A Closed-Form EM Algorithm for Sparse Coding

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Abstract
We define and discuss the first sparse coding algorithm based on closed-form EM updates and continuous latent variables. The underlying generative model consists of a flexibly parameterized ‘spike-and-slab’ prior and a standard Gaussian noise model. Closed-form solutions for E- and M-step equations are derived by generalizing probabilistic PCA. The resulting EM algorithm infers all model parameters including the level of sparsity, and it takes all modes of a potentially multi-modal posterior into account. The computational cost of the algorithm scales exponentially with the number of hidden dimensions. However, with current computational resources, efficient training of model parameters is still possible for medium-scale problems (involving up to 20 hidden dimensions). Thus the model can be applied to the typical range of source separation tasks. In numerical experiments on artificial data we verify likelihood maximization and show that the derived algorithm recovers the sparse directions of standard sparse coding distributions. On source separation benchmarks comprising realistic data we show that the algorithm is competitive with other recent methods.

1 Introduction
Probabilistic generative models are a standard approach to model data distributions and to infer instructive information about the data generating process. Furthermore, they have been very successful in providing a common framework for relating different Machine Learning algorithms previously considered as being largely independent (see, for instance, [Roweis and Ghahramani, 1999; Bishop, 2006]). As examples, principle component analysis, factor analysis, or sparse coding (e.g., [Olshausen and Field, 1996]) have all been formulated in the form of probabilistic generative models. In the limit of zero observation noise (see, e.g., [Dayan and Abbott, 2001]), sparse coding recovers a form of independent component analysis (ICA), and ICA itself is a very popular approach to blind source separation.

A standard procedure to optimize parameters in generative models is the application of Expectation Maximization (EM; [Dempster et al., 1977; Neal and Hinton, 1998]). However, for many generative models the optimization using EM is analytically intractable. For stationary data only the most elementary models such as mixture models and factor analysis (which contains probabilistic PCA as special case) have closed-form solutions for E- and M-step equations. EM for more elaborate models requires approximations. In particular, sparse coding models (Olshausen and Field, 1996; Lee et al. 2007; Seeger, 2008, and many more) require approximations because integrals over the latent variables do not have closed-form solutions.

In this work we study a generative model that combines the Gaussian prior of probabilistic PCA (p-PCA) with a binary prior distribution. This combination will maintain the p-PCA property of having closed-form solutions for EM optimization but it extends the model to include sparse coding like distributions. At the same time, composing the prior with two parts allows for a higher flexibility of the prior which can potentially model the true generating process more precisely. Distributions combining binary and continuous parts have been discussed and used as priors before (e.g.

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Consider a set of independent data points \( \{ \mathbf{y}^{(n)} \}_{n=1,...,N} \) with \( \mathbf{y}^{(n)} \in \mathbb{R}^D \). For these data we seek parameters \( \Theta = (\mathbf{\pi}, W, \sigma) \) that maximize the data likelihood \( \mathcal{L} = \prod_{n=1}^{N} p(\mathbf{y}^{(n)} | \Theta) \) under the GSC generative model. To maximize the likelihood we apply Expectation Maximization in the form studied by (Neal and Hinton 1998). That is, instead of maximizing the likelihood directly, we maximize the free-energy given by:

\[
\mathcal{F}(\Theta^{\text{old}}, \Theta) = \sum_{n=1}^{N} \sum_{s, \mathbf{z}} p(s, \mathbf{z} | \mathbf{y}^{(n)}, \Theta^{\text{old}}) \left[ \log p(\mathbf{y}^{(n)} | \mathbf{z}) + \log (p(s | \mathbf{\pi})) + \log (p(\mathbf{z} | \Theta)) \right] d\mathbf{z} + H(\Theta^{\text{old}}),
\]

where \( \mathbf{\pi} \) parameterizes the probability of non-zero entries. After generation, both vectors are combined using a pointwise multiplication:

\[
(s \odot \mathbf{z})_{h} = s_{h} z_{h} \quad \text{for all } h.
\]

The resulting hidden variable (which we will keep denoting by \( \mathbf{s} \odot \mathbf{z} \)) follows a spike-and-slab distribution, and where \( \mathbf{\pi} \) parameterizes the data noise (assumed to be independent per observed variable). The prior \( p(\mathbf{s}, \mathbf{z} | \Theta) \) and the noise distribution \( p(\mathbf{z} | \Theta) \) define the generative model under consideration. As a special case, the model contains probabilistic PCA. This can easily be seen by setting all \( \pi_{h} \) equal to one and mean vector \( \mu \) of the continuous latent part to zero. In fact, for the sake of brevity and readability, we will derive the algorithm holding the latter true and give the general case for non-zero \( \mu \) afterwards.

The model \( (\mathbf{1}) \) to \( (4) \) is capable of generating a broad range of distributions including sparse coding like distributions. This is illustrated by Fig. \( \mathbf{1} \) which also demonstrates that the parameters \( \pi_{h} \) allow for continuously changing PCA-like to a SC-like distribution. As such distributions are generated using Gaussian distributions for all continuous variables, we will refer to the generative model \( (1) \) to \( (4) \) as the Gaussian Sparse Coding (GSC) model.

### 2 Expectation Maximization (EM) for Parameter Optimization

Figure 1: Distributions generated by the GSC generative model. The left column shows the distributions generated for \( \pi_{h} = 1 \) for all \( h \). In this case the model generates p-PCA distributions. The middle column shows an intermediate value of \( \pi_{h} \). The generated distributions are not Gaussians anymore but have a slight star shape. The right column shows distributions for small values of \( \pi_{h} \). The generated distributions have a salient star shape similar to standard sparse coding distributions.

[Image 108x561 to 327x711]
where $H(\Theta^{old})$ is an entropy term only depending on parameter values held fixed during the optimization of $F$ w.r.t. $\Theta$. Note that integration over the hidden space involves an integral over the continuous part and a sum over the binary part.

Optimizing the free-energy consists of two steps: given the current parameters $\Theta^{old}$ the posterior probability is computed in the E-step; and given the posterior, $F(\Theta^{old}, \Theta)$ is maximized w.r.t. $\Theta$ in the M-step. Iteratively applying E- and M-steps locally maximizes the data likelihood.

**Parameter update equations.** Let us first consider the maximization of the free-energy in the M-step before considering expectation values w.r.t. to the posterior in the E-step. For brevity and readability we will show derivations for the case of a zero-mean prior distribution ($\bar{\mu} = 0$ in Eqn. 1), and only give results for the general case at the end of this section. Given a generative model, conditions for a maximum free-energy are canonically derived by setting the derivatives of $F(\Theta^{old}, \Theta)$ w.r.t. the second argument to zero. For the GSC model we obtain for the parameters $\bar{\pi}$, $W$, and $\sigma^2$:

**M-step Equations**

$$\bar{\pi} = \frac{1}{N} \sum_{n=1}^{N} \langle \tilde{s} \rangle_n,$$

$$W = \left( \sum_{n=1}^{N} \tilde{y}^{(n)} \langle \tilde{s} \odot \tilde{z} \rangle_n^T \right) \left( \sum_{n=1}^{N} \langle (\tilde{s} \odot \tilde{z}) (\tilde{s} \odot \tilde{z})^T \rangle_n \right)^{-1},$$

$$\sigma^2 = \frac{1}{ND} \sum_{n=1}^{N} \left[ \langle (\tilde{y}^{(n)})^T \tilde{y}^{(n)} \rangle - 2 \langle (\tilde{y}^{(n)})^T W \tilde{s} \odot \tilde{z} \rangle_n + \text{Tr} \left( W^T W \langle (\tilde{s} \odot \tilde{z}) (\tilde{s} \odot \tilde{z})^T \rangle_n \right) \right].$$

where

$$\langle f(\tilde{s}, \tilde{z}) \rangle_n = \sum_{\tilde{s}} \int_{\tilde{z}} p(\tilde{s}, \tilde{z} | \tilde{y}^{(n)}, \Theta^{old}) f(\tilde{s}, \tilde{z}) \, d\tilde{z}.$$  

**Proof**

The derivations of equations (6) and (7) follow the same lines as the derivations of update rules for p-PCA or sparse coding (see, e.g., Olshausen and Field, 1996; Dayan and Abbott, 2001; Bishop, 2006).

Equations (6) to (7) define a new set of parameter values $\Theta = (\bar{\pi}, W, \sigma)$ given the current values $\Theta^{old}$. These ‘old’ parameters only enter the equations through the expectation values $\langle \tilde{s} \rangle_n$, $\langle \tilde{s} \odot \tilde{z} \rangle_n$, and $\langle (\tilde{s} \odot \tilde{z}) (\tilde{s} \odot \tilde{z})^T \rangle_n$. The expectations themselves are therefore often referred to as the sufficient statistics of the model.

**Expectation Values.** Although the derivation of M-step equations can be analytically intricate, it is the E-step that, for most generative models, poses the major challenge. Source of the problems involved are analytically intractable integrals required for posterior distributions and for expectation values w.r.t. the posterior. The true posterior is therefore often replaced by an approximate distribution (see, e.g., Bishop, 2006; Seeger, 2008) or in the form of factored variational distributions (Jordan et al., 1998; Jaakkola, 2000). The most frequently used approximation remains, however, the maximum-a-posterior approximations (MAP; see, e.g., Olshausen and Field, 1996; Lee et al., 2007) which replaces the true posterior by a delta-function around the posterior’s maximum value. Alternatively, analytically intractable expectation values are often approximated using sampling approaches. Using approximations always implies, however, that many analytical properties of exact EM are not maintained. Approximate EM iterations may, for instance, decrease the likelihood or may not recover (local or global) likelihood optima in many cases. The limited set of models with exact EM solutions contains p-PCA (and more generally factor analysis) and mixture models such as the mixture-of-Gaussians. Here we extend the set of known models with exact EM solutions by showing that the analytical tractability of p-PCA can be maintained for the GSC model.

To facilitate later derivations first note that the discrete latent variable of the GSC model can be combined with the matrix of basis functions such that

$$W (\tilde{s} \odot \tilde{z}) = \tilde{W}_{xz} \tilde{z},$$

with $(\tilde{W}_{xz})_{dh} = W_{dh} \delta_{hl}.  

By applying (8), Bayes’ rule, and Gaussian identities we can show that the posterior $p(\tilde{s}, \tilde{z} | \tilde{y}^{(n)}, \Theta)$ decomposes into two factors as follows:

**Posterior Probability**

$$p(\tilde{s}, \tilde{z} | \tilde{y}^{(n)}, \Theta) = p(\tilde{s} | \tilde{y}^{(n)}, \Theta) \mathcal{N}(\tilde{z}; \kappa_{\tilde{z}}^{(n)}, \Lambda_{\tilde{z}}),$$
where \( p(\hat{s} | \hat{y}^{(n)}, \Theta) = \frac{\mathcal{N}(\hat{y}^{(n)}; \hat{0}, C_\Sigma) p(\hat{s} | \Theta)}{\sum_{\hat{s}^{'}} \mathcal{N}(\hat{y}^{(n)}; \hat{0}, C_\Sigma) p(\hat{s}^{'} | \Theta)} \) with \( C_\Sigma = \hat{W}_s \hat{W}_s^T + \sigma^2 \mathbb{I}_D \), (11)

and \( M_s = \hat{W}_s^T \hat{W}_s + \sigma^2 \mathbb{I}_H \), \( \pi_s^{(n)} = (M_s)^{-1} \hat{W}_s^T \hat{y}^{(n)} \), \( \Lambda_s = \sigma^2 (M_s)^{-1} \). (12)

**Proof**

First note that we can (in analogy to p-ACA) re-write:

\[
\mathcal{N}(\hat{y}^{(n)}; W(\hat{s} \odot \hat{z}), \sigma^2 \mathbb{I}_D) \mathcal{N}(\hat{z}; \hat{0}, \mathbb{I}_H) = \mathcal{N}(\hat{y}^{(n)}; \hat{0}, C_\Sigma) \mathcal{N}(\hat{z}; \hat{r}_s^{(n)}), \Lambda_s),
\]

where we have used (9) and Gaussian identities. Using Bayes’ rule we thus get:

\[
p(\hat{s}, \hat{z} | \hat{y}^{(n)}, \Theta) = \frac{\mathcal{N}(\hat{y}^{(n)}; \hat{0}, C_\Sigma) \mathcal{N}(\hat{z}; \hat{r}_s^{(n)}), \Lambda_s) p(\hat{s} | \Theta) \sum_{\hat{s}^{'}} \mathcal{N}(\hat{y}^{(n)}; \hat{0}, C_\Sigma) \mathcal{N}(\hat{z}; \hat{r}_s^{(n)}), \Lambda_s) p(\hat{s}^{'} | \Theta) \}
\]

Marginalizing over \( \hat{z} \) then yields:

\[
p(\hat{s} | \hat{y}^{(n)}, \Theta) = \int p(\hat{s}, \hat{z} | \hat{y}^{(n)}, \Theta) d\hat{z} = \frac{\mathcal{N}(\hat{y}^{(n)}; \hat{0}, C_\Sigma) p(\hat{s} | \Theta) \sum_{\hat{s}^{'}} \mathcal{N}(\hat{y}^{(n)}; \hat{0}, C_\Sigma) p(\hat{s}^{'} | \Theta), (13)
\]

and we thus obtain (10) and (11).

Equations (10) and (11) represent the crucial result for the computation of the E-step below because, first, they show that the posterior does not involve analytically intractable integrals and, second, for fixed \( \hat{s} \) and \( \hat{y}^{(n)} \) the dependency on \( \hat{z} \) follows a Gaussian distribution. This special form allows for the derivation of analytical expressions for the expectation values as required for the M-step (6) to (7). They are given by:

**E-step Equations**

\[
\langle \hat{s} \rangle_n = \sum_{\hat{s}} p(\hat{s} | \hat{y}^{(n)}, \Theta) \hat{s}
\]

\[
\langle \hat{s} \odot \hat{z} \rangle_n = \sum_{\hat{s}} p(\hat{s} | \hat{y}^{(n)}, \Theta) \hat{s} \odot \hat{r}_s^{(n)}
\]

\[
\langle (\hat{s} \odot \hat{z})(\hat{s} \odot \hat{z})^T \rangle_n = \sum_{\hat{s}} p(\hat{s} | \hat{y}^{(n)}, \Theta) (\hat{s} \hat{z}) \odot (\hat{s} \hat{z})^T
\]

Note that we have to use the current values \( \Theta = \Theta^{old} \) for all parameters on the right-hand-side.

**Proof**

For all derivations we used the posterior form (10) and solve standard forms of Gaussian integrals:

\[
\langle \hat{s} \rangle_n = \sum_{\hat{s}} p(\hat{s} | \hat{y}^{(n)}, \Theta) \hat{s} \langle \int_{\mathbb{R}^D} \mathcal{N}(\hat{z}; \hat{r}_s^{(n)}, \Lambda_s) d\hat{z} \rangle
\]

\[
\langle \hat{s} \odot \hat{z} \rangle_n = \sum_{\hat{s}} p(\hat{s} | \hat{y}^{(n)}, \Theta) \hat{s} \odot \langle \int_{\mathbb{R}^D} \mathcal{N}(\hat{z}; \hat{r}_s^{(n)}, \Lambda_s) d\hat{z} \rangle
\]

\[
\langle (\hat{s} \odot \hat{z})(\hat{s} \odot \hat{z})^T \rangle_n = \sum_{\hat{s}} p(\hat{s} | \hat{y}^{(n)}, \Theta) (\hat{s} \hat{z}) \odot (\hat{s} \hat{z})^T \odot \langle \int_{\mathbb{R}^D} \mathcal{N}(\hat{z}; \hat{r}_s^{(n)}, \Lambda_s) d\hat{z} \rangle
\]

where \( A \odot B \) denotes pointwise multiplication of matrix elements.

The E-step equations (14) to (16) represent a closed-form solution for expectation values required for the closed-form M-step (6) to (7).

**Generalization.** A generalization to include prior-distributions with non-zero mean marginals \( (\mathbb{N} \neq \hat{0}) \) in Eqn (1) is straight-forward. M- and E-step equations are derived following the same steps as for \( \hat{v} = \hat{0} \). First the posterior of the generalized model is given by:

\[
p(\hat{s}, \hat{z} | \hat{y}^{(n)}, \Theta) = p(\hat{s} | \hat{y}^{(n)}, \Theta) \mathcal{N}(\hat{z}; \hat{v}_s^{(n)}, \Lambda_s),
\]

(17)
where \[ p(\tilde{s} | \tilde{y}^{(n)}, \Theta) = \frac{\mathcal{N}(\tilde{y}^{(n)}; \tilde{\mu}_s, C_s) p(\tilde{s} | \Theta)}{\sum_{\tilde{s}'} \mathcal{N}(\tilde{y}^{(n)}; \tilde{\mu}_s', C_{s'}) p(\tilde{s}' | \Theta)} \] (18)

with \( \tilde{\mu}_s = \tilde{W}_s \tilde{\nu}, \) and \( \tilde{\nu}_s^{(n)} = (\tilde{s} \odot \tilde{\nu}) + (M_s)^{-1} \tilde{W}_s^T (\tilde{y}^{(n)} - \tilde{\mu}_s). \) (19)

The parameters \( C_s, M_s \) and \( \Lambda_s \) given in (11) and (12) remain unchanged. The derivation of equation (17) follows along the same lines as the derivation for (10). Also note that equations (17) and (18) get reduced to (10) and (11) as \( \tilde{\nu} \to 0. \) Finally the E-step equations (15) and (16) have to be revised by replacing \( \tilde{\nu}_s^{(n)} \) with \( \tilde{\nu}_s^{(n)}. \)

The M-step equations (6) to (7) remain unchanged. By setting the derivative of free-energy (5) w.r.t. \( \tilde{\nu} \) to zero the additional update is given by:
\[ \tilde{\nu} = \sum_{n=1}^{N} \langle \tilde{s} \odot \tilde{z} \rangle_n / \sum_{n=1}^{N} \langle \tilde{s} \rangle_n. \] (20)

3 Numerical Experiments

As for all algorithms based on closed-form EM and unlike algorithms involving approximations, the GSC algorithms will never decrease the data likelihood. In practice, EM increases the likelihood at least to local likelihood maxima. As parameter optimization is non-convex, we have to numerically investigate, however, how frequently local optima are obtained and how well the derived algorithm performs on more practical tasks.

Model verification. First, we verify on artificial data that the algorithm increases the likelihood and that it can recover the parameters of the generating distribution. For this, we generated \( N = 500 \) data points \( \tilde{y}^{(n)} \) from the GSC generative model (1) to (4) with \( H = 2 \) hidden dimensions and \( D = 2 \) observed dimensions. Using a fixed parameter set \( \Theta^{gen} \), we generated one fixed set of \( N \) data points (11). To the generated data, we then applied the GSC algorithm and ran 300 EM iterations. As initial parameters \( \Theta \) for each run we randomly and uniformly initialized \( \pi_h \) between 0.05 and 10, set \( \sigma^2 \) to the variance across the data points, and the elements of \( W \) we chose to be independently drawn from a normal distribution with zero mean and unit variance. The algorithm was run 250 times on the data (each time with another random initialization). In all runs the generating parameter values were recovered with high accuracy. Runs with different generating parameters \(^1\) produced essentially the same results.

![Comparison of standard sparse coding and GSC](image)

Figure 2: Comparison of standard sparse coding and GSC. **Left panels**: Cauchy distribution (along one hidden dimension) as a standard SC prior\(^{(1)}\) and data generated by it. **Right panels**: Spike-and-slab distribution (one of the hidden dimensions) inferred by the GSC algorithm along with inferred sparse directions (solid red lines).

Recovery of sparse directions. The previous experiment used data generated by the GSC generative model itself. While such data is useful for the verification of the algorithm, it is in practice rarely the case that a data distribution is exactly matched by the distribution assumed by an algorithm. Data distributions could thus have heavier tails than the distribution assumed by the GSC model, or the spike-and-slab property of the prior might not be reflected by the data. To test the model’s robustness w.r.t. a relaxation of the GSC assumptions, we applied the GSC algorithm to a standard sparse coding model. We used a standard Cauchy prior and a Gaussian noise model\(^{(1)}\) for data generation. Fig.2 second panel shows data generated by this sparse coding model while the first panel shows the prior density along one of its hidden dimensions. We generated \( N = 500 \)

\(^{1}\)We obtained \( W^{gen} \) by independently drawing each matrix entry from a normal distribution with zero mean and standard deviation 3. \( \pi^{gen}_h \) values were drawn from a uniform distribution between 0.05 and 1, and \( (\sigma^{gen})^2 \) was uniformly drawn from values between 0.05 and 10.
The experiments show that the GSC algorithm can recover sparse directions. Thus, for any data that is assumed to follow a standard SC distribution, we can expect the GSC algorithm to find the heavy tail directions (i.e., the sparse directions) of standard SC were recovered. As a measure for the match between actual and recovered directions we use the Amari index \( A(W) = \frac{1}{2H(H-1)} \sum_{h,h' = 1}^{H} \left( \frac{|O_{h,h'}|}{\max_{i,j} |O_{h,i,j}|} + \frac{|O_{h,h'}^{gen}|}{\max_{i,j} |O_{h,i,j}|^{gen}} \right) - \frac{1}{H-1} \) \( (21) \), which shows a very accurate recovery of the sparse directions. Fig. 2B (right) visualizes the distribution recovered by the GSC algorithm in a typical run. The dotted red lines show the density contours of the learned distribution \( p(\tilde{y} | \Theta) \). The high accuracy in the recovery of the generating sparse directions (solid black lines) can be observed by comparison with the recovered directions (solid red lines). The results of experiments are qualitatively the same if we increase the number of hidden and observed dimensions. E.g., for \( H = 4 \) and \( D = 4 \), we found the algorithm converged to high likelihood values in 91 of 100 runs. The average Amari index accross all 91 runs with high likelihoods was again below \( 10^{-2} \).

Other than data generated by standard SC with Cauchy prior, we also ran the algorithm on data generated by SC with Laplace prior \( (\text{Olshausen and Field, 1996}; \text{Lee et al., 2007}) \). The qualitative results are the same: in most case the algorithm converged to high likelihood values, and for high likelihood values we recovered the sparse directions with high accuracy. For \( H = 2 \) and \( D = 2 \) dimensions we converged to high likelihood values in 99 of 100 runs. The average Amari index of all runs with high likelihood values was 0.06 which is slightly higher than for the Cauchy prior. In the same experiment but with \( H = 4 \) and \( D = 4 \) the algorithm converged to high likelihood values in 97 of 100 runs. The average Amari index of all runs with high likelihoods was 0.07 in this case.

The experiments show that the GSC algorithm can recover sparse directions. Thus, for any data that is assumed to follow a standard SC distribution, we can expect the GSC algorithm to find the heavy tail directions. For most tasks these sparse directions are the only relevant parameters. Source separation represent a prominent such example, which makes this task a natural application domain of the GSC algorithm.

**Source separation.** We applied both standard and generalized versions of the GSC algorithm to publicly available datasets. We used the non-artificial benchmarks of a recent paper \( (\text{Suzuki and Sugiyama, 2011}) \). The datasets 10halo, Sergio7, Speech4, c3signals contain mainly acoustic data obtained from \( (\text{ICALAB}; \text{Cichocki et al., 2007}) \). As is customary for source separation.
Sparse Coding (SC) algorithms seek to represent data points by combinations of, on average, few hidden components. In their generative formulation, individual hidden dimensions are typically assumed independent and distributed according to a super-Gaussian probability density (heavy-tail distribution). In applications, e.g., to source separation, SC algorithms can infer heavy-tail direction and thus allow for the unmixing of signals from different sources. The GSC algorithms falls into the same class as standard SC algorithms. However, instead of a heavy-tail distribution a spike-and-slab distribution is used. In the direction of any hidden dimension the spike-and-slab density converges to zero as a Gaussian distribution but the property of on average few non-zero components is maintained. In numerical experiments we have shown that this allows for the recovery of sparse directions for data distributed according to standard SC distributions (see Fig. 2). Another class of...
models that is often used to recover sparse directions is independent component analysis (Comon, 1994; Hyvärinen and Oja, 1997). ICA algorithms are more general than most SC algorithms in the sense that they seek non-Gaussian direction rather than only super-Gaussian ones. If data is sub-Gaussian, ICA can usually recover hidden dimensions where standard SC algorithms can not. On the other hand, ICA does assume a deterministic relation between the observed and hidden vectors and is thus not modeling observation noise (see, e.g., Dayan and Abbott, 2001, for a discussion).

The GSC algorithm belongs to the class of SC algorithms that can not find the hidden directions when they are sub-Gaussian. Comparison on source separation benchmarks (Tab. 1) therefore shows poor performance if the data contains sub-Gaussian sources c5signals. Among SC models, the most wide-spread approach to train model parameters is the MAP approximation to compute the expectation values required in the M-step. Various version with different sophisticated methods to efficiently find the MAP estimate have been studied in the literature (e.g. Tibshirani, 1996). MAP estimates, however, only represent the maximal value of the desired full posterior probability. Furthermore, MAP optimizations usually require regularization parameters that have to be set by hand or have to be inferred by applying cross-validation. Approximations that take more aspects of the posterior into account than just its maximum are therefore actively investigated (e.g. Seeger, 2008; Mohamed et al., 2010). The GSC algorithm studied in this paper uses the full posterior for parameter optimization, which represents a distinguishing feature amongst all previously studied SC algorithms. Furthermore, the prior is flexibly parameterized and automatically adjusts to different levels of sparsity. No regularization parameter has to be set by hand or has to be determined using cross-validation. Furthermore, using the full posterior avoids potential biases introduced by approximations. MAP and Laplace estimations assume monomodal posterior distributions, for instance. If posteriors are not monomodal, these approximations can negatively effect performance. As we can compute posterior distributions in our model exactly, all modes of the posteriors are taken into account for the parameter updates.

Closed-form EM learning of the algorithm also comes with a cost, however. Without approximations, the computational resources required scale exponentially with the number of hidden dimensions. This can be seen by considering (11) where the partition function requires a summation over all binary vectors $\tilde{s}$ (similar for expectation values w.r.t. the posterior). Based on current computational resources, the GSC algorithms can be run efficiently on tasks with up to 20 hidden dimensions, and is thus well applicable to the typical range of source separation tasks. More hidden dimensions resulted in unreasonable amounts of computation times. In terms of required computational resources the GSC algorithm can therefore not compete with ICA or other SC approaches. The example of source separation shows, however, that many well-established task domains only require a relatively low dimensional hidden space. In such domains the GSC algorithm benefits from taking potentially multimodal posterior into account and it allows from inferring the full set of model parameters including the sparsity. Furthermore, when using a number of hidden dimensions larger than the number of sources, the model can discard dimensions by setting $\pi_h$ parameters to zero. The studied model could thus be considered as treating parameter inference in a more Bayesian way than, e.g., SC with MAP estimates. However, the most rigorous way would be a fully Bayesian approach that not only infers the hidden dimensions using Bayesian non-parameterics but also selects the optimal noise model (compare Lee et al., 2007) which can in general be non-Gaussian. A model that is currently under development goes in this direction (Mohamed et al., 2010). It also shares the spike-and-slab prior with the GSC approach. In (Mohamed et al., 2010) the Bayesian methodology with conjugate priors and hyperparameters in combination with Laplace approximation and different sampling schemes is used to infer the parameters. While the aim in (Mohamed et al., 2010) is a large flexibility, the approach discussed in this paper aims at keeping the generative model in a form that still allows for closed-form EM solutions. Nevertheless, further generalizations of the GSC model are still possible. While for p-PCA a change of the Gaussian prior distribution does not result in a larger set of generating distribution, the same does not apply for the GSC model. Replacing (2) by a general multi-variate Gaussian thus extends the model to genuinely new distributions. Also the prior distribution (2) can be changed, e.g., to include dependent binary variables. Likewise, we can generalize the noise distribution (4) to have different noise variances for the different observed variables as in factor analysis. All these extensions involve more parameters with more intricate E- and M-step equations. However, the principal property of closed-form EM solutions can be maintained. This is because the posterior can maintain its principle factored structure (10). In these cases, the procedure for deriving closed-form update equations remains essentially unchanged (compare Sec. 2).
To summarize, we have studied a novel sparse coding algorithm and have shown its competitiveness on source separation benchmarks. The algorithm assumes a continuous latent space and parameter optimization is based on closed-form EM updates. For continuous latent variables, a sparse coding algorithm with closed-form EM solution has not been reported previously, maybe because sparse coding has always been assumed to require approximations. Along with our numerical results, the main contribution of this study is therefore the demonstration that, based on the appropriate assumptions, a closed-form EM solution for sparse coding can indeed be derived.

References


