On the stability and convergence of the time-fractional variable order telegraph equation

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Abstract

In this work, we have generalized the time-fractional telegraph equation using the concept of derivative of fractional variable order. The generalized equation is called time-fractional variable order telegraph equation. This new equation was solved numerically via the Crank–Nicholson scheme. Stability and convergence of the numerical solution were presented in details. Numerical simulations of the approximate solution of the time-fractional variable order telegraph equation were presented for different values of the grid point.

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1. Introduction

It would be difficult to imagine a world without communication systems. A plethora of guided fixed line telephones as well as a multitude of unguided systems to serve cellular phones are evident in our surrounding world. The telegraph equations are a pair of linear differential equations which describe the voltage and current on an electrical transmission line with distance and time. The equations come from Oliver Heaviside, who in the 1880s developed the transmission line model. The model demonstrates that, the electromagnetic waves can be reflected on wire, and that wave patterns can appear along the line. The theory applies to transmission lines of all frequencies including high-frequency transmission line such as telegraph wires, radio frequency conductors, audio frequency such as telephone lines, low frequency such as power lines and direct current. Tomasi 2004 [3] supplied the theoretical background on transmission and transmission lines, including open-wire lines and cables [39]. In [39] a loss power transmission line excited by a step voltage at the sending-end and shorted at the receiving-end is analyzed. In 1995, Wylie [6] gave some derivations for the telegraph equations. The classical telegraph equation is originally deduced by the variational relationship between the voltage wave and the current wave on the well-proportioned transmission line. So it is also called the transmission line equation. However, the classical telegraph equation cannot well describe the abnormal diffusion phenomena during finite long transmissions, where the voltage wave or the current wave possibly exist [4–6]. Therefore, it is necessary to investigate the fractional telegraph equation, including the time and (or) space fractional derivatives.

The fractional telegraph equation has recently been considered by many authors. For instance Cascaval et al. [7] discussed the time-fractional telegraph equations, dealing with well-posedness, using the Riemann–Liouville approach and presenting a study involving asymptotic concepts. Orsingher and Begh in [10] discussed the time-fractional telegraph equation and telegraph processes with Brownian time, showing that some processes are governed by time-fractional telegraph equations.

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Chen et al. [9] also discussed and derived the solution of the time-fractional telegraph equation with three kinds of non-homogeneous boundary conditions, making use of the method of variables separation. Orsingher and Zhao [8] considered the space-fractional telegraph equations, by obtaining the Fourier transform of its fundamental solution, and presenting a symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation. Momani [11] discussed analytic and approximate solutions of the space- and time-fractional telegraph differential equations by means of the so-called Adomian decomposition method. Dehghan et al. [12] investigated the so-called general space–time fractional telegraph equations by the method of differential and integral calculus, discussing the solution using Laplace and Fourier transforms in variables $t$ and $x$ respectively. However, the variable order fractional derivative is an extension of constant order-fractional derivative. Variable-order fractional derivative are very useful when investigating the memory properties which change with time and spatial location [40]. Several innovative studies by some eminent researchers [40–43] have shown that, many complex physical problems can be described with great success via variable order derivatives. A novel study underlining the advantages of using these derivatives rather than constant order fractional derivative had been presented in [40]. The area of variable-order derivatives is more recent, it is worthwhile to note that the definition given below as RL1 was considered in the 1990s only [1]. The evaluation of dissimilar potential definitions is concerned in [2]. The paper addresses mostly variable negative orders, considering positive orders only in transient. Applications are given in [21–25] to viscoelasticity, processing of geographical data in [35], to signature verification in [36] and diffusion processes in [37,38]. Since the equations described by the variable order derivatives are highly complex, difficult to handle analytically, it is therefore advisable to investigate their solutions numerically. Possible numerical implementations of variable-order fractional derivatives are given in [21–25].

The aim of this work is to further investigate the possibility to extent the telegraph equation with derivatives of fractional order, to the concept of variable order derivative. In this paper we will study the stability and the convergence analysis of the following telegraph equation:

$$D_{\alpha(t,x)}^{\alpha(x,t)} w + k \frac{\partial w}{\partial t} \frac{\partial w}{\partial x^2} + bw = 0, \quad 1 < \alpha(x, t) \leq 2, \quad k > 0 \text{ and } b > 0,$$

with the initial conditions

$$u(x, 0) = \delta(x), \quad \frac{\partial u(x, 0)}{\partial t} = 0$$

and boundary conditions

$$u(0, t) = u(k \pi, t) = 0,$$

where $\delta(x)$ is a real continuous function, $0 \leq t \leq T$, $0 \leq x \leq L$, and $k$ is a natural number.

2. Brief history of the variable order derivative

Variable-order fractional derivatives, which are an extension of constant-order fractional derivatives, have been introduced in several physical fields [13–15]. The variable-order fractional derivative is good at depicting memory properties that change with time or spatial location [16–18].

2.1. Preliminaries

For fractional analysis and its applications we refer to the book in [19]. Let’s recall the relevant definitions for variable order fractional calculus [20].

**Definition 2.1.** Left and right Riemann–Liouville integrals of variable order: Let $0 < \alpha(x, t) < 1$ for all $(x, t) \in [a, b]$ and $f \in L_1[a, b]$ then,

$$a_{t}^{\alpha(x,t)}(f(t)) = \int_{a}^{t} \frac{1}{\Gamma[\alpha(x,t)]} (t-x)^{\alpha(x,t)-1} f(x)dx \quad (t > a)$$

is called the left Riemann–Liouville integral of variable fractional order $\alpha(.,.)$. While

$$b_{t}^{\alpha(x,t)}(f(t)) = \int_{t}^{b} \frac{1}{\Gamma[\alpha(x,t)]} (x-t)^{\alpha(x,t)-1} f(x)dx \quad (t < b)$$

is referred to as the right Riemann–Liouville integral of variable fractional order $\alpha(.,.)$.

**Definition 2.2.** Left and right Riemann–Liouville derivatives of variable fractional order: Let $0 < \alpha(t,x) < 1$ for all $(x, t) \in [a, b]$. If $a_{t}^{1-\alpha(x,t)} f \in AC[a, b]$ then, the left Riemann–Liouville derivative of variable fractional order $\alpha(.,.)$ is given as:
\[ aD_t^{\alpha(t)}(f(t)) = \frac{d}{dt} \int_a^t \frac{1}{\Gamma[1 - \alpha(t, x)]} (t - x)^{-\alpha(t, x)} f(x) \, dx \quad (t > a) \]  

(5)

Likewise we have the following expression referred to as the right Riemann–Liouville derivative of variable fractional order \( \alpha(t, .) \).

\[ bD_t^{\alpha(t)}(f(t)) = \frac{d}{dt} \int_t^b \frac{1}{\Gamma[1 - \alpha(t, x)]} (x - t)^{-\alpha(t, x)} f(x) \, dx \quad (t < b) \]  

(6)

**Definition 2.3.** Left and right Caputo derivatives of variable fractional order: Let \( 0 < \alpha(t, x) < 1 \) for all \((t, x) \in [a, b] \). If \( aD_t^{1-\alpha(t)} f \in AC[a, b] \) then, the left Caputo derivative of variable fractional order \( \alpha(t, .) \) is given as:

\[ aD_t^{\alpha(t)}(f(t)) = \int_a^t \frac{1}{\Gamma[1 - \alpha(t, x)]} (t - x)^{-\alpha(t, x)} \frac{d}{dx} f(x) \, dx \quad (t > a) \]  

(7)

Likewise, we have the following expression referred to as the right Caputo derivative of variable fractional order \( \alpha(t, .) \).

\[ bD_t^{\alpha(t)}(f(t)) = \int_t^b \frac{1}{\Gamma[1 - \alpha(t, x)]} (x - t)^{-\alpha(t, x)} \frac{d}{dx} f(x) \, dx \quad (t < b) \].  

(8)

However, we will use throughout this work the following definition:

**Definition 2.4.** Let \( f : \mathbb{R} \rightarrow \mathbb{R}, x \rightarrow f(x) \) denote a continuous and necessary differentiable function, let \( \alpha(x) \) be a continuous function in \((0, 1]\). Then its variable order differential in \([0, M]\) is defined as:

\[ D_t^{\alpha(x)}(f(x)) = \frac{1}{\Gamma(1 - \alpha(x))} \int_0^x (x - t)^{-\alpha(t)} \frac{df(t)}{dt} \, dt \]  

(9)

The above derivative is called the Caputo variable order differential operator, and computes the derivative of a constant to be zero.

3. Numerical solutions

The finite difference schemes for constant-order time or space fractional diffusion equations have been widely studied [21–25]. For constant-order time fractional diffusion equations, the implicit difference scheme was proposed in [26]. The weighted average finite difference method was suggested in the work by [27]. A matrix approach for fractional diffusion equations was presented in [28] and a flexible numerical scheme for the discretization of the space–time fractional diffusion equation in [29]. Recently, a numerical scheme for VO space fractional advection–dispersion equation was investigated in [30]. Lin investigated the explicit scheme for VO nonlinear space fractional diffusion equation [31].


To establish numerical schemes for the above equation, we let \( x_l = lh, 0 \leq l \leq M, \ Mh = L, \ t_k = k\tau, 0 \leq k \leq N, \ N\tau = T, \ h \) is the step size and \( \tau \) is the time, \( M \) and \( N \) are grid points.

We introduce the Crank–Nicholson scheme as follows. Firstly, the discretization of first and second order time derivative is stated as:

\[ \frac{\partial w}{\partial t} = \frac{1}{2} \left( \frac{w(x_{l+1}, t_{k+1}) - w(x_{l-1}, t_{k+1})}{2(\tau)} + \frac{w(x_{l+1}, t_k) - w(x_{l-1}, t_k)}{2(\tau)} \right) + O(\tau^2) \]  

(10)

\[ \frac{\partial^2 w}{\partial t^2} = \frac{1}{2} \left( \frac{w(x_{l+1}, t_{k+1}) - 2w(x_{l+1}, t_k) + w(x_{l-1}, t_{k-1})}{(\tau)^2} \right) \]  

\[ + \frac{w(x_{l+1}, t_k) - 2w(x_{l}, t_k) + w(x_{l}, t_{k-1})}{(\tau)^2} \right) + O(\tau^2) \]  

(11)

\[ w = \frac{1}{2} \left( w(x_l, t_{k+1}) + w(x_l, t_k) \right) \]  

(12)
\[
\frac{\partial^2 w}{\partial x^2} = \frac{1}{2} \left( \left( (w(x_{i+1}, t_{k+1}) - 2w(x_i, t_{k+1}) + w(x_{i-1}, t_{k+1})) \right) \right)
+ \left( \left( w(x_{i-1}, t_{k+1}) - 2w(x_i, t_k) + w(x_{i+1}, t_k) \right) \right) + O(h^2)
\]

The Crank–Nicholson scheme for the VO time fractional diffusion model can be stated as follows:

\[
\frac{\partial_t^{\alpha_k+1} w(x_i, t_{k+1})}{\partial_t^{\alpha_k+1}} = \frac{\tau^{-\alpha_k+1}}{\Gamma(2-\alpha_k+1)} \left( w(x_i, t_{k+1}) - w(x_i, t_k) \right)
+ \sum_{j=1}^{k} \left[ [w(x_i, t_{k+1-j}) - w(x_i, t_{k-j})] \right] (j+1)^{1-\alpha_k+1} - (j)^{1-\alpha_k+1} \right)
\]

Now replacing Eqs. (10), (11), (12), (13) in (1) we obtain the following:

\[
\frac{\tau^{-\alpha_k+1}}{\Gamma(2-\alpha_k+1)} \left( w(x_i, t_{k+1}) - w(x_i, t_k) \right)
+ \sum_{j=1}^{k} \left[ [w(x_i, t_{k+1-j}) - w(x_i, t_{k-j})] \right] (j+1)^{1-\alpha_k+1} - (j)^{1-\alpha_k+1} \right)
\]

For simplicity:

\[
w(x_i, t_k) = w^k_i, \quad T^{k+1}_i = \frac{k}{4\tau} \Gamma(2-\alpha_k+1) \tau^{\alpha_k+1}, \quad A^{k+1}_k = \frac{a}{2h^2} \Gamma(2-\alpha_k+1) \tau^{\alpha_k+1},
\]

\[
B^{k+1}_l = \frac{b}{2} \Gamma(2-\alpha_k+1) \tau^{\alpha_k+1}, \quad b^{l,k+1} = (j+1)^{1-\alpha_k+1} - (j)^{1-\alpha_k+1}, \quad \text{and}
\]

\[
\lambda^{l,k+1} = b^{l,k+1} - b^{l,k+1}
\]

Then Eq. (15) can be reduced to:

\[
w^{k+1}_l - w^k_l + \sum_{j=1}^{k} \left[ w^{k+1}_{l+j} - w^k_{l+j} \right] b^{l,k+1} = -T^{k+1}_i \left[ w^{k+1}_{l+1} - w^k_{l+1} + w^k_l - w^k_{l-1} \right]
\]

\[
+ A^{k+1}_l \left[ w^{k+1}_{l+1} - 2w^k_{l+1} + w^k_{l+1} - 2w^k_{l+1} + w^k_{l-1} - B^{k+1}_l \right] \left[ w^{k+1}_l + w^k_l \right]
\]

We can rearrange the above equation in the following form:

\[
w^{k+1}_l \left[ 1 + 2A^{k+1}_l + B^{k+1}_l \right] = w^{k+1}_{l+1} \left[ A^{k+1}_l - T^{k+1}_i \right] + w^k_{l+1} \left[ T^{k+1}_i + A^{k+1}_l \right] + w^k_l \left[ 1 - 2A^{k+1}_l - B^{k+1}_l \right]
\]

\[
+ w^k_{l+1} \left[ T^{k+1}_i - T^{k+1}_l \right] - \sum_{j=1}^{k} \left[ w^{k+1}_{l+j} - w^k_{l+j} \right] b^{l,k+1}
\]

4. Stability analysis

In this section, we will investigate the stability conditions of the Crank–Nicholson scheme for the time-fractional VO telegraph equation.

Let \( \xi^k = w^k_i - W^k_i \), here \( W^k_i \) is the approximate solution at the point \( (x_i, t_k) \), \( k = 1, 2 \ldots N, l = 1, 2 \ldots M - 1 \) and in addition \( \xi^k = [\xi^k_{1, 2, \ldots, \xi^k_{M-1}}] \) and the function \( \xi^k(x) \) is chosen to be:

\[
\xi^k(x) = \begin{cases} 
\xi^k & \text{if } x_i - \frac{h}{2} < x \leq x_i + \frac{h}{2}, \quad l = 1, 2 \ldots M - 1 \\
0 & \text{if } L - \frac{h}{2} < x \leq L.
\end{cases}
\]
Then, the function $\zeta^k(x)$ can be expressed in Fourier series as follows:

$$
\zeta^k(x) = \sum_{m=-\infty}^{m=\infty} \delta_k(m) \exp[2i\pi m k/L]
$$

$$
\delta_k(x) = \frac{1}{L} \int_0^L \rho_k(x) \exp[2i\pi mx/L] dx.
$$

(19)

It was established by [26] that

$$
\|\rho^2\|^2 = \sum_{m=-\infty}^{m=\infty} \|\delta_k(m)\|^2
$$

(20)

examine that for all $k, l \geq 1, 0 \leq 1 - \alpha^{l+1}_k < 1$. In addition,

Remarks 1. The following properties of the coefficients $T_i^{k+1}, A_1^{k+1}, \lambda_j^{l,k+1}, b_i^{k+1}$ and $b_i^{l+1}$ can be established.

1. $A_i^{k+1}, T_i^{k+1}$ and $B_i^{k+1}$ are positive for all $l = 1, 2 \ldots M - 1$

2. $0 < \lambda_j^{l,k} \leq \lambda_j^{l,k-1} \leq 1$ for all $l = 1, 2 \ldots M - 1$

3. $0 \leq b_j^{l,k} \leq 1, \sum_{j=0}^{k-1} b_j^{l,k+1} = 1 - \lambda_k^{l,k+1}$ for all $l = 1, 2 \ldots M - 1$

(21)

The error done while employing the Crank–Nicholson scheme to solve our VO telegraph equation can be expressed as follows:

$$
\zeta_i^{k+1}[1 + 2A_i^{k+1} + B_i^{k+1}] = \zeta_i^{k+1}[A_i^{k+1} - T_i^{k+1}] + \zeta_i^{k+1}[T_i^{k+1} + A_i^{k+1}] + \zeta_i^{k+1}[1 - 2A_i^{k+1} - B_i^{k+1}]
$$

$$
+ \zeta_i^{k+1}[A_i^{k+1} - T_i^{k+1}] - \sum_{j=1}^{k} \zeta_i^{k+1-j} - \zeta_i^{k-j} b_j^{l,k+1}
$$

(22)

If we assume that: $\zeta_i^{k}$ in Eq. (22) can be put in the delta-exponential form as follows:

$$
\zeta_i^{k} = \delta_k \exp[i\phi k]
$$

(23)

where $\phi$ is a real spatial wave number, replacing the above equation (23) in (22) we obtain the following expressions:

$$
\delta_1 \left[1 + 4 \sin^2 \left(\frac{\phi h}{2}\right) A_i^{k+1} + 2 \sin^2 \left(\frac{\phi h}{2}\right) B_i^{k+1}\right] = \delta_0 \left[1 - 4 \sin^2 \left(\frac{\phi h}{2}\right) A_i^{k+1} - 2 \sin^2 \left(\frac{\phi h}{2}\right) B_i^{k+1}\right]
$$

for $k = 0$

(24)

and

$$
\delta_{k+1} \left[1 + 4 \sin^2 \left(\frac{\phi h}{2}\right) A_i^{k+1} + 2 \sin^2 \left(\frac{\phi h}{2}\right) B_i^{k+1}\right] = \delta_k \left[1 - 4 \sin^2 \left(\frac{\phi h}{2}\right) A_i^{k+1} - 2 \sin^2 \left(\frac{\phi h}{2}\right) B_i^{k+1}\right]
$$

$$
- \sum_{j=0}^{k-1} \lambda_{j+1}^{l,k+1} \delta_{k-j} + \lambda_k^{l,k+1} \delta_0
$$

for $k = 1, 2 \ldots N - 1$.

(25)

Eqs. (24) and (26) can be reformulated as follows:

$$
\delta_1 = \delta_0 \left[1 - 4 \sin^2 \left(\frac{\phi h}{2}\right) A_i^{k+1} - 2 \sin^2 \left(\frac{\phi h}{2}\right) B_i^{k+1}\right]
$$

$$
[1 + 4 \sin^2 \left(\frac{\phi h}{2}\right) A_i^{k+1} + 2 \sin^2 \left(\frac{\phi h}{2}\right) B_i^{k+1}]
$$

(26)

$$
\delta_{k+1} = \delta_k \left[1 - 4 \sin^2 \left(\frac{\phi h}{2}\right) A_i^{k+1} - 2 \sin^2 \left(\frac{\phi h}{2}\right) B_i^{k+1}\right]
$$

$$
- \sum_{j=0}^{k-1} \lambda_{j+1}^{l,k+1} \delta_{k-j} + \lambda_k^{l,k+1} \delta_0
$$

(27)

To complete the proof of the stability of the Crank–Nicholson scheme for solving the time-fractional VO telegraph equation, we shall prove that, for all $k = 1, 2 \ldots N - 1$ Eqs. (26) and (27) satisfy the following condition:

$$
|\delta_k| \leq |\delta_0|.
$$
We use induction on the natural number \( k \) to establish this. For \( k = 1 \) and from Eq. (27) we have that:

\[
|\delta_1| = |\delta_0| \left| \frac{1 - 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^1 - 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^1}{1 + 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^1 + 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^1} \right| \leq |\delta_0| \left| \frac{1 + 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^1 + 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^1}{1 + 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^1 + 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^1} \right| = |\delta_0|.
\]  

(28)

The inequality is then true for \( k = 1 \). Let us assume now that the formula is satisfied for \( m = 2, 3 \ldots k \), then we have that:

\[
|\delta_{k+1}| = \left| \delta_k \left[ 1 - 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^k - 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^k \right] \right| + \sum_{j=0}^{k-1} |\delta_{j+1}| \left| \delta_{k-j} \right| + |\delta_{k+1}||\delta_0|.
\]

(29)

Applying the triangle inequality, we obtain the following expression:

\[
|\delta_{k+1}| \leq |\delta_k| \left| 1 - 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^k - 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^k \right| + \sum_{j=0}^{k-1} |\delta_{j+1}| \left| \delta_{k-j} \right| + |\delta_{k+1}|\left| \delta_0 \right|.
\]

(30)

Using now the induction hypothesis leads to

\[
|\delta_{k+1}| \leq |\delta_0| \left| 1 - 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^{k+1} - 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^{k+1} \right| + \sum_{j=0}^{k-1} \left| \delta_{j+1} \right| \left| \delta_{k-j} \right| + \left| \delta_{k+1} \right| \left| \delta_0 \right|.
\]

(31)

If we factorize \( |\delta_0| \) in the above equation (31) we obtain the following:

\[
|\delta_{k+1}| \leq \left| \left( 1 - 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^{k+1} - 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^{k+1} \right) \right| \left| \delta_0 \right|.
\]

(32)

Making use of 3 of Remark 1, we arrive at the following simplified expression

\[
|\delta_{k+1}| \leq \left| \left( 1 - 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^{k+1} - 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^{k+1} \right) \right| \left| \delta_0 \right|
\]

\[
\leq \left| \left( 1 + 4 \sin^2 \left( \frac{\psi}{2} \right) A_1^{k+1} + 2 \sin^2 \left( \frac{\psi}{2} \right) B_1^{k+1} \right) \right| \left| \delta_0 \right|
\]

then,

\[
|\delta_{k+1}| \leq |\delta_0|
\]

which completes the proof.

5. Convergence analysis

If we assume that, \( w(x_i, t_k) \) \((l = 1, 2, \ldots M, k = 1, 2 \ldots N - 1)\) is the exact solution of our main problem at the point \((x_i, t_k)\) in space and time, then by putting \( \Omega^2_i = w(x_i, t_k) - w\) and \( \Omega^\delta = (0, \Omega^\delta_1, \Omega^\delta_2, \ldots, \Omega^\delta_{M-1}) \) substituting this in Eq. (22), we obtain

\[
\Omega^2_i \left[ 1 + 2A^1_i + B^1_i \right] - \Omega^1_{i+1} \left[ A^1_i - T^1_i \right] - \Omega^1_{i-1} \left[ T^1_i + A^1_i \right] = E^1_i \quad \text{for } k = 0
\]

(32)

and

\[
\Omega^k_{i+1} \left[ 1 + 2A^{k+1}_i + B^{k+1}_i \right] - \Omega^{k+1}_{i+1} \left[ A^{k+1}_i - T^{k+1}_i \right] - \Omega^{k+1}_{i-1} \left[ T^{k+1}_i + A^{k+1}_i \right]
\]

\[
- \sum_{j=1}^{k} \left[ \Omega^{k-1-j}_i \Omega^{j-1}_i \right] b_{j+1} \quad \text{for } k \geq 1
\]

(33)

The new expression \( E^{k+1}_i \) introduced here is defined as:

\[
E^{k+1}_i = \left( w(x_i, t_{k+1}) - w(x_i, t_k) + \sum_{j=1}^{k} \left[ w(x_i, t_{k+1-j}) - w(x_i, t_{j-1}) \right] \right)
\]

\[
+ k \left[ \frac{1}{2} \left( \frac{w(x_{i+1}, t_{k+1}) - w(x_{i-1}, t_{k+1})}{2(\tau)} \right) + \frac{w(x_{i+1}, t_k) - w(x_{i-1}, t_k)}{2(\tau)} \right] - a \left[ \frac{1}{2} \left( \frac{w(x_{i+1}, t_{k+1}) - 2w(x_i, t_{k+1}) + w(x_{i-1}, t_{k+1})}{(h)^2} \right) \right]
\]

\[
+ b \left( \frac{1}{2} \left( w(x_{i+1}, t_{k+1}) + w(x_i, t_k) \right) \right) = E^{k+1}_i
\]
From Eqs. (11) to (14) we have the following:

\[
\frac{\partial w}{\partial t} + \tau M = \frac{1}{2} \left( \left( \frac{w(x_{i+1}, t_{k+1}) - w(x_i, t_{k+1})}{2(\tau)} \right) + \left( \frac{w(x_{i+1}, t_k) - w(x_i, t_k)}{2(\tau)} \right) \right) \tag{34}
\]

\[
\frac{\partial^2 w}{\partial x^2} + h^2 M_1 = \frac{1}{2} \left( \left( \frac{w(x_{i+1}, t_{k+1}) - 2w(x_i, t_{k+1}) + w(x_{i-1}, t_{k+1})}{(h)^2} \right) + \left( \frac{w(x_{i-1}, t_{k+1}) - 2w(x_i, t_k) + \Phi(t_{i-1}, t_k)}{(h)^2} \right) \right) \tag{35}
\]

\[
\frac{\partial^{\alpha_{k+1}} w(x_i, t_{k+1})}{\partial t^{\alpha_{k+1}}} + \tau M_2 = \frac{\tau^{\alpha_{k+1}}}{\Gamma(2 - \alpha_{k+1})} \left( w(x_i, t_{k+1}) - w(x_i, t_k) \right) + \sum_{j=1}^{k} \left[ w(x_i, t_{k+1-j}) - w(x_i, t_{k-j}) \right] \left[ (j + 1)^{1-\alpha_{k+1}} - (j)^{1-\alpha_{k+1}} \right] \tag{36}
\]

From the above equations (33), (34) and (35) we have that,

\[E_i^{k+1} \leq H(2^{1+\alpha_{k+1}} + h^2\tau^{\alpha_{k+1}}) \]

where \( M_1, M_2, \) and \( M \) are constants. Taking into account Caputo type fractional derivative, the detailed error analysis on the above scheme can refer to the work by Diethelm et al. [33] and further work by Li and Tao [34].

**Lemma 1.** In the case of Crank–Nicholson scheme for solving the time-fractional VO telegraph equation, \( \| \Omega^{k+1} \|_\infty \leq H(2^{1+\alpha_{k+1}} + h^2\tau^{\alpha_{k+1}})(\Omega_j^{k+1})^{-1} \) is true for \( k = 0, 1, 2 \ldots N - 1 \) where \( \| F \|_\infty = \max_{1 \leq t \leq M-1} (\Omega^k) \), \( H \) is a constant. In addition,

\[\alpha_{k+1} = \begin{cases} 
\min_{1 \leq t \leq M-1} \alpha_{k+1}, & \text{if } \tau < 1 \\
\max_{1 \leq t \leq M-1} \alpha_{k+1}, & \text{if } \tau > 1.
\end{cases} \]

This can be proved using induction on the natural number \( k \). When \( k = 0 \), we have the following expression:

\[|\Omega_i^1| \leq |\Omega_i^1[1 + 2A_i^1 + B_i^1]| - |\Omega_{i+1}^1[1 - T_i^1]| - |\Omega_{i-1}^1[T_i^1 + A_i^1]| = |E_i^1| \leq H(2^{1+\alpha_i^1} + h^2\tau^{\alpha_i^1})(\Omega_j^{1,1})^{-1} \]

that is verified for \( k = 0 \).

Assuming by induction that, for \( i = 1 \ldots N - 2 \), \( \| \Omega_i^{k+1} \|_\infty \leq H(2^{1+\alpha_{k+1}} + h^2\tau^{\alpha_{k+1}})(\Omega_j^{k+1})^{-1} \) is verified then,

\[|\Omega_i^{k+1}| \leq |\Omega_i^{k+1}[1 + 2A_i^{k+1} + B_i^{k+1}] - \Omega_{i+1}^{k+1}[A_i^{k+1} - T_i^{k+1}] - \Omega_{i-1}^{k+1}[T_i^{k+1} + A_i^{k+1}]| \]

\[= \sum_{j=1}^{k} \left[ |\Omega_j^{k+1-j} - \Omega_j^{k-j}|b_j^{k+1} \right]. \tag{37}
\]

Again making use of the triangle inequality we can write the following expression:

\[|\Omega_i^{k+1}| \leq |\Omega_i^{k+1}[1 + 2A_i^{k+1} + B_i^{k+1}]| + |\Omega_{i+1}^{k+1}[A_i^{k+1} - T_i^{k+1}]| + |\Omega_{i-1}^{k+1}[T_i^{k+1} + A_i^{k+1}]| \]

\[+ \sum_{j=1}^{k} \left[ |\Omega_j^{k+1-j} - \Omega_j^{k-j}|b_j^{k+1} \right]. \]

Using the fact that \( \| F \|_\infty = \max_{1 \leq t \leq M-1} (\Omega^k) \), we reduce the above equation to:

\[|\Omega_i^{k+1}| \leq |\Omega_i^{k+1}[1 + 2A_i^{k+1} + B_i^{k+1}]| + |\Omega_{i+1}^{k+1}[A_i^{k+1} - T_i^{k+1}]| + |\Omega_{i-1}^{k+1}[T_i^{k+1} + A_i^{k+1}]| + \sum_{j=1}^{k} \| F \|_\infty b_j^{k+1} \]

\[|\Omega_i^{k+1}| \leq |E_i^{k+1}| + \sum_{j=1}^{k} \| F \|_\infty b_j^{k+1} \leq H(2^{1+\alpha_{k+1}} + h^2\tau^{\alpha_{k+1}}) + \sum_{j=1}^{k} \| F \|_\infty b_j^{k+1} \]

\[|\Omega_i^{k+1}| \leq H(2^{1+\alpha_{k+1}} + h^2\tau^{\alpha_{k+1}})(\lambda_j^{l,k+1} + \lambda_j^{l,k+1} - \lambda_j^{l,k+1})^{-1}. \tag{39}
\]
Finally we can obtain the following:

$$|\Omega_{i}^{k+1}| \leq H(2 \tau^{1+\alpha_{i}^{k+1}} + h^{2} \tau^{\alpha_{i}^{k}})\left(\lambda_{0}^{nk+1}\right)\left(\lambda_{j}^{nk+1}\right)^{-1}.$$

However we recall that, $\lambda_{0}^{nk+1} = 1$ and thus:

$$|\Omega_{i}^{k+1}| \leq H(2 \tau^{1+\alpha_{i}^{k+1}} + h^{2} \tau^{\alpha_{i}^{k}})\left(\lambda_{j}^{nk+1}\right)^{-1}.$$

This completes the proof.

6. Numerical approximation

In this section, we present some numerical simulations for the approximate solution of the time-fractional VO telegraph equation.
Let us consider the time-fractional VO derivative telegraph equation of the following form:

\[
\begin{cases}
D_t^{2-\sin(x+t)(\frac{1}{100})} u(x, t) + 4 \frac{\partial u(x, t)}{\partial t} + 4u(x, t) - 9 \frac{\partial^2 u(x, t)}{\partial x^2} = 0 \\
u(0, t) = u(3\pi, t) = \frac{\partial u(x, 0)}{\partial t} = 0 \\
u(x, 0) = 1 \text{ if } \pi \leq x \leq 2\pi, \text{ and 0 otherwise}
\end{cases}
\]

Here, \( \alpha(x, t) = 2 - \sin(x + t)(\frac{1}{100}) \), \( k = 4 \), \( b = 4 \) and \( a = 9 \). But for \( \alpha(x, t) = 2 \).

Approximate solutions of the main problem have been depicted in Figs. 1, 2, 3, 4, and 5. They are plotted according to different values of \( N \). Fig. 1 to Fig. 5 show the solution for \( N = 10 \), \( N = 20 \), \( N = 40 \), \( N = 60 \) and \( N = 100 \) respectively. We chose in this case \( h = 0.02 \) and \( \tau = 0.01 \).
7. Conclusion

We have extended the time-fractional variable order telegraph equation to the concept of the variable order derivative. The extended equation called time-fractional variable order telegraph was solved numerically via the Crank–Nicholson scheme. The stability and convergence of the numerical solution was examined in detail. Numerical simulations of the approximate solution of the time-fractional variable order telegraph equation were presented for different values of $N$.

References


