An approach to singular modules by indeterminacy concept

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Abstract

In this article, we characterize a singular module and present several new strong relationships with neutrosophic (NC) properties. Some properties and characterizations neutrosophic of singular modules are given. Also, different basic results about these modules are considered. Moreover; for a (R ∪ I)-module; if NC(M∪I) over NC(R∪I), then NC(Z(M∪I)) ≤ NC(M∪I). Any neutrosophic simple module is either nonsingular or singular. On the other hand, if NC(RUI) has no zero divisors then NC(Z(MUI)) = NC(T(MUI)) where NC(T(MUI)) is a neutrosophic torsion module. Finally, some definitions and properties of neutrosophic singular module have been presented in this article.

Keywords: Singular module; Free module Annihilator module; Neutrosophic ring; Neutrosophic submodule.

1. Introduction

Smarandache [1] in 1999 introduced a new mathematical idea called neutrosophic set as an extension of fuzzy set [2]. This idea aroused the attention of numerous researchers around the world, by merging it with other branches of mathematics. For example, Salama and Alblowi [3] offered the notion of neutrosophic topological spaces utilizing idea of neutrosophic sets and many of their applications see [4,5]. Kandasamy and Smarandache [6] first established algebraic structures with neutrosophic ideas, following them Abed et al. [7,8] merged neutrosophic with module theory. Al-Sharqi et al. [9,10] applied the idea of neutrosophic with complex values and used this idea to solve some real-life applications. On the other hand, algebraic structures is a fertile environment that has attracted the attention of many researchers around the world, Al-Jumaili et al. [11] investigated a new class of mappings with strongly closed graphs in topological algebraic structures. Al-Hamido [12] discussed a new approach to a neutrosophic algebraic structure called a neutrosophic groupoid. Zail et al. [13] studied some new results of a generalization of BCK-algebra (Ω-BCK-algebra). and other contributions, see [14-16].
Throughout this paper will be assumed any ring is a commutative with identity and every module is unitary. Any submodule A of an R-module M is called essential [17] if there exists B is a submodule of M such that A∩B≠0. An element m belong to module M over the ring R[18] is called singular element if the right of annihilator m: ann(m)= {r∈ R: mr =0} is essential in R. We denote Z(M) to the set of all singular elements:

$$Z(M)=\{m \in M: ann(M) \leq_{ess} R\}.$$

A module M is called singular [19] if Z(M)=M and is called nonsingular if Z(M)=0. On the other hand, Kasch [20] defined torsion free module by; if T(M)=M and is called torsion free module if T(M)=0 where T(M) refer to all torsion element in M.

This paper is organized as follows: In Section 2 recall different notions and fundamental results which play vital role in this study. In Section 3, we display several characterizations and essential properties concerning of

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Indeterminacy of one important module namely singular module. Finally, the difficulties and importance of the study were discussed in the conclusion section in Section 4.

2. Preliminaries

Definition 2.1. [1] We refer to neutrosophic set NC(S) by

B={≺u, (μ_B(u), β_B(u), δ_B(u)≻) : u ∈ U}

where U the universe of discourse and u is an element of U with μ, β, δ : → [0, 1] are define respectively the degree of membership (True), the degree of indeterminacy and the degree of non-membership (False) of u in U such that

{$0 \leq (\mu_B(u) + \beta_B(u) + \delta_B(u) \leq 3$}

Note: We denote NC(R ∪ I) to neutrosophic ring and NC(M ∪ I) neutrosophic module.

Definition 2.2. [6] Let I be neutrosophic of element with I^2=I and let (G, *) be a group such that ≺ G ∪ I ≻={x+yI : x, y ∈ G}. The set NC(G)={ ≺ G ∪ I ≻, *} is called neutrosophic group and generate by G and I**.**

Definition 2.4. [6] The set NC(R)={ ≺ R ∪ I ≻, +, *} = {x+yI x, y ∈ R} is called neutrosophic ring and generate by R and I where R is a ring with two binary operation + and *.

Examples 2.5. [6] The ring of integer number Z, the ring of integer modulo n Z_n and the ring of rational numbers Q (≺ Z ∪ I ≻= x+yI x, y ∈ Z, ≺ Z_n ∪ I ≻ = x+yI x, y ∈ Z_n and ≺ Q ∪ I ≻) = x+yI x, y ∈ Q) are called NC(Z), NC(Zn) and NC(Q) respectively**.**

Definition 2.6. [13] Any R-module (M, +, .) or (I(M), +, .) is called weak neutrosophic of M (NC(M)) if M(I)= ≺ M ∪ I ≻ is NC(S) and generate by M and I.

Remark 2.7. **[13]** If M(I) is neutrosophic module over R =≺ R ∪ I ≻, then M(I)= ≺ M ∪ I ≻ is called strong eutrosophic (R ∪ I)-module.

Example 2.8. (R ∪ I) is very important example of NC(R ∪ I). Also, NC(G ∪ I) is another example.

Definition 2.9. [14] If we have NC(M ∪ I) over NC(R ∪ I), then NC((N ∪ I)) is a strong NC(N ∪ I) submodule of NC(M ∪ I) if NC(N ∪ I) is itself strong too.

3. The main Results

In this section we study the Indeterminacy of one important module namely singular module. Also, some new results have been presented in this paper.

Definition 3.1. (Im) in the NC(M∪I) over NC(R∪I) means the right of annihilator (Im): ann(Im)={Ir∈ NC((R∪I)): (Imr) =I0} is essential in (R∪I).

Remark 3.2. We denote NC(Z(M∪I)) to the set of all neutrosophic singular module:

$$NC(Z(M\cup I))=\{Im \in NC((M\cup I)): ann(M\cup I)\leq_{ess} NC((R\cup I))\}.$$

Proposition 3.3. Suppose that we have NC(M∪I) over NC(R∪I). Then NC(Z(M∪I)) ≤ NC (M∪I).

Proof:

To prove NC(Z(M∪I)) is a submodule of NC((M∪I)), we need to show that NC(Z(M∪I)) ≠ φ. It is clear that (I0) ∈ NC(Z(M∪I)) and hence NC(Z(M∪I)) ≠ φ. Assume that Ix, Iy are two elements in Z(M∪I). So

$$ann(Ix)\leq_{ess} NC((R\cup I)) \text{ and } ann(Iy))\leq_{ess} NC((R\cup I)).$$

But

$$ann(I(x-y))\supset ann(Ix) \cap ann(Iy) \leq_{ess} NC((R\cup I)).$$

So

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$$Ix\text{-}Iy=I(x\text{-}y) \in NC(Z(M\cup I)).$$

Let $Ix \in NC(Z(M\cup I))$ and $Ir \in NC(Z(R\cup I))$. We must prove that $I(xr) \in NC(Z(M\cup I))$, that is to show $ann(Ix) \leq_{ess} NC((R\cup I))$. Given any element $Ia \in NC((R\cup I))$ - $ann(I(xr))$. So, $I(xr)Ia \neq I0$. But $ann(Ix) \leq_{ess}(R\cup I)$. Then

$$\exists\ Ib \in NC((R\cup I)) \ni I(ra)\ Ib \in ann(Ix) \text{ and } I(ra)Ib \neq I0.$$

So

$$I(x(rab))=I0,\ Ib \in NC((R\cup I)) \ni I(rab) \neq I0.$$

We have $I(ab) \neq I0$ and $I(ab) \in ann(I(xr))$. Thus, $ann(I(xr)) \leq_{ess} NC((R\cup I))$.

Remarks 3.4.

a) We say $NC(Z(M\cup I))=\{Im \in NC(M\cup I): (Im)A=(I0)$ for some $NC(A\cup I) \leq_{ess} NC(R\cup I)\}$; because: Let $NC(H\cup I) =\{Im \in NC(M\cup I): (Im)A=(I0), NC(A\cup I) \leq_{ess} NC(R\cup I)\}$. We must prove that $NC(HUI)=NC(Z(MUI))$.

Let $Im \in NC(Z(MUI))$. Then $ann(Im) \leq_{ess} NC(RUI)$. So $(Im).(ann(Im))=0$. Hence

$$Im \in NC(HUI). \text{ Therefore } NC(HUI) \supseteq NC(Z(MUI)).$$

Now, the other direction, suppose that $Im \in NC(HUI)$. So,

$$(Im)A=0,\ NC(AUI) \leq_{ess} NC(RUI).$$

But $NC(AUI) \subseteq ann(Im)$. Then $ann(Im) \leq_{ess} NC(RUI)$. Therefore $Im \in NC(Z(MUI))$. Hence,

$$NC(HUI) \subseteq NC(Z(MUI)).$$

Then

$$NC(Z(MUI))) = NC(HUI).$$

Thus

$$NC(HUI)= NC(Z(MUI)).$$

a)Z(Im). Soc(NC(RUI))=0, because: let $Im \in NC(Z(MUI))$. Then $ann(Im) \leq e_{ss} NC(RUI)$. But Soc (NC(RUI)) equal the intersection of all neutrosophic of essential ideals NC(AUI). Hence $Soc(NC(RUI)) \subseteq ann(Im)$. So,

$$(Im).(Soc(NC(RUI))=0\ \forall\ Im \in NC(Z(MUI))$$

Thus

$$NC(Z(MUI)).(Soc(NC(RUI))) =0.$$

Corollary 3.5. Let $f:NC(MUI) \rightarrow NC(NUI)$ be a neutrosophic (RUI)-homonorphism. Then $f(NCZ(MUI))) \subseteq NC(Z(NUI))$.

Proof:

Suppose that $f:NC(MUI) \rightarrow NC(NUI)$ be a neutrosophic (RUI)-homonorphism. Let $Iy \in f(NC(Z(MUI)))$ such that $Iy=f(Im)$; $Im \in Z(MUI)$. Hence $Ann(Im) \leq_{ess} NC(RUI)$. But

$ann(Im) \subseteq ann(f(Im))$. Since $\forall\ Ir \in ann(Im)$,so $I(mr)=0$.Then $f(Imr)=0$. Hence $f(Im)(Ir)=I0$. Therefore $Ir \in ann(f(Im))$. So $ann(f(Im)) \leq e_{ss} NC(RUI)$. Thus $Iy=f(Im) \in NC(Z(NUI))$.

Corollary 3.6. Let NC(MUI) be neutrosophic module and $NC(NUI) \leq NC(MUI)$. Then $NC(Z(NUI)) = NC(Z(MUI)) \cap NC(NUI)$.

Since any $Im \in NC(Z(NUI))$. So, $Im \in NC(NUI)$ with $ann(Im) \leq_{ess} NC(RUI)$. Hence

$$NC(Z(NUI)) \leq NC(Z(NUI)) \cap NC(NUI).$$

Clear that; $NC(Z(NUI)) \cap NC(NUI) \leq NC(Z(NUI))$. Thus, the result is obtained.

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Examples 3.7.

a) Let NC(ZUI) be neutrosophic as a Z-module. The neutrosophic singular submodule of NC(ZUI) equal (IO). Since ∀ Ix ∈NC(ZUI) and IX≠ I0, so ann(Ix) $\not\leq_{ess}$NC(ZUI).

b) Let NC(Z_{12}UI) be neutrosophic as (Z UI)-module. ∀ IX ∈ NC(Z_{12}UI) with $ann_{(ZUI)}$ (IX) ≠0 and so ann(IX)$\leq_{ess}$NC(ZUI). Hence NC(Z(Z_{12}UI))=NC(Z_{12}UI) and then ∀ In ∈ NC(ZUI); In>1), NC(Z(Z_nUI))=NC(Z_nUI).

C) Let NC(Z_4UI) be neutrosophic as NC(Z_4UI)-modul. So nc(Z($_4$UI))={$\overline{I0}, \overline{I2}$}<NC($Z_4$UI).

d) For the (Z_4UI)-module (QUI), NC(Z(QUI))=0

Now we study the meutrosophic of singular module and non-singular module, but before that we need to present the following definition.

Definition 3.8. An NC(RUI)-module NC(MUI) is called neutrosophic singular if NC(Z(MUI))=NC(MUI) and it is called neutrosophic non-singular if NC(Z(MUI))= I0.

Remarks 3.9 .

a) NC(RUI) is called neutrosophic singular(non-singular) if NC(RUI) as NC(RUI)-module is neutrosophic singular (neutrosophic non-singular) .

b) ∀ (In) ∈ NC(ZUI); (In) > (I1), imply that NC(Z_nUI) as (ZUI)-module is neutrosophic singular, while each of (ZUI)-module NC(ZUI), NC(QUI) are neutrosophic non-singular and NC(Z_nUI) as (ZUI)-module is neutrosophic non-singular.

c) Let NC(NUI) be neutrosophic submodule of NC(MUI). If NC(MUI) is singular implies NC(NUI) is also singular and if NC(MUI) is nonsingular, so NC(NUI) is also nonsingular.

d) any neutrosophic simple module is either nonsingular or singular. On the other hand, if NC(RUI) has no zero divisors then NC(Z(MUI)) = NC(T(MUI)) where NC(T(MUI)) is a neutrosophic torsion module.

Proof (d)

Since NC(T(MUI)) ={Ix∈NC(T(MUI)): ann(Ix)≠I0} and NC((RUI)) has no zero divisors, then ∀ Ix ∈ NC(T(MUI)). Henc ann(Ix)$\leq_{ess}$NC(R∪ I). Thus

NC(T(M∪ I))≤NC(Z(M ∪I) ∀ Ix ∈ NC(Z(MUI)).

So,

ann(Ix) ≠I0 and hence Ix ∈ NC(T(MUI)).

Therefore,

NC(Z(MUI)) ⊆ NC(T(MUI)).

Thus

NC(Z(MUI)) = NC(T(MUI)).

Proposition 3.10. Let NC(MUI) be a module over NC(RUI). Then NC(MUI) is nonsingular if and only if Hom[NC(M_1UI), NC(MUI)]= where NC(M_1UI) is a singular module.

Proof:

Suppose that NC(MUI) is nonsingular. Then NC(Z(MUI)) =I0. Let NC(M_1UI) be a singular module. So NC(Z(MUI))= NC(M_1UI).

Take NC(M_1UI) → NC(MUI) as NC(RUI)-homomorphism. Then

f(NC(Z(M_1UI)) ⊆ NC(Z(MUI)).

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Hence

$$f(NC(M_1UI) \subseteq (I0).$$

Thus,

$$f(NC(M_1UI)=I0\ (f=i0).$$

Conversely: to prove NC(MUI) is a nonsingular. Since

$$NC(Z(NC(Z(MUI)))=NC(Z(MUI))$$

$$=NC(Z(MUI)) \cap NC(MUI)$$

$$=NC(Z(MUI))$$

with NC(ZNC(Z(MUI)))=NC(Z((MUI)), then NC(Z(MUI)) is a is singular module. Hence

$$Hom(NC(Z(MUI)), NC(MUI)=I0.$$

Also, NC(Z(MUI)) $\leq$ NC(MUI)). Therefore, i $\in$ Hom(NC(Z(MUI)), NC(MUI)=I0, i is inclusion mapping. Hence i=I0 with NC(Z((MUI))=I0. Thus NC(MUI) is a nonsingular.

Proposition 3.11. Let NC(MUI) be neutrosophic NC(RUI)-module. Then NC(MUI) is singular if and only if there exists I0 $\rightarrow NC(M_1\cup I) \xrightarrow{f} NC(M_2\cup I) \xrightarrow{g}$ I0, where f is a neutrosophic essential monomorphism.

Proof:

Suppose that NC(MUI) is a singular. Take an exact sequence

$$I0 \rightarrow NC(M_1\cup I) \xrightarrow{i} NC(M_2\cup I) \xrightarrow{g} I0$$

with $NC(M_1\cup I) \subseteq NC(M_2\cup I)$ and $NC(M_2\cup I)$ is a neutrosophic free module.

Let $\{Im_\beta\}$ be a neutrosophic basis for $NC(M_2\cup I)$. Then $g(Im_\beta)$ $(A_\beta\cup)$=I0 $\ni$ $(A_\beta\cup I)$ is a neutrosophic essential ideai ($NC(M_2\cup I)$ is a singular). Hence

$$g(Im_\beta A_\beta)=I0, \text{ so } m_\beta A_\beta \subseteq Ker(g).$$

But ker(g)=Im(i)=$NC(M_1\cup I)$. So,

$$I(m_\beta A_\beta) \subseteq NC(M_1\cup I)\ \forall\ I_\beta.$$

Since $IA_\beta \leq_{ess} NC(R\cup I)$, we obtain $I(m_\beta A_\beta) \leq_{ess} NC((R\cup I)$.

Therefore

$$\oplus(Im_\beta A_\beta) \leq_{ess} \oplus(Im_\beta NC(R\cup I))=NC(M_2\cup I).$$

But $I(m_\beta A_\beta) \leq NC((M_1\cup I) \subseteq NC((M_2\cup I)$. Then, $NC(M_1\cup I) \leq_{ess} NC(M_2\cup I)$. Thus $f:NC(M_1\cup I) \rightarrow NC(M_2\cup I)$ is essential monomorphism.

Conversely: Assume that we have neutrosophic exact sequence. Let $Im_2 \in (M_2\cup I)$ and let $h:NC(R\cup I) \rightarrow NC(M_2\cup I)$ $\ni$ h(Ir)=I(m_2r); Ir $\in$ NC(R∪I).

Since $f(M_1\cup I) \leq_{ess} NC((M_2\cup I)$, then $h^{-1}(f(M_1 \cup I) \leq_{ess} NC((R\cup I)$.

But

$$h^{-1}(f(M_1 \cup I)=\{Ir \in NC(R\cup I):h(Ir) \in f(M_1\cup I)\}$$

$$= Ir \in NC(R\cup I):I(m_2 r) \in f(M_1\cup I)\}.$$

Let NC(A∪I)= $h^{-1}(f(M_1\cup I)$. So,

$NC(A\cup I) \leq_{ess} NC((R\cup I)$ and $I(m_2A) \leq f(M_1\cup I)$=Ker(g).

Then $g(Im_2)A$=I0 and hence $g(Im_2) \in NC(Z(M\cup I))$. But g is an epi. Hence

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$$\forall \text{ Im} \in \text{NCM}\cup \text{ I}) \ni \text{Im}_2 \in \text{NC}(\text{M}\cup\text{I}) \text{ with } g(\text{Im}_2)=\text{Im}.$$

Hence NC(Z(M∪ I)). So, ∀ Im ∈ NC(M∪I) ∋Im_2∈NC(R∪I) with g(Im_2)=Im, we obtain the following

$$\text{NC}(\text{M}\cup\text{I})= \text{NC}(\text{Z}(\text{M}\cup \text{ I})).$$

Thus NC(M∪ I) is a singular.

2. Conclusion

The concepts of neutrosophy and neutrosophic sets have putted the foundation for a whole family of novel mathematical theories to generalizing both their classical and fuzzy counterparts, such as a neutrosophic set theory . The main target of our manuscript is to display and study a new idea of neutrosophic of singular modules Also, different basic results about these modules are considered. Moreover; for a (R ∪ I)-module; if NC(M∪I) over NC(R∪I), then NC(Z(M∪I)) ≤ NC(M∪I). Any neutrosophic simple module is either nonsingular or singular. Finally, some definitions and properties of neutrosophic singular module have been investigated in this article.

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