A new iterative method for variational inequalities

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Abstract

It is well known that the variational inequalities are equivalent to the fixed point problems. Using this equivalence, we suggest and consider a new three-step iterative method for solving variational inequalities. The new iterative method is obtained by using three steps under suitable conditions. We prove that the new method is globally convergent. Our results can be viewed as significant extensions of the previously known results for variational inequalities. Preliminary numerical experiments are included to illustrate the advantage and efficiency of the proposed method.

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1. Introduction

Variational inequality has become a rich of inspiration in pure and applied mathematics. In recent years, classical variational inequality problems have been extended and generalized to study a large variety of problems arising in structural analysis, economics, optimization, operations research and engineering sciences, see [1–19] and the references therein. Using the projection technique, it has been shown that the variational inequalities are equivalent to the fixed point problem. This alternative formulation has played a fundamental and significant part in developing several numerical methods for solving variational inequalities and related optimization problems. The convergence of these methods requires that the operator is both strongly monotone and Lipschitz continuous. Secondly, it is very difficult to evaluate the projection of the operator except for very simple cases. This fact has motivated many authors to develop other techniques including the technique of updating the solution. Using essentially the idea and technique of [2], we suggest and analyze a new iterative method for solving variational inequalities. The proposed method consists of three steps, uses in the step 1 and step 2 the method in [2] and we propose a new third step. It is shown that the convergence of the
proposed method requires only pseudomonocity, which is a weaker condition than monotonicity. An example is given to illustrate the efficiency and advantage of the proposed new iterative method.

2. Preliminaries

Let \( H \) be a real Hilbert space, whose inner product and norm are denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \| \). Let \( K \) be a closed convex set in \( H \). Let \( T : K \to H \) be a nonlinear operator. We consider the problem of finding \( u^* \in K \) such that

\[
\langle T(u^*), u - u^* \rangle \geq 0, \quad \forall u \in K,
\]

which is known as the variational inequality problem, introduced and studied by Stampacchia [18] in 1964. For the applications, numerical methods, sensitivity analysis and physical formulations, see [1–19] and the references therein.

**Lemma 2.1.** For a given \( z \in H \), \( u \in K \) satisfies the inequality

\[
\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,
\]

if and only if

\[
u = P_K z,
\]

where \( P_K \) is the projection operator of \( H \) onto the convex set \( K \).

It follows from Lemma 2.1 that:

\[
\langle z - P_K z, P_K z - v \rangle \geq 0, \quad \forall v \in K.
\]

And

\[
\|P_K(z) - v\| \leq \|z - v\|, \quad \forall z \in \mathbb{R}^n, \quad v \in K.
\]

It is well-known that \( P_K \) is nonexpansive, that is,

\[
\|P_K(u) - P_K(v)\| \leq \|u - v\|, \quad \forall u, \ v \in H.
\]

Using Lemma 2.1, one can easily show that the problem (2.1) is equivalent to the fixed-point problem, see, for example, [13].

**Lemma 2.2.** \( u^* \in K \) is solution of the variational inequality (2.1) if and only if \( u^* \in K \) satisfies the relation:

\[
u^* = P_K[u^* - \rho T(u^*)],
\]

where \( \rho > 0 \) is a constant.

From Lemma 2.2, it is clear that \( u \) is solution of (2.1) if and only if \( u \) is a zero point of the function

\[
r(u, \rho) = u - P_K[u - \rho T(u)].
\]

Throughout this paper, we make following assumptions.

**Assumptions**

- \( H \) is a finite dimension space.
- \( T \) is continuous and pseudomonotone operator on \( K \), that is,
  \[
  \langle T(u), u' - u \rangle \geq 0 \Rightarrow \langle T(u'), u' - u \rangle \geq 0, \quad \forall u', \ u \in K.
  \]
- The solution set of problem (2.1), denoted by \( \Omega \), is nonempty.

3. Basic results

In this section, we prove some basic properties, which will be used to establish the sufficient and necessary conditions for the convergence of the proposed method. The following results can be proved by similar arguments as those in [2,14]. Hence the proofs will be omitted.
Lemma 3.1 [2]. For all \( u \in K \) and \( \rho' \geq \rho > 0 \), it holds that
\[
\|r(u, \rho')\| \geq \|r(u, \rho)\| \tag{3.1}
\]
and
\[
\frac{\|r(u, \rho')\|}{\rho'} \leq \frac{\|r(u, \rho)\|}{\rho}. \tag{3.2}
\]

Lemma 3.2 [2]. If \( u \) is not a solution of problem (2.1), then there exist \( \delta \in (0, 1) \) and \( \epsilon' > 0 \), such that for all \( \rho \in (0, \epsilon'] \),
\[
\rho\|T(u) - T(P_K[u - \rho T(u)])\| \leq \delta\|r(u, \rho)\|. \tag{3.3}
\]

Lemma 3.3 [14]. \( \forall u \in K, u^* \in \Omega^* \) and \( \rho > 0 \) we have
\[
\langle u - u^*, d(u, \rho) \rangle \geq \phi(u, \rho), \tag{3.4}
\]
where
\[
d(u, \rho) := r(u, \rho) - \rho T(u) + \rho T(P_K[u - \rho T(u)]). \tag{3.5}
\]
and
\[
\phi(u, \rho) := \|r(u, \rho)\|^2 - \rho\|r(u, \rho), T(u) - T(P_K[u - \rho T(u)])\|. \tag{3.6}
\]

From Lemmas 3.2 and 3.3 we have
\[
\langle u - u^*, d(u, \rho) \rangle \geq \phi(u, \rho) \geq (1 - \delta)\|r(u, \rho)\|^2. \tag{3.7}
\]

Taking the above inequality into consideration, we suggest and consider a new three-step iterative method for solving variational inequality (2.1).

Algorithm 3.1. For a given \( u^k \in K \), compute the approximate solution \( u^{k+1} \) by the iterative schemes:

Step 1
\[
\tilde{u}^k = P_K[u^k - \rho_k T(u^k)], \tag{3.8}
\]
where \( \rho_k \) satisfies
\[
\|\rho_k(T(u^k) - T(\tilde{u}^k))\| \leq \delta\|u^k - \tilde{u}^k\|, \quad 0 < \delta < 1. \tag{3.9}
\]

Step 2
\[
\hat{u}^k = P_K[u^k - \alpha_k d(u^k, \rho_k)], \tag{3.10}
\]
where
\[
\varepsilon^k := \rho_k(T(\hat{u}^k) - T(u^k)), \tag{3.11}
\]
\[
d(u^k, \rho_k) := u^k - \hat{u}^k + \varepsilon^k, \tag{3.12}
\]
and
\[
\alpha_k := \frac{\phi(u^k, \rho_k)}{\|d(u^k, \rho_k)\|^2}. \tag{3.13}
\]

Step 3. For \( \tau > 0 \), the new iterate \( u^{k+1}(\tau) \) is defined by
\[
u^{k+1}(\tau) = P_K[\hat{u}^k - \tau(u^k - \hat{u}^k)]. \tag{3.14}
\]

How to choose a suitable step length \( \tau > 0 \) to force convergence will be discussed in Section 3.
Remark 3.1. (3.9) implies that
\[ |\langle u^k - \tilde{u}^k, e^k \rangle| \leq \delta \|u^k - \tilde{u}^k\|^2, \quad 0 < \delta < 1. \]  
(3.15)

The next lemma shows that \( x_k \) and \( \phi(u^k, \rho_k) \) are lower bounded away from zero.

Lemma 3.4. For given \( u^k \in \mathbb{R}^n \) and \( \rho_k > 0 \), let \( \tilde{u}^k \) and \( e^k \) satisfy (3.8) and (3.10), then
\[ \phi(u^k, \rho_k) \geq (1 - \delta)\|u^k - \tilde{u}^k\|^2 \]  
(3.16)

and
\[ x_k \geq \frac{1}{2}. \]  
(3.17)

Proof. It follows from (3.11) and (3.15) that:
\[ \phi(u^k, \rho_k) := \langle u^k - \tilde{u}^k, d(u^k, \rho_k) \rangle = \|u^k - \tilde{u}^k\|^2 + \langle u^k - \tilde{u}^k, e^k \rangle \geq (1 - \delta)\|u^k - \tilde{u}^k\|^2. \]

Otherwise, we have
\[ \langle u^k - \tilde{u}^k, d(u^k, \rho_k) \rangle = \|u^k - \tilde{u}^k\|^2 + \langle u^k - \tilde{u}^k, e^k \rangle \geq \frac{1}{2}\|u^k - \tilde{u}^k\|^2 + \langle u^k - \tilde{u}^k, e^k \rangle + \frac{1}{2}\|e^k\|^2 = \frac{1}{2}\|d(u^k, \rho_k)\|^2, \]
where the inequality follows from (3.9) and:
\[ x_k \geq \frac{1}{2}. \]

we can get the assertion of this lemma. \( \square \)

We now consider the criteria of \( \tau \), which ensures that \( u^{k+1}(\tau) \) is closer to the solution set than \( u^k \). For this purpose, we define
\[ \Gamma(\tau) := \|u^k - u^*\|^2 - \|u^{k+1}(\tau) - u^*\|^2. \]  
(3.18)

Theorem 3.1. Let \( u^* \in \Omega^* \). Then we have
\[ \Gamma(\tau) \geq \tau\{\|u^k - \tilde{u}^k\|^2 + \Theta_k(x_k)\} - \tau^2\|u^k - \tilde{u}^k\|^2, \]  
(3.19)

where
\[ \Theta_k(x_k) := \|u^k - u^*\|^2 - \|u^k - \tilde{u}^k\|^2. \]  
(3.20)

Proof. It follows from (3.14) and (2.5) that:
\[ \Gamma(\tau) \geq \|u^k - u^*\|^2 - \|u^k - \tau(u^k - \tilde{u}^k) - u^*\|^2 = 2\tau\langle u^k - u^* - \tilde{u}^k, u^k - \tilde{u}^k \rangle - \tau^2\|u^k - \tilde{u}^k\|^2 \]
\[ = 2\tau\{\|u^k - \tilde{u}^k\|^2 - \|u^k - \tilde{u}^k, u^k - \tilde{u}^k\|^2\} - \tau^2\|u^k - \tilde{u}^k\|^2. \]  
(3.21)

Using the following identity:
\[ \langle u^* - \tilde{u}^k, u^k - \tilde{u}^k \rangle = \frac{1}{2}\left(\|u^k - u^*\|^2 - \|u^k - u^*\|^2\right) + \frac{1}{2}\|u^k - \tilde{u}^k\|^2, \]
we have
\[ \|u^k - \tilde{u}^k\|^2 - 2\langle u^* - \tilde{u}^k, u^k - \tilde{u}^k \rangle = (\|u^k - u^*\|^2 - \|u^k - u^*\|^2). \]  
(3.22)

Substituting (3.22) into (3.21) and using the notation of \( \Theta_k(x_k) \), we obtain (3.19), the required result. \( \square \)
Let \( u^* \in K \) be a solution of problem \((2.1)\) and let \( \phi(u^k, \rho_k) \) and \( \Theta_k(z_k) \) be defined by \((3.12)\) and \((3.20)\) respectively. Then we get
\[
\Theta_k(z_k) \geq z_k \phi(u^k, \rho_k).
\]

**Proof.** Let \( u^* \in H \) be a solution of problem \((2.1)\). Then from \((3.7)\) and \((3.13)\) we obtain
\[
\Theta_k(z_k) = ||u^k - u^*||^2 - ||\hat{u}^k - u^*||^2 \geq ||u^k - u^*||^2 - ||\hat{u}^k - u^* - z_k d(u^k, \rho_k)||^2 \\
= 2z_k \langle \hat{u}^k - u^*, d(u^k, \rho_k) \rangle - z_k^2 ||d(u^k, \rho_k)||^2 \geq z_k \phi(u^k, \rho_k).
\]

Using Theorems 3.1 and 3.2, we get
\[
\Gamma(\tau) \geq A(\tau),
\]
where
\[
A(\tau) = \tau \{ ||u^k - \hat{u}^k||^2 + z_k \phi(u^k, \rho_k) \} - \tau^2 ||u^k - \hat{u}^k||^2.
\]
The above inequality tells us how to choose a suitable \( \tau_k \). Since \( A(\tau_k) \) is a quadratic function of \( \tau_k \) and it reaches its maximum at
\[
\tau_k^* = \frac{||u^k - \hat{u}^k||^2 + z_k \phi(u^k, \rho_k)}{2 ||u^k - \hat{u}^k||^2}
\]
and
\[
A(\tau_k^*) = A(\tau_k^*) = \tau_k^* \{ ||u^k - \hat{u}^k||^2 + z_k \phi(u^k, \rho_k) \}.
\]

Then from Lemma 3.1, we get
\[
\tau_k^* \geq 1 - \frac{\delta}{2} \left( \frac{||u^k - \hat{u}^k||^2 + ||u^k - \hat{u}^k||^2}{2 ||u^k - \hat{u}^k||^2} \right) \geq \frac{1 - \delta}{4},
\]
and
\[
A(\tau_k^*) \geq \frac{\tau_k^*(1 - \delta)}{4} ||u^k - \hat{u}^k||^2 \geq \left( \frac{1 - \delta}{16} \right) ||u^k - \hat{u}^k||^2.
\]

For fast convergence, we take a relaxation factor \( \gamma \in [1,2) \) and the step-size \( \tau_k \) by \( \tau_k = \gamma \tau_k^* \). Simple calculations show that
\[
A(\gamma \tau_k^*) = \gamma \tau_k^* \{ ||u^k - \hat{u}^k||^2 + z_k \phi(u^k, \rho_k) \} - \left( \gamma^2 \tau_k^* \right) \left( \tau_k^* ||u^k - \hat{u}^k||^2 \right) = \gamma(2 - \gamma) A(\tau_k^*) \quad \Box
\]

**4. Convergence analysis**

In this section, we consider the convergence analysis of Algorithm 3.1 and this is the main motivation.

**Theorem 4.1.** Let \( u^* \) be a solution of problem \((2.1)\) and let \( u^{k+1} \) be the sequence obtained from Algorithm 3.1. Then \( u^k \) is bounded and
\[
||u^{k+1} - u^*||^2 \leq ||u^k - u^*||^2 - \frac{1}{16} \gamma(2 - \gamma)(1 - \delta)^2 ||u^k - \hat{u}^k||^2.
\]

**Proof.** Let \( u^* \) be a solution of the variational inequality \((2.1)\). Then, from \((3.25)\), \((3.30)\) and \((3.29)\), we have
\[
||u^{k+1} - u^*||^2 = ||u^k - u^*||^2 - \Gamma(\gamma \tau_k^*) \leq ||u^k - u^*||^2 - \gamma(2 - \gamma) A(\tau_k^*) \\
\leq ||u^k - u^*||^2 - \frac{1}{16} \gamma(2 - \gamma)(1 - \delta)^2 ||u^k - \hat{u}^k||^2.
\]
Since $c \in [1, 2)$ and $\delta \in (0, 1)$ we have
\[ \|u^{k+1} - u^*\| \leq \|u^k - u^*\| \leq \cdots \leq \|u^0 - u^*\|. \]
This shows that the sequence $u^k$ is bounded. \qed

We now prove the convergence of Algorithm 3.1 and this is the main motivation of our next result.

**Theorem 4.2.** The sequence $\{u^k\}$ generated by Algorithm 3.1 converges to a solution of the variational inequality (2.1).

**Proof.** It follows from (4.1) that:
\[ \sum_{k=0}^{\infty} \|u^k - \tilde{u}\|^2 < +\infty, \]
which means that
\[ \lim_{k \to +\infty} \|u^k - \tilde{u}\| = 0. \quad (4.2) \]
Consequently $\{\tilde{u}^k\}$ is also bounded. Since $\|r(u^k, \rho)\|$ is a non-decreasing function of $\rho$, it follows from $\rho_k \geq \rho_{\text{min}}$ that:
\[ \|r(\tilde{u}^k, \rho_{\text{min}})\| \leq \|r(\tilde{u}^k, \rho_k)\| \]
\[ = \|\tilde{u}^k - P_K(\tilde{u}^k - \rho_k T(\tilde{u}^k))\| \]
\[ (\text{using (3.8) and (3.10)}) = \|P_K(\tilde{u}^k - \rho_k T(\tilde{u}^k)) - P_K(\tilde{u}^k - \rho_k T(\tilde{u}^k))\| \]
\[ (\text{using (2.6)}) \leq \|u^k - \tilde{u}^k + \varepsilon\| \]
\[ (\text{using (3.9)}) \leq (1 + \delta)\|u^k - \tilde{u}^k\| \]
and from (4.2), we get
\[ \lim_{k \to +\infty} r(\tilde{u}^k, \rho_{\text{min}}) = 0. \quad (4.3) \]
Let $\bar{u}$ be a cluster point of $\{\tilde{u}^k\}$ and the subsequence $\{w^k\}$ converges to $\bar{u}$. Since $r(u, \rho)$ is a continuous function of $u$, it follows from (4.3) that:
\[ r(\bar{u}, \rho_{\text{min}}) = \lim_{j \to +\infty} r(w^j, \rho_{\text{min}}) = 0. \]
According to Lemma 2.2, $\bar{u}$ is a solution of (2.1). Note that inequality (4.1) is true for all solutions of problem (2.1), hence we have
\[ \|u^{k+1} - \bar{u}\| \leq \|u^k - \bar{u}\|, \quad \forall k \geq 0. \quad (4.4) \]
Since $\{w^k\} \to \bar{u}$ and $u^k - \tilde{u} \to 0$, for any given $\varepsilon > 0$, there is an $l > 0$, such that
\[ \|w^k - \bar{u}\| < \varepsilon/2 \quad \text{and} \quad \|u^k - w^k\| < \varepsilon/2. \quad (4.5) \]
Therefore, for any $k \geq k_0$, it follows from (4.4) and (4.5) that:
\[ \|u^k - \bar{u}\| \leq \|u^k - \bar{u}\| \leq \|u^k - w^k\| + \|w^k - \bar{u}\| < \varepsilon \]
and thus the sequence $\{u^k\}$ converges to $\bar{u}$.

We now prove that the sequence $\{u^k\}$ has exactly one cluster point. Assume that $\tilde{u}$ is another cluster point and satisfies
\[ \delta := \|\bar{u} - \tilde{u}\| > 0. \]
Since $\bar{u}$ is a cluster point of the sequence $\{u^k\}$, there is a $k_0 > 0$ such that
\[ \|u^{k_0} - \bar{u}\| \leq \frac{\delta}{2}. \]
On the other hand, since \( \bar{u} \in \Omega^* \) and from (4.1), we have
\[
\| u^k - \bar{u} \| \leq \| u^{k_0} - \bar{u} \| \quad \text{for all } k \geq k_0,
\]
it follows that:
\[
\| u^k - \bar{u} \| \geq \| \bar{u} - \bar{u} \| - \| u^k - \bar{u} \| \geq \frac{\delta}{2} \quad \forall k \geq k_0.
\]
This contradicts the assumption that \( \bar{u} \) is cluster point of \( \{ u^k \} \), thus the sequence \( \{ u^k \} \) converges to \( \bar{u} \in \Omega^* \). \( \square \)

The details of Algorithm 3.1 are as follows.

Algorithm 4.1

Step 0. Let \( \rho_0 = 1, \; \delta : = 0.95 < 1, \; \gamma = 1.95, \; \epsilon > 0, \; k = 0 \) and \( u^0 \in K \).
Step 1. If \( \| r(u^k, \rho_k) \|_{\infty} \leq \epsilon \), then stop. Otherwise, go to Step 2.
Step 2.
\[
\begin{align*}
\bar{u}^k &= P_K[u^k - \rho_k T(u^k)], \quad \bar{u}^k = \rho_k (T(\bar{u}^k) - T(u^k)), \\
\bar{r} &= \frac{1}{\| \bar{u}^k - \bar{u} \|}, \\
\textbf{While} (r > \delta) \quad &\rho_k = \frac{\rho_k}{r} * \rho_k, \quad \bar{u}^k = P_K[\bar{u}^k - \rho_k T(u^k)], \\
&\bar{e}^k = \rho_k (T(\bar{u}^k) - T(u^k)), \quad r = \frac{\| \bar{r} \|}{\| \bar{u}^k - \bar{u} \|}.
\end{align*}
\]
end While
Step 3. Set
\[
\begin{align*}
d(u^k, \rho_k) &:= u^k - \bar{u}^k + \bar{e}^k, \\
\phi(u^k, \rho_k) &:= \langle u^k - \bar{u}^k, d(u^k, \rho_k) \rangle, \quad \alpha_k = \frac{\phi(u^k, \rho_k)}{\| d(u^k, \rho_k) \|^2}, \\
\bar{u}^k &= P_K[u^k - \alpha_k d(u^k, \rho_k)], \\
\bar{r}^k &= \frac{\| u^k - \bar{u}^k \|^2 + 2 \alpha_k \phi(u^k, \rho_k)}{2 \| u^k - \bar{u}^k \|^2}, \\
\tau_k &= \gamma \bar{r}^k, \\
u^{k+1} &= P_K[u^k - \tau_k (u^k - \bar{u}^k)].
\end{align*}
\]
Step 4. \( \rho_{k+1} = \begin{cases} \frac{\rho_k \cdot 0.7}{r} & \text{if } r \leq 0.5; \\ \rho_k & \text{otherwise.} \end{cases} \)
Step 5. \( k := k + 1; \) go to Step 1.

5. Computational results

In this section, we apply the new method to a traffic equilibrium problem, which is a classical and important problem in transportation science, see [11,19]. The numerical results show that the new method is attractive in practice.

Consider a network \([N, L]\) of nodes \(N\) and directed links \(L\), which consists of a finite sequence of connecting links with a certain orientation. Let \(a, b, \) etc., denote the links; \(p, q, \) etc., denote the paths; \(\omega\) denote an origin/destination (O/D) pair of nodes of the network; \(P_\omega\) denotes the set of all paths connecting O/D pair \(\omega\); \(u_p\) represent the traffic flow on path \(p\); \(d_\omega\) denote the traffic demand between O/D pair \(\omega\), which must satisfy
\[
d_\omega = \sum_{p \in P_\omega} u_p,
\]
where \(u_p \geq 0, \forall p;\) and \(f_a\) denote the link load on link \(a\), which must satisfy the following conservation of flow equation:
\[
f_a = \sum_{p \in P} \delta_{ap} u_p,
\]
where
\[ \delta_{ap} = \begin{cases} 1, & \text{if } a \text{ is contained in path } p; \\ 0, & \text{otherwise}. \end{cases} \]

Let \( A \) be the path-arc incidence matrix of the given problem and \( f = \{ f_a, a \in L \} \) be the vector of the link load. Since \( u \) is the path-flow, \( f \) is given by
\[ f = A^T u. \]

In addition, let \( t = \{ t_a, a \in L \} \) be the row vector of link costs, with \( t_a \) denoting the user cost of travelling link \( a \) which is given by
\[ t_a(f_a) = t_0^a \left[ 1 + 0.15 \left( \frac{f_a}{C_a} \right)^4 \right], \quad (5.1) \]

where \( t_0^a \) is the free-flow travel cost on link \( a \) and \( C_a \) is designed capacity of link \( a \). Then \( t \) is a mapping of the path-flow \( u \) and its mathematical form is
\[ t(u) := t(f) = t(A^T u). \]

Note that the travel cost on the path \( p \) denoted by \( \theta_p \) is
\[ \theta_p = \sum_{a \in p} \delta_{ap} t_a(f_a). \]

Let \( P \) denote the set of all the paths concerned. Let \( \theta = \{ \theta_p, p \in P \} \) be the vector of (path) travel cost. For given link travel cost vector \( t \), \( \theta \) is a mapping of the path-flow \( u \), which is given by
\[ \theta(u) = At(u) = At(A^T u). \]

Associated with every O/D pair \( \omega \), there is a travel disutility \( \lambda_{\omega}(d) \), which is defined as following:
\[ \lambda_{\omega}(d) = -m_\omega \log(d_\omega) + q_\omega. \quad (5.2) \]

Note that both the path costs and the travel disutilities are functions of the flow pattern \( u \). The traffic network equilibrium problem is to seek the path flow pattern \( u^* \), which induces a demand pattern \( d^* = d(u^*) \), for every O/D pair \( \omega \) and each path \( p \in P_\omega 
\[ T_p(u) = \theta_p(u) - \lambda_{\omega}(d(u)). \]

The problem can be reduced to a variational inequality in the space of path-flow pattern \( u \):
\[ \text{Find } u^* \geq 0 \text{ such that } (u - u^*)^T T(u^*) \geq 0, \quad \forall u \geq 0, \quad (5.3) \]

which is exactly the variational inequality (2.1).

![Fig. 1. The network used for the numerical test.](image-url)
In particular, we test the example studied in [11,19]. The network is depicted in Fig. 1. The free-flow travel cost and the designed capacity of links (5.1) are given in Table 1, the O/D pairs and the coefficient $m$ and $q$ in the disutility function (5.2) are given in Table 2. For this example, there are together 12 paths for the four given O/D pairs listed in Table 4.

In all tests we take $\delta = 0.95$ and $\gamma = 1.95$. All iterations start with $u^0 = (1, \ldots, 1)^T$ and $\rho_0 = 1$, and stopped whenever $\|r(u, \rho)\|_\infty \leq \varepsilon$. All codes are written in Matlab and run on a P4-2.00G notebook computer. The

<table>
<thead>
<tr>
<th>Link</th>
<th>Free-flow travel time $\hat{t}_a^0$</th>
<th>Capacity $C_a$</th>
<th>Link</th>
<th>Free-flow travel time $\hat{t}_a^0$</th>
<th>Capacity $C_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6</td>
<td>200</td>
<td>7</td>
<td>5</td>
<td>150</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>200</td>
<td>8</td>
<td>10</td>
<td>150</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>200</td>
<td>9</td>
<td>11</td>
<td>200</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>200</td>
<td>10</td>
<td>11</td>
<td>200</td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>100</td>
<td>11</td>
<td>15</td>
<td>200</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>100</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2
The O/D pairs and the coefficient $m$ and $q$ in (5.2)

<table>
<thead>
<tr>
<th>No. of the pair</th>
<th>O/D pair</th>
<th>$m_x$</th>
<th>$q_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1,7)</td>
<td>25</td>
<td>25 log 600</td>
</tr>
<tr>
<td>2</td>
<td>(2,7)</td>
<td>33</td>
<td>33 log 500</td>
</tr>
<tr>
<td>3</td>
<td>(3,7)</td>
<td>20</td>
<td>20 log 500</td>
</tr>
<tr>
<td>4</td>
<td>(6,7)</td>
<td>20</td>
<td>20 log 400</td>
</tr>
</tbody>
</table>

Table 3
Numerical results for for different $\varepsilon$

<table>
<thead>
<tr>
<th>Different $\varepsilon$</th>
<th>The method in [2]</th>
<th>Algorithm 4.1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No. it.</td>
<td>CPU (s)</td>
</tr>
<tr>
<td>$10^{-4}$</td>
<td>141</td>
<td>0.07</td>
</tr>
<tr>
<td>$10^{-5}$</td>
<td>171</td>
<td>0.08</td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>201</td>
<td>0.1</td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>230</td>
<td>0.11</td>
</tr>
<tr>
<td>$10^{-8}$</td>
<td>261</td>
<td>0.12</td>
</tr>
</tbody>
</table>

Table 4
The optimal path follow

<table>
<thead>
<tr>
<th>O/D pair</th>
<th>Path no.</th>
<th>Link of path</th>
<th>Optimal path-flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>O/D pair (1,7)</td>
<td>1</td>
<td>(1,3)</td>
<td>165.3145</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>(2,4)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>(11)</td>
<td>138.5735</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>(5,1,3)</td>
<td>82.5281</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>(5,2,4)</td>
<td>0</td>
</tr>
<tr>
<td>O/D pair (2,7)</td>
<td>6</td>
<td>(5,11)</td>
<td>55.7871</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>(8,6,4)</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>(8,9)</td>
<td>87.0260</td>
</tr>
<tr>
<td>O/D pair (3,7)</td>
<td>9</td>
<td>(7,3)</td>
<td>19.7549</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>(10)</td>
<td>229.9747</td>
</tr>
<tr>
<td>O/D pair (6,7)</td>
<td>11</td>
<td>(9)</td>
<td>178.5600</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>(6,4)</td>
<td>0</td>
</tr>
</tbody>
</table>
iteration numbers and the computational time for Algorithm 4.1 and the method in [2] for different \( \varepsilon \) are reported in Table 3. For the case \( \varepsilon = 10^{-8} \), the optimal path flow and link flow are given in Tables 4 and 5, respectively.

The numerical experiments show that the new method is more flexible and efficient to solve the traffic equilibrium problem.

References