{1 + 2\tau}{1 + \tau \exp\left(\frac{\tau}{\tau}(V_5 - V_{\min})\right)} \right]
\]

through simulation of the model Equations (56)-(60) for \(\tau = 0.001\). The important observation from this case-study is that (see Figure 8), the probability constraint \(\Pr\{ V_5 \geq V_{\min} \} \geq \alpha \) cannot be satisfied with the reliability level \(\alpha = 1\) if \(V_{\min} > 0.96\).

8. Comments and conclusion

Uncertainties are prevalent in practical engineering applications. In a dynamic process input uncertainties propagate through the process and cause uncertainties in the predicted outputs. As a result deterministic constraints on outputs of the process may become infeasible. In practical applications a degree of constraint violation is tolerated. As a result strategies for the optimal and reliable performance of a system can be designed based on the CCOPT approach, by formulating chance constraints on the process output variables.

This article attempts to give the state-of-the-art in chance-constrained nonlinear optimisation problems. It is indicated that, there is a growing interest in the application of CCOPT methods in various fields of engineering. The use of the CCNMPC scheme is an advancement of traditional nonlinear model predictive strategies to cope with possible constraint violations due to the uncertainty of predicted states of the system. In the CCNMPC scheme, it is required to solve chance-constrained nonlinear optimisation problem (CCOPT) on each prediction horizon. The type of algorithm to be used to solve the CCOPT depends on the structural properties of the chance constraints. In general, evaluation of chance constraints becomes difficult when process models are nonlinear. Therefore, there are various approaches in the literature to surmount this and other difficulties. Specifically, analytic approximation approaches ameliorate some of the difficulties both by guaranteeing feasibility of chance constraints and their simpler computations. Nonetheless, at each step of the optimisation algorithm for CCOPT, evaluation of probability integrals of higher dimensions is, in most cases, unavoidable. Numerically, this can be achieved either through the QMC or sparse-grid

![Figure 8. Probability of holding the constraint \(V_5 > V_{\min}\) depending on \(V_{\min}\) and for different integration routines, numbers in brackets denote the number of grid points.](image-url)
integration methods, depending on the distribution of the uncertain variables and the properties of the integrand.

Two case-studies, one with Gaussian normal and the other with product-beta distributed uncertainties, show applicability of chance-constrained models in the solution of modern engineering problems.

Since the inception of CCOPT methods, so much has been achieved in the theoretical analysis and computational strategies for linear CCOPT problems. At this time, there are only a few contributions in the numerical methods of nonlinear chance-constrained problems. MPC stability in the presence of chance constraints needs to be investigated, particularly for nonlinear dynamic process. Despite the fact that there are several modern engineering applications with non-Gaussian uncertainties, there is a scant consideration for such uncertainties in chance constraints. Chance-constrained problems with partial differential model equations have not been touched yet. For the reduction of computation time, the sparse-grid integration technique is highly promising, but it has some drawbacks: presence of non-negative integration weights resulting in numerical cancelations, poor performance for integrands of lower smoothness and sparsity of grid points causing some important scenarios to be missed. These and other related issues are open for future investigations.

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Note
1. When precisely written the function \( \varphi_{\alpha}(\xi) \) should be \( \varphi(\alpha, \xi, \xi) \). The integrals in Equations (23) and (24) should be understood in this context.

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