The moment generating function of a bivariate gamma-type distribution

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\textbf{ABSTRACT}

A bivariate gamma-type density function involving a confluent hypergeometric function of two variables is being introduced. The inverse Mellin transform technique is employed in conjunction with the transformation of variable technique to obtain its moment generating function, which is expressed in terms of generalized hypergeometric functions. Its cumulative distribution function is given in closed form as well. Many distributions such as the bivariate Weibull, Rayleigh, half-normal and Maxwell distributions can be obtained as limiting cases of the proposed gamma-type density function. Computable representations of the moment generating functions of these distributions are also provided.

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\textbf{1. Introduction}

The univariate gamma distribution plays a major role in statistics and is involved in countless applications. The first form of its bivariate extension has been introduced by McKay [13]. An interesting application of this distribution was considered by Clarke [3] in connection with the joint distribution of annual streamflow and areal precipitation. Cherian [2] proposed another form of the bivariate gamma distribution, and Prekopa and Szantai [18] utilized it to study the streamflow of a river.

Kibble [10] and Moran [14] discussed a symmetrical bivariate gamma distribution whose joint characteristic function is given by

\begin{equation}
\frac{1}{(1-it_1)(1-it_2) + w^2 t_1 t_2^2}, \quad \alpha > 0,
\end{equation}

where \(i = \sqrt{-1}\). Asymmetrical extensions were introduced by Sarmanov [21,22], and another generalization was proposed by Jensen [8] and Smith et al. [23]. Jensen [8] also extended Moran’s bivariate gamma distributions.

Recently Nadarajah and Gupta [15] introduced two bivariate gamma distributions based on a characterizing property involving products of gamma and beta random variables. They provided certain representations of their joint densities, product moments, conditional densities and moments. Some of these representations involve special functions such as the complementary incomplete gamma and Whittaker’s functions. The Farlie–Gumbel–Morgenstern type bivariate gamma distribution was studied by D’Este [4] and Gupta and Wong [7]. Dussauchoy and Berland [5] introduced a joint distribution in the form of a confluent hypergeometric function of two dependent gamma random variables \(X_1\) and \(X_2\) with the property that \(X_2 = \beta X_1\) and \(X_1\) are independent. Nakhi and Kalla [17] defined a probability density function involving a generalized \(r\)-Gauss hypergeometric function and discussed its associated statistical functions. Saxena and Kalla [24] studied a new mixture distribution associated with the Fox–Wright generalized hypergeometric function and discussed some associated...
statistical functions. Nakhi and Kalla [16] discussed further generalizations involving mixture distributions. Provost et al. [19] defined the gamma–Weibull distribution by introducing an additional shape parameter, which also acts somewhat as a location parameter. They provided closed form representations for many associated statistical functions. Saboor and Ahmad [20] defined a bivariate gamma function and its associated density in terms of a confluent hypergeometric function of two variables and discussed some properties of related mathematical functions as well as its probability density function.

In Section 2, we define a bivariate probability density function involving a confluent hypergeometric function of two variables. We provide a representation of its cumulative distribution function. Closed form representations of its moment generating function are derived in the Appendix by making use of an innovative approach that combines the inverse Mellin transform and transformation of variable techniques. The joint moments and marginal pdf’s are provided in Section 3. Several particular cases of interest are pointed out in Section 4. The proposed bivariate model is applied to a data set in Section 5 where a parameter estimation technique is being described. The use of the proposed bivariate distribution should lead to modeling improvements in various fields of scientific investigation that rely on Weibull or gamma-type distributions. These include reliability engineering, extreme value theory and failure analysis in the former case, and life testing, industrial engineering (manufacturing times and distribution processes), risk management (probability of ruin) and queuing systems in the latter.

The remainder of this section is devoted to the derivation of the moment generating function of the proposed bivariate gamma-type random vector. First, the Mellin transform of a function and its inverse are defined.

If \( f(x) \) is a real piecewise smooth function that is defined and single valued almost everywhere for \( x > 0 \) and such that \( \int_0^\infty x^{-1} |f(x)| \, dx \) converges for some real value \( k \), then \( M_f(s) = \int_0^\infty x^{-s} f(x) \, dx \) is the Mellin transform of \( f(x) \). Whenever \( f(x) \) is continuous, the corresponding inverse Mellin transform is

\[
\mathcal{F}^{-1}(f; x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s f(s) \, ds,
\]

which, together with \( M_f(s) \), constitute a transform pair. The path of integration in the complex plane is called the Bromwich path. Eq. (1.2) determines \( f(x) \) uniquely if the Mellin transform is an analytic function of the complex variable \( s \) for \( c_1 \leq \Re(s) = c \leq c_2 \) where \( c_1 \) and \( c_2 \) are real numbers and \( \Im(s) \) denotes the real part of \( s \). In the case of a real continuous non-negative random variable whose density function is \( f(x) \), the Mellin transform is its moment of order \( (s-1) \) and the inverse Mellin transform yields \( f(x) \). When

\[
M_f(s) = \left\{ \prod_{i=0}^{m-1} \Gamma(b_i + B) \right\} \left\{ \prod_{i=0}^{n-1} \Gamma(1 - a_i + A) \right\} = h(s),
\]

where \( m, n, p, q \) are nonnegative integers such that \( 0 \leq n \leq p, 1 \leq m \leq q \), \( a_i, i = 1, \ldots, p \), \( b_j, j = 1, \ldots, q \), are positive numbers and \( a_i, i = 1, \ldots, p \), \( b_j, j = 1, \ldots, q \), are complex numbers such that \( -A_i < b_j + v \neq B_j(1 - a_i + \lambda) \) for \( v, \lambda = 0, 1, 2, \ldots, j = 1, \ldots, m, i = 1, \ldots, n \), and \( h(s) \) is defined in (1.2) and the Bromwich path \( (c - i\infty, c + i\infty) \) separates the points \( s = -(b_j + v)/b_j, j = 1, \ldots, m, v = 0, 1, 2, \ldots, \), which are the poles of \( \Gamma(b_j + B) \), \( j = 1, \ldots, m \), from the points \( s = -(1 - a_i)/a_i, i = 1, \ldots, n \), \( s = -(1 - a_i + \lambda)/a_i, i = 1, \ldots, n \), \( \lambda = 0, 1, 2, \ldots, \), which are the poles of \( \Gamma(1 - a_i + A) \), \( i = 1, \ldots, n \). Thus, one must have

\[
\max_{1 \leq i \leq m} \Re\{-(b_i/B_i)\} < c < \min_{1 \leq i \leq n} \Re\{-(1 - a_i)/A_i\}.
\]

The integral (1.4) converges absolutely when \( m + n - p - 2/q - 2/\lambda > 0 \). When \( A_i = B_j = 1 \) for \( i = 1, \ldots, p \) and \( j = 1, \ldots, q \), the \( H \)-function reduces to Meijer’s \( G \)-function, that is,

\[
G_{p,q}^{m,n}(x^{1/n} a_1, \ldots, a_p ; b_1, \ldots, b_q) = H_{p,q}^{m,n}(x^{1/n} a_1, \ldots, a_p ; b_1, \ldots, b_q).
\]

The following analytic continuation formula can prove useful:

\[
G_{p,q}^{m,n}(x^{1/n} a_1, \ldots, a_p ; b_1, \ldots, b_q) = G_{q,p}^{m,n}(1/x^{1/n} a_1, \ldots, 1 - a_p).
\]
2. A bivariate extension of the gamma distribution

2.1. Introduction

The probability density function of a gamma distribution with parameters \( k \) and \( \theta \) can be expressed as follows:

\[
f(x; k, \theta) = \frac{x^{k-1}e^{-x/\theta}}{\theta^k \Gamma(k)}I_{\mathbb{R}^+}(x), \tag{2.1}
\]

where \( I_{\mathbb{R}^+}(x) \) denotes the indicator function on the set of positive real numbers, \( \theta > 0 \) is a scale parameter and \( k > 0 \) is a shape parameter.

Kalla et al. [9] studied a density function of the form,

\[
f(x) = \frac{\beta x^{m-1}}{\nu_x(0)}x^\beta e^{-\frac{x}{\beta}}F_1\left(\lambda, b; c - \frac{2\beta}{\beta}; \frac{x}{\beta}\right)I_{\mathbb{R}^+}(x), \tag{2.2}
\]

where \( \beta \geq 0, \ m + \beta > 0, \) \( \delta \) and \( \alpha \) are positive real numbers, and \( \nu_x(0) \) is as defined below. The \( h \)th moment of this distribution is

\[
\mu'_h(h) = \frac{\nu_x(h)}{\nu_x(0)}, \tag{2.3}
\]

\[
\nu_x(h) = 2^{m-1}I(c)\left\{\frac{\delta^{\frac{mh+b+\alpha}{\beta}}}{\nu_x(0)}I\left(\frac{m+b}{\beta}, \frac{m+h+b+\alpha}{\beta}; \frac{x}{\beta}\right)\right. \times_2 F_2\left(b, b-c+1; b - \frac{m+h}{\beta}, b - \lambda + 1; \frac{x}{\beta}\right) \\
+ \left. \left(\frac{m+b}{\beta}I\left(\frac{m+h+b+\alpha}{\beta}; \frac{x}{\beta}\right) \times_2 F_2\left(b, b-c+1; b - \frac{m+h}{\beta}, b - \lambda + 1; \frac{x}{\beta}\right) \right) \right. \\
+ \left. \left(\frac{m+h+b+\alpha}{\beta}I\left(\frac{m+b+\alpha}{\beta}; \frac{x}{\beta}\right) \times_2 F_2\left(b, b-c+1; b - \frac{m+h}{\beta}, b - \lambda + 1; \frac{x}{\beta}\right) \right) \right) \right) \right) \right) \right) \right)
\]

for \( h = 0, 1, \ldots, \) whenever \( h + m + \beta > 0 \). We obtained the closed form representation given in (2.4) with respect the density Kalla et al. [9] by making use of the symbolic computational package Mathematica.

Al-Saqabi et al. [1] defined the following univariate generalized gamma-type probability function in terms of a certain confluent hypergeometric function of two variables:

\[
f(x) = Ax^{\lambda-1}e^{-px^\alpha}\Phi_1(a, b; c; -2x\alpha^{-\beta}, \beta x^\beta)I_{\mathbb{R}^+}(x), \tag{2.5}
\]

where

\[
A = \sum_{j=0}^{\infty} \frac{(a)_j b^j}{(c)_j j!} \frac{\Gamma(\frac{j}{2} + 1) \Gamma(c + j)}{\Gamma(\frac{1}{2} + \frac{j}{2}) \Gamma(b + j + 1)} F_2\left(a + j, b + 1; c + j + \frac{1}{2}; \frac{x}{\beta}\right) \\
+ \sum_{j=0}^{\infty} \frac{(a)_j b^j}{(c)_j j!} \frac{\Gamma(\frac{j}{2} + 1) \Gamma(c + j)}{\Gamma(\frac{1}{2} + \frac{j}{2}) \Gamma(b + j + 1)} F_2\left(a + j, b + 1; c + j + \frac{1}{2}; \frac{x}{\beta}\right), \tag{2.6}
\]

with, for example, \( (a)_j = \Gamma(a + j)/\Gamma(a) \) and

\[
\Phi_1(a, b; c; -2x\alpha^{-\beta}, \beta x^\beta) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_j (b)_j (-2x\alpha^{-\beta})^j (\beta x^\beta)^m}{(c)_j m! j!}, \tag{2.7}
\]

where \( |x\alpha^{-\beta}| < 1 \).

See, for instance, Erdélyi et al. [6].

We introduce the following bivariate gamma-type density function which involves another type of confluent hypergeometric function of two variables:

\[
f(x, y) = Cx^{\lambda-1}y^{\lambda-1}e^{-px^\alpha}e^{-py^\alpha} \Psi_2(a; b, c; 2x^\alpha, \beta y^\beta)I_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y), \tag{2.8}
\]

where

\[
\Psi_2(a; b, c; 2x^\alpha, \beta y^\beta) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_j (x^\alpha)_j \beta (y^\beta)_j m! j!}{(b)_j (c)_j m! j!}, \tag{2.9}
\]

and
\( C^{-1} = \int_0^\infty \int_0^\infty x^{i-1} y^{j-1} e^{-\frac{x}{p_1} + e^{-\frac{x}{p_2} y^2}} y_2 (a; b; c; \alpha x^i, \beta y^j) \, dx \, dy \)

\[
= \int_0^\infty \int_0^\infty x^{i-1} y^{j-1} e^{-\frac{x}{p_1} t^i} e^{-p_2 y^2} \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{a_{j+m} \alpha x^i \beta y^j}{(b_j c) m!} \, dx \, dy.
\]

(2.10)

Since the double sum in (2.10) is uniformly convergent, the summations and integrals can be interchanged. Then applying some basic calculus, we obtain

\[
C^{-1} = \frac{\Gamma(k_1/\delta_1) \Gamma(k_2/\delta_2)}{\delta_1 \delta_2 p_1^{k_1/\delta_1} p_2^{k_2/\delta_2}} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{j+m} (k_1/\delta_1) (k_2/\delta_2) (\alpha x^i \beta y^j)}{(b_j c) m!}.
\]

(2.11)

where

\[
F_2 (a; k_1, k_2; b, c; \alpha, \beta) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_{j+m} (k_1) (k_2) \beta y^j}{(b_j c) m!}.
\]

(2.12)

see Erdélyi et al. [6, p. 224].

The cdf of the bivariate gamma-type distribution specified by (2.8) is given by

\[
F(u, v) = \frac{C}{\delta_1 \delta_2 p_1^{k_1/\delta_1} p_2^{k_2/\delta_2}} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha x^i \beta y^j)}{(b_j c) m!} \times \left( \Gamma \left( j + \frac{k_1}{\delta_1} \right) - \Gamma \left( j + \frac{k_1}{\delta_1} + \frac{1}{v_1} \right) \right) \times \left( \Gamma \left( m + \frac{k_2}{\delta_2} \right) - \Gamma \left( m + \frac{k_2}{\delta_2} + \frac{1}{v_2} \right) \right) I_{R_1 \times R_2}(u, v).
\]

(2.13)

where \( C \) is as specified in (2.11), \( \Gamma(a, x) = \int_0^\infty t^{a-1} e^{-t} \, dt \) is an incomplete gamma function, and \( \Re(p_i v^k) > 0, \Re(\delta_i) \geq 0, \ i = 1, 2. \)

The joint pdf and cdf of the proposed generalized distribution are respectively plotted in Figs. 1 and 2 for certain combinations of the parameters.

2.2. The moment generating function

Let \( X \) and \( Y \) be random variables whose joint pdf is specified by (2.8). By definition, their joint moment generating function is

\[
M_{X,Y}(t_1, t_2) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a_{j+m} \beta y^j)}{(b_j c) m!} \int_0^\infty x^{i+1} e^{-px^i} e^{tx_1} \, dx \times \int_0^\infty y^{j+1} e^{-p_2 y^2} e^{tx_2} \, dy.
\]

(2.14)

It is shown in the Appendix that when \( \delta_1 = p/q \) and \( \delta_2 = f/g \),

\[
M_{X,Y}(t_1, t_2) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a_{j+m} \beta y^j)}{(b_j c) m!} \times \frac{q^{1/2} g^{1/2} p_1^{1/2} q^{1/2} f^{1/2} q^{1/2} / \Gamma(j+1/2+m) \Gamma(m+1/2)}{(-t_1)^{j+1/2+m} (-t_2)^{j+1/2+m} (2\pi)^{1/4}} \times G^{d/2}_{d+2} \left( \frac{p}{t_1}, \frac{p}{q} \right) \frac{1 - d_2 / q}{d_1, 1, q - 1} \times G^{d/2}_{d+2} \left( \frac{f}{t_2}, \frac{f}{g} \right) \frac{1 - d_4 / g}{d_4, 1, f - 1}.
\]

(2.15)

Fig. 1. Plot of \( f(x, y) \) for \( a = 4, b = 3, c = 15, \alpha = 0.4, \delta_1 = p/q, \delta_2 = f/g, \beta = 0.6, \lambda_1 = 2.9, \lambda_2 = 3.5, p_1 = 2.6, p_2 = 3.3, p = 2, q = 3, f = 3 \) and \( g = 2 \).
3. Joint moments and marginals

Explicit representations of the joint moments and marginal density functions of the random variables $X$ and $Y$ whose joint density is specified by (2.8), are provided in this section.

The $(r,s)$th moment of $X$ and $Y$ is given by

$$\mu_{rs} = \frac{\Gamma((\lambda_1 + r)/\delta_1)\Gamma((\lambda_2 + s)/\delta_2)}{\Gamma(\lambda_1/\delta_1)\Gamma(\lambda_2/\delta_2)p_1^{\lambda_1/\delta_1}p_2^{\lambda_2/\delta_2}} \times \frac{F_2(\lambda_1 + r)/\delta_1, (\lambda_2 + s)/\delta_2; c, b; a/p_1, \beta/p_2)}{F_2(\lambda_1/\delta_1, \lambda_2/\delta_2; c, b; a/p_1, \beta/p_2)},$$

where $\Re(p_i) > 0$, $\Re(\delta_i) > 0$, $i = 1, 2$, $\Re(r + k\delta_1 + \lambda_1) > 0$ and $\Re(s + m\delta_2 + \lambda_2) > 0$.

Thus, the covariance between $X$ and $Y$ is given by
\[ \text{Cov}(X, Y) = \frac{\Gamma(\frac{1}{\delta_1})}{\Gamma(\frac{1}{\delta_1})} \frac{1}{\Gamma(\frac{1}{\delta_1} + 1)} \frac{1}{\Gamma(\frac{1}{\delta_2} + 1)} F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) \\
\times \left( F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) - F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) \right) \]
\[ \times F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) \].
\] (3.2)

Denoting X and Y, whose joint density is given in (2.8), by \( X_1 \) and \( X_2 \), respectively, the marginal density function of \( X_i \) is
\[ f_i(x) = \frac{\delta_1 p_1^{1/\delta_1} \Psi_i(a; \lambda_1, \lambda_2 + 1; \alpha; b; \beta)}{\Gamma(\frac{1}{\delta_1})} F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) \]
for \( i = 1, 2 \) and where
\[ \Psi_i(a; \lambda; b, c; \alpha, \beta) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a)_j (\lambda)_m}{(b(c)_m) j! m!} \]
Thus,
\[ E(X_i) = \frac{\Gamma(\frac{1}{\delta_1} + 1)}{\Gamma(\frac{1}{\delta_1})} F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) \]
and
\[ \text{Var}(X_i) = \frac{1}{\Gamma(\frac{1}{\delta_1})} \frac{1}{\Gamma(\frac{1}{\delta_2})} F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) \\
\times \left( \Gamma(\frac{1}{\delta_1} + 2) F_2(a; \lambda_1 + 2, \lambda_2 + 1; \alpha p_1, \beta p_2) - \Gamma(\frac{1}{\delta_1} + 1) F_2(a; \lambda_1 + 1, \lambda_2 + 1; \alpha p_1, \beta p_2) \right) \]
\[ = 1, 2. \] (3.5)

4. Related distributions

Some particular cases of interest are presented in this section. First, we note that as \( \alpha \to 0 \) and \( \beta \to 0 \), the joint density given in (2.8) becomes
\[ f_i(x, y) = \delta_1 \delta_2 p_1^{1/\delta_1} p_2^{1/\delta_2} x^{\lambda_1 - 1} y^{\lambda_2 - 1} e^{-x^{\delta_1} - y^{\delta_2}} \]
\[ I_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y). \] (4.1)

It is seen that the resulting random variables denoted by \( X_i \) and \( Y_i \), are independently distributed. The corresponding cdf is then
\[ F_i(u, v) = p_1^{1/\delta_1} p_2^{1/\delta_2} u^{\lambda_1} (p_2 u^{\lambda_2} - 1) \]
\[ \times \left( \Gamma(\frac{1}{\delta_1}) - \Gamma(\frac{1}{\delta_1} + 1; p_2 u^{\lambda_2}) \right) \]
\[ = 1, 2. \] (4.2)

Some special cases of interest of the bivariate gamma-type density given by (4.1) are enumerated below.

(i) The bivariate Weibull distribution function which can be expressed as
\[ f_w(x, y) = \theta_1 \theta_2 k_1 k_2 x^{\delta_1 - 1} y^{\delta_2 - 1} e^{-x^{\delta_1} y^{\delta_2}} I_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y). \] (4.4)

the corresponding cdf and mgf being
\[ F_w(u, v) = \left( 1 - e^{-u^{\delta_1}} \right) \left( 1 - e^{-v^{\delta_2}} \right) \] (4.5)
and
\[ M_{X,Y}^{(W)}(t_1, t_2) = \frac{(2\pi)^{2-(q+1)/2} \theta_1 \theta_2 q^{-1/2} g^{-1/2} p^{q+1} f_{1/2}}{(-t_1)^q (-t_2)^{q+1}} \times C_{q+1} \left( \left( \frac{p}{t_1} \right)^{-p} \left( \frac{q}{t_2} \right)^{-q} \right) \left( 1 - \frac{d_4 / g}{d_3} \right), \]
\[ \text{where } d_2 = 0, 1, \ldots, q - 1, \quad d_4 = 0, 1, \ldots, g - 1. \]

(4.6)

respectively. The pdf (4.4), cdf (4.5) and mgf (4.6) can respectively be obtained from (4.1)-(4.3) by letting \( p_1 = \theta_1, \ p_2 = \theta_2, \ \lambda_1 = \lambda_1 = k_1 = p / q \) and \( \lambda_2 = \lambda_2 = k_2 = f / g \).

(ii) The bivariate Maxwell density function
\[ f_M(x, y) = \frac{2 \pi x y^2 e^{-x^2/(2\phi_1^2)} e^{-y^2/(2\phi_2^2)}}{\phi_1^2 \phi_2^2} \mathcal{I}_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y), \]

as well as its associated cdf
\[ F_M(u, v) = \left( \sqrt{x \pi} - 2 \Gamma \left( \frac{1}{2}, \frac{u^2}{2 \pi} \right) \right) \left( \sqrt{y \pi} - 2 \Gamma \left( \frac{1}{2}, \frac{v^2}{2 \pi} \right) \right) \]

(4.7)

and moment generating function
\[ M_{X,Y}^{(M)}(t_1, t_2) = \frac{-32}{\pi^{q+1} t_1^q t_2^q \phi_1^2 \phi_2^2} G_{q+1} \left( \frac{(\phi_1 t_1)^2}{2} \left( \frac{1}{3/2, 2} \right) G_{q+1} \left( \frac{(\phi_2 t_2)^2}{2} \right) \left( \frac{1}{3/2, 2} \right) \right) \]
\[ \times \left( \frac{\phi_1}{\sqrt{\pi}} \right)^{q+1} + e^{\phi_1^2 t_1^2 / 2} \left( t_1^2 \phi_1^2 + 1 \right) \left( 1 + \text{erf} \left( \frac{t_1 \phi_1}{\sqrt{2}} \right) \right) \]

(4.9)

are particular cases of (4.1)-(4.3) with \( p_1 = 1 / \phi_1^2, \ p_2 = 1 / \phi_2^2, \ p = 2, \ q = 1, \ f = 2, \ \lambda_1 = \lambda_2 = 2, \ \delta_1 = \delta_2 = 2, \) and \( g = 1, \)

where \( \text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt. \)

(iii) The bivariate half-normal density function,
\[ f_h(x, y) = \frac{4 \phi_1 \phi_2}{\pi^{q+2}} e^{-x^2/4 \phi_1^2} e^{-y^2/4 \phi_2^2} \mathcal{I}_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y), \quad \phi_1, \phi_2 > 0, \]

(4.10)

its associated cdf
\[ F_h(u, v) = \text{erf} \left( \frac{2u \phi_1}{\sqrt{\pi}} \right) \text{erf} \left( \frac{2v \phi_2}{\sqrt{\pi}} \right) \]

(4.11)

and its moment generating function,
\[ M_{X,Y}^{(h)}(t_1, t_2) = \frac{4 \phi_1 \phi_2}{\pi^{q+1} t_1^q t_2^q \phi_1^2 \phi_2^2} G_{q+1} \left( \frac{\pi t_1^2}{4 \phi_1^2} \right) \left( \frac{1}{1/2, 1} \right) G_{q+1} \left( \frac{\pi t_2^2}{4 \phi_2^2} \right) \left( \frac{1}{1/2, 1} \right) \]
\[ \times \left( \frac{\phi_1}{\sqrt{\pi}} \right)^{q+1} + e^{\phi_1^2 t_1^2 / 2} \left( t_1^2 \phi_1^2 + 1 \right) \left( 1 + \text{erf} \left( \frac{\sqrt{2 \pi} / 2}{\phi_1^2} \right) \right) \]

(4.12)

are also particular cases of (4.1)-(4.3) with \( \lambda_1 = \lambda_2 = 2, \ p_1 = 1/2 \phi_1^2, \ p_2 = 1/2 \phi_2^2, \ \delta_1 = \delta_2 = 2, \ p = 2, \ q = 1, \ f = 2 \) and \( g = 1. \)

(iv) The bivariate Rayleigh density function,
\[ f_r(x, y) = \frac{xy e^{-x^2/(2a_1^2)} e^{-y^2/(2a_2^2)}}{a_1^2 a_2^2} \mathcal{I}_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y), \]

(4.13)

as well as its associated cdf,
\[ F_r(u, v) = \left( 1 - e^{-u^2/(2a_1^2)} \right) \left( 1 - e^{-v^2/(2a_2^2)} \right) \]

(4.14)

and moment generating function,
\[ M_{X,Y}^{(R)}(t_1, t_2) = \frac{4}{\pi (a_1 t_1)^q (a_2 t_2)^q} G_{q+1} \left( \frac{(a_1 t_1)^2}{2} \right) \left( \frac{1}{1/3/2, 2} \right) \times G_{q+1} \left( \frac{(a_2 t_2)^2}{2} \right) \left( \frac{1}{1, 3/2} \right) \]
\[ \times \left( \sqrt{\pi} \right)^{q+1} a_1 a_2 e^{a_1^2 t_1^2 / 2} \left( \frac{a_1 t_1}{\sqrt{2}} \right) \left( \text{erf} \left( \frac{a_1 t_1}{\sqrt{2}} \right) + 1 \right) \]

(4.15)
are particular cases of (4.1)–(4.3) with \( p_1 = 1/(2a_i^2), \) \( p_2 = 1/(2a_j^2), \) \( \lambda_1 = \delta_1 = 2, \) \( \lambda_2 = \delta_2 = 2, \) \( p = 2, \) \( q = 1, \) \( f = 2 \) and \( g = 1. \)

(v) The bivariate gamma distribution with density function

\[
    f_G(x, y) = \frac{x^{v_1 - 1}y^{v_2 - 1}e^{-x/\phi_1}e^{-y/\phi_2}}{(\phi_1)^{v_1}(\phi_2)^{v_2}}I_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y), \quad v_1, v_2, \phi_1, \phi_2 > 0
\]

is also a particular case of (4.1) with \( \lambda_1 = v_1, \) \( \lambda_2 = v_2, \) \( p_1 = 1/\phi_1, \) \( p_2 = 1/\phi_2 \) and \( \delta_1 = \delta_2 = 1. \)

(vi) The bivariate Erlang distribution with density function

\[
    f_E(x, y) = \frac{\beta_1^{(n_1 + 1)}\beta_2^{(n_2 + 1)}x^{n_1+1}y^{n_2+1}e^{-x/\beta_1}e^{-y/\beta_2}}{I_{\beta_1}^{(n_1 + 1)}I_{\beta_2}^{(n_2 + 1)}}I_{\mathbb{R}^+ \times \mathbb{R}^+}(x, y)
\]

is too a special case of (4.1) with \( \xi_1, \xi_2 = 0, 1, 2, \ldots \) and \( \delta_1 = \delta_2 = 1. \)

One could also consider the symmetrized versions of the above distributions whose density functions are given by

\[
    f_s(x, y) = \frac{f(|x|, |y|)}{4}I_{\mathbb{R}^2}(x, y).
\]

For instance the bivariate normal distribution whose density function is

\[
    f_N(x, y) = \frac{e^{-x^2/2\sigma_1^2}e^{-y^2/2\sigma_2^2}}{2\pi \sigma_1 \sigma_2}I_{\mathbb{R}^2}(x, y), \quad \sigma_i, i = 1, 2
\]

is the symmetrized form of the bivariate half-normal density specified by (4.10) with \( \sigma_i^2 = \pi/(2\phi_i^2), \) \( i = 1, 2. \)

5. Parameter estimation and application

The bivariate model (2.8) is fitted to a data set previously analyzed by Yue [25], which consists of flood peaks (variable \( u \)) and volumes (variable \( w \)) as observed in the Madawask basin, Quebec, Canada from 1919 to 1995. A histogram of this data set is shown in Fig. 3.

This data is first normalized by means of the standardizing transformation

\[
    V^{-1/2}\left(\begin{array}{c}
    u - \bar{u} \\
    w - \bar{w}
    \end{array}\right),
\]

where \( \bar{u} = 254.74, \) \( \bar{w} = 9184.03 \) and \( V^{-1/2} = (V_{ii}^{-1}) \) denotes the inverse of the symmetric square root of the estimated covariance matrix denoted by \( V, \) with
Then, constants $\gamma_1$ and $\gamma_2$ are respectively added to the first and second coordinates so that the support of the transformed distribution lies within the first quadrant (i.e., $u > 0$ and $w > 0$). These constants are determined as follows:

$$
\gamma_1 = \text{Absolute value of } \min \left( V_{11}^{-1/2} (u - \bar{u}) + V_{12}^{-1/2} (w - \bar{w}) \right) \\
+ \frac{1}{2} \left[ \max \left( V_{11}^{-1/2} (u - \bar{u}) + V_{12}^{-1/2} (w - \bar{w}) \right) - \min \left( V_{11}^{-1/2} (u - \bar{u}) + V_{12}^{-1/2} (w - \bar{w}) \right) \right];
$$

$$
\gamma_2 = \text{Absolute value of } \min \left( V_{21}^{-1/2} (u - \bar{u}) + V_{22}^{-1/2} (w - \bar{w}) \right) \\
+ \frac{1}{2} \left[ \max \left( V_{21}^{-1/2} (u - \bar{u}) + V_{22}^{-1/2} (w - \bar{w}) \right) - \min \left( V_{21}^{-1/2} (u - \bar{u}) + V_{22}^{-1/2} (w - \bar{w}) \right) \right],
$$

with

$$
\min \left( V_{11}^{-1/2} (u - \bar{u}) + V_{12}^{-1/2} (w - \bar{w}) \right) = -2.4579,
$$

$$
\max \left( V_{11}^{-1/2} (u - \bar{u}) + V_{12}^{-1/2} (w - \bar{w}) \right) = 2.22751,
$$

$$
\min \left( V_{12}^{-1/2} (u - \bar{u}) + V_{22}^{-1/2} (w - \bar{w}) \right) = -2.0631
$$

and

$$
\max \left( V_{12}^{-1/2} (u - \bar{u}) + V_{22}^{-1/2} (w - \bar{w}) \right) = 2.0550
$$

Thus, the following transformation is applied to the original data:

$$
\begin{pmatrix} x \\
 y \end{pmatrix} = V^{-1/2} \begin{pmatrix} u - \bar{u} \\
 w - \bar{w} \end{pmatrix} + \begin{pmatrix} \gamma_1 \\
 \gamma_2 \end{pmatrix}.
$$

In order to estimate the parameters of the proposed bivariate gamma-type distribution as specified by the density function appearing in Eq. (2.8), the loglikelihood of the transformed sample is then maximized with respect to the parameters by making use of the \texttt{NMaximize} command in the symbolic computational package \textit{Mathematica}. Given the transformed data $(x_i, y_i), i = 1, \ldots, 77$, the loglikelihood function is actually given by

$$
\ell(\lambda_1, \lambda_2, p_1, p_2, a, b, c, \alpha, \delta_1, \beta, \delta_2) = \sum_{i=1}^{n} \log \left( f(x_i, y_i; \lambda_1, \lambda_2, p_1, p_2, a, b, c, \alpha, \delta_1, \beta, \delta_2) \right)
$$

$$
= n \log(\mathcal{C}) + \sum_{i=0}^{n} \log \left( \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{\left( a \right)_k m \left( a \right)_m (2 \pi)^{3/2}}{c \left( b \right)_k !} \frac{\left( b \right)_m !}{m!} \frac{\mathcal{C}_k}{\mathcal{C}_m} \sum_{i=1}^{\infty} x_i^{k-1} y_i^{m-1} e^{-x_i} e^{-y_i} x_i^{k} y_i^{m} \right), \tag{5.1}
$$
where \( f(x, y; \lambda_1, \lambda_2, p_1, p_2, a, b, c, z, \delta_1, \beta, \delta_2) \) is as given in (2.8). The parameter estimates were determined to be \( \bar{a} = 0.0229287, \bar{b} = 0.0135392, \bar{c} = 0.014, \bar{p}_1 = 0.013, \bar{p}_2 = 0.199541, \bar{z} = 0.0011, \bar{\beta} = 0.00815306, \bar{\delta_1} = 3.28416, \bar{\delta_2} = 2.16555, \bar{\lambda_1} = 5.80589 and \bar{\lambda_2} = 10. \) Finally, the inverse transformation,

\[
\begin{pmatrix} u \\ w \end{pmatrix} = \sqrt{V} \begin{pmatrix} x - \gamma_1 \\ y - \gamma_2 \end{pmatrix} + \begin{pmatrix} \bar{u} \\ \bar{w} \end{pmatrix}
\]

is applied to the variables \((x, y)\) in order obtain a density function that can be utilized as a model for the distribution of the original data. Noting that the Jacobian of the inverse of that transformation is the determinant of \( V^{-1/2} \) or, equivalently, the inverse of the determinant of \( V^{1/2} \), the resulting joint density function is

\[
h(u, w) = \frac{1}{|V|^{1/2}} f\left(V_{11}^{-1/2}(u - \bar{u}) + V_{12}^{-1/2}(w - \bar{w}) + V_{21}^{-1/2}(u - \bar{u}) + V_{22}^{-1/2}(w - \bar{w}) + \gamma_2\right).
\]

It is seen that the resulting bivariate density estimate, which is plotted in Fig. 4, generally reflects the main features of the original data.

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**Appendix A. Derivation of the moment generating function**

The inverse Mellin transform and transformation of variable techniques are employed to derive the moment generating function of the proposed bivariate gamma-type distribution. Let \(X\) and \(Y\) be random variables whose joint density is specified by (2.8). By definition, their joint moment generating function is

\[
M_{X,Y}(t_1, t_2) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{a_j b_m e^{j t_1} e^{m t_2}}{(j! m!)} \int_{0}^{\infty} x^{j+m-1} e^{-p_x e^x} e^{t_1 x} dx \times \int_{0}^{\infty} y^{j+m-1} e^{-p_y e^y} e^{t_2 y} dy.
\]

Now consider an integral of the type

\[
\int_{0}^{\infty} x^v e^{-\alpha x} dx.
\]

We show that (A.2) is proportional to the density of the ratio of random variables \(X_1\) and \(X_2\) whose pdf’s are

\[
g_1(x_1) = c_1 x^{\delta_1-1} e^{-x^\gamma_1} I_{\gamma_1}(x_1)
\]

and

\[
g_2(x_2) = c_2 x^{\delta_2-1} e^{-x^\gamma_2} I_{\gamma_2}(x_2),
\]

respectively, \(c_1\) and \(c_2\) being normalizing constants. Let \(u = x_1/x_2\) and \(v = x_2\) so that \(x_1 = u v\) and \(x_2 = v\), the absolute value of the Jacobian of the inverse transformation being \(v\). Thus, the joint density of the random variables \(U\) and \(V\) is \(v g_1(u v)g_2(v)\) and the marginal density of \(U = X_1/X_2\) is

\[
h_1(u) = \int_{0}^{\infty} v g_1(u v)g_2(v) dv,
\]

that is,

\[
h_1(u) = c_1 c_2 \int_{0}^{\infty} e^{-(u v)^{\delta_1}} v^{\delta_2-2} e^{v^{\delta_2}} dv,
\]

which, on letting \(u = \theta^{1/k}\) and \(v = x\), becomes

\[
h_1(\theta^{1/k}) = c_1 c_2 \int_{0}^{\infty} e^{x^{\delta_2-1}} e^{-x^\delta_1} dx.
\]

Alternatively, the density of \(X_1/X_2\) can be obtained by means of the inverse Mellin transform technique. The inverse Mellin transform of \(U = X_1/X_2\) is then

\[
h_1(u) = \frac{c_1 c_2}{k(-s)^s} \frac{1}{2\pi i} \int_{C} (-u/s)^{-s} \Gamma(t/k) \Gamma(\eta - t) dt,
\]

\(s, k, \varphi\) being such that \(\eta > s > 0\) and \(0 < k < 1\) and \(s\) and \(k\) are not simultaneously zero.
where \( C \) denotes the Bromwich path described in the introduction. Thus, in terms of an \( H \)-function as defined in (1.4), one has

\[
h_1(u) = \frac{c_1c_2}{k(-s)^k} H_1^{(1)} \left( \frac{u}{s} \left( \frac{1 - \eta, 1}{0, 1/k} \right) \right), \quad s < 0. \tag{A.5}\]

On replacing \( \eta \) with \( \lambda_1 + \delta_1j.k \) with \( \delta_1 \) and \( s \) with \( t_1 \) in the first integral appearing in the R.H.S. of (A.1) and \( \eta \) with \( \lambda_2 + \delta_2m.k \) with \( \delta_2 \) and \( s \) with \( t_2 \) in the second, it is seen that the moment generating function of the bivariate gamma-type density function appearing in (2.8) can be expressed as

\[
h(u_1, u_2) = \frac{C}{\delta_1(-t_1)^k\delta_2(-t_2)^m} \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(-j)^m} \left( \frac{\theta_jm \alpha^j \beta^m}{H_{1,1}(1)} \left( \frac{1 - \lambda_1 - \delta_1j.1}{\theta_jm \alpha^j \beta^m} \right) \right)\left( \frac{1 - \lambda_2 - \delta_2m.1}{\theta_jm \alpha^j \beta^m} \right).	ag{A.6}\]

When \( k = p/q \) where \( p \) and \( q \) are positive integers, one can express the integral in (A.4) as a Meijer's \( G \)-function by setting \( z = t/p \) and making use of the Gauss–Legendre multiplication formula,

\[
\Gamma(r + qz) = (2\pi)^{1/2} q^{-1/2} \sum_{k=0}^{\infty} \frac{1}{1/2} \left( \frac{k + r}{q} \right)\text{.} \tag{A.7}\]

Then, on letting \( u = \theta^{1/k} = (d_1^{1/k}) \), one has

\[
h_1(\theta^{1/k}) = \frac{c_1c_2(2\pi)^{1/2} q^{-1/2} \theta^{1/2} \theta p^{1/2}}{\left(1 - \frac{1}{\theta^{p/q}} \right)^{1/2}} G_{p,q}^{q,p} \left( \frac{\theta}{\theta^{p/q}} \left( \frac{1 - \frac{1}{\theta^{p/q}}}{\theta^{p/q}} \right) \right).	ag{A.8}\]

Since the expressions in (A.8) and (A.3) are equal when \( k = p/q \), one has

\[
\int_0^\infty x^{q-1} e^{-x} dx = (2\pi)^{1/2} q^{-1/2} \theta^{1/2} \theta p^{1/2} \left(1 - \frac{1}{\theta^{p/q}} \right)^{1/2} G_{p,q}^{q,p} \left( \frac{\theta}{\theta^{p/q}} \left( \frac{1 - \frac{1}{\theta^{p/q}}}{\theta^{p/q}} \right) \right).	ag{A.9}\]

On replacing \( \eta \) with \( \lambda_1 + \delta_1j. \theta \) with \( p_1.k \) with \( \delta_1 = p/q \) and \( s \) with \( t_1 \) in the first integral appearing in the R.H.S. of (A.1) and \( \eta \) with \( \lambda_2 + \delta_2m. \theta \) with \( p_2.k \) with \( \delta_2 = f/g \) and \( s \) with \( t_2 \) in the second, and then making use of (1.7), we obtain the moment generating function (2.15) of the bivariate gamma-type density function specified by (2.8).

References