Nonlinear sliding mode control of an induction motor

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SUMMARY

This paper proposes a new non-linear sliding mode controller for induction motors. The controller assumes that only the motor speed and stator currents are measured and seeks to provide asymptotic tracking of speed and flux. The control law incorporates a sliding mode observer and uses backstepping ideas to synthesise the non-linear controller sliding surfaces. Good results have been obtained in the benchmark simulations. Copyright © 2000 John Wiley & Sons, Ltd.

1. INTRODUCTION

Among high-performance motors, induction motors have the simplest mechanical construction because there are no brushes, no commutator and no permanent magnet in contrast to the DC motor. They can also be used in harsh environments since there are no problems with sparks and corrosion. Induction motors are, however, the most difficult to control because of their nonlinear dynamics, the electric rotor variables are not measurable and the physical parameters are most often imprecisely known. Presently, there are no technological or cost obstacles to the implementation of complex control algorithms, consequently, an important problem in the manufacturing of high-performance induction motor drives is the design of the control algorithm itself.

The control of induction motors has attracted much attention in the last few decades. An overview of the important control techniques has been given in References [1, 2]. One of the most significant developments in this area has been field oriented control, References [3–5]. The method essentially comprises partial feedback linearization together with PI control to regulate the motor states. This technique is very useful except that it is very sensitive to parameter variations.

To improve on the method of field oriented control, full linearizing state feedback control based on differential geometric theory has been proposed in References [6–8]. These methods require relatively complicated non-linear calculations in the control algorithm and suppose that all the parameters are well known. To improve this technique, in Reference [9], the authors added...
to the control algorithm an adaptive law to estimate the rotor resistance and the load torque which are both assumed to be constant.

Another technique to control induction motors is called direct torque control (DTC), see for example References [10–12]. In contrast to field oriented control, DTC does not tend to reproduce the electromechanical behaviour of a DC machine drive, but aims at a complete exploitation of the flux and motor torque capabilities of an induction motor fed by a PWM inverter. This technique requires only knowledge of the stator resistance, however at low speed the stator flux estimation deteriorates due to the effect of inaccuracies in the value of the stator resistance. The magnitude of the stator flux exhibits undesirable oscillations which in turn produces torque fluctuations.

In Reference [13], a passivity-based approach is proposed for induction motor control. This approach exploits the system energy dissipation property to solve the underlying control problem, however, the physical motor parameters are assumed to be well known. For more recent developments in this area see References [14, 15].

Recently, extensive research has been conducted in the area of so-called backstepping, see References [16, 17], which is based on Lyapunov theory. Backstepping relies on establishing a certain canonical form and identifying inductively a hierarchy of virtual control signals which eventually become the real control signals. At each step a Lyapunov methodology identifies a stabilization function for each virtual control and an associated Lyapunov function which is inductively augmented to form a global Lyapunov function for the complete system. It is important to note that the backstepping technique has been applied to the induction motor in Reference [16]. The control law provides global asymptotic stability of the system assuming once again that the physical parameters are well known.

Sliding mode control ideas, References [18, 19], have been investigated for induction motor control, motivated by the fact that enforcing a sliding mode leads to low sensitivity with respect to a class of disturbances and plant parameter variations. In Reference [19], a methodology to design a nonlinear switching manifold is described. In addition, decoupling is also discussed to reduce the complexity of the control design problem. In Reference [20], the authors propose a sliding mode rotor flux observer with a nonlinear sliding mode controller. The experimental results obtained from implementing this scheme on a DSP-based system, have shown the applicability and the robust performance of this algorithm. However, global stability of the approach has not been theoretically proven. In Reference [21], a real-time comparison of field oriented control, the input–output linearization method and the sliding-mode-based technique has been undertaken. The authors have shown that in terms of rotor resistance variation and at low speed, the sliding mode controller gave the best results. In Reference [22], the backstepping method was shown to be an elegant method for the design of non-linear sliding manifolds.

This paper develops a non-linear sliding mode controller for an induction motor building on the work in Reference [16]. Assuming that only the stator currents and the rotor speed are available, a sliding mode observer is designed to estimate the rotor flux, then a non-linear sliding mode controller is derived taking into account the estimated values of the flux with a systematic design of the controller sliding manifolds. Under certain mild assumptions the closed-loop stability of the algorithm (observer + controller) is shown.

The paper is organised as follows. The modelling of the induction motor is reviewed in Section 2. The sliding mode flux observer with the nonlinear sliding mode controller are discussed in Section 3. Section 4, deals with the simulation results in case of the benchmark problem. Finally some concluding remarks end the paper.
2. INDUCTION MOTOR MODEL

The motor model, under the assumption of linear magnetic circuits, is a fifth-order non-linear system [15], given by

\[
\Sigma_1 = \begin{cases} 
\frac{\dot{x}_1}{x_1} &= -a_1 x_1 + a_3 x_4 x_5 + a_2 x_3 + bu_a \\
\frac{\dot{x}_2}{x_2} &= -a_1 x_2 - a_3 x_4 x_5 + a_2 x_4 + bu_b \\
\frac{\dot{x}_3}{x_3} &= -a_5 x_3 - a_6 x_4 x_5 + a_4 x_1 \\
\frac{\dot{x}_4}{x_4} &= -a_5 x_4 + a_6 x_5 x_5 + a_4 x_2 \\
\frac{\dot{x}_5}{x_5} &= a_7 (x_2 x_3 - x_1 x_4) - a_8 - a_9 x_5 - a_{10} x_5^2
\end{cases}
\]

The state vector is \(X = [x_1, x_2, x_3, x_4, x_5]^T\) and the output vector \(Y = [i_a, i_b, \varphi_a, \varphi_b, \omega]^T\), where \(i_a\) and \(i_b\) are the stator currents, \(\varphi_a\) and \(\varphi_b\) are, respectively, the rotor flux, and \(\omega\) is the rotor speed. The parameters \(a_1 = \gamma, a_2 = K/T_r, a_3 = pK, a_4 = L_m/T_r, a_5 = 1/T_r, a_6 = p, a_7 = pL_m/\sigma L_r, a_8 = T_1/J, a_9 = (f + a_{11})/J, a_{10} = a_{12}/J\) and \(\gamma = R_s/\sigma L_c + R_s L_m^2/\sigma L_c L_r, K = L_m/\sigma L_c L_r, T_r = L_n/R_s\), where \(R_s\) and \(R_r\) are, respectively, the rotor and stator resistances; \(L_c\) and \(L_r\) are, respectively, the rotor and stator inductances; \(L_m\) is the mutual inductance; \(p\) is the number of pairs of poles; \(f\) is the friction coefficient; \(J\) is the moment of inertia and \(T_1\) is the constant term of the load torque \(T_L\) given by

\[
T_L = T_1 + a_{11} x_5 + a_{12} x_5^2
\]

and \(a_{11} a_{12}\) are a constant terms.

3. CONTROLLER FORMULATION

This section proposes a new control strategy for speed and flux tracking assuming that the stator currents and motor speed are measured. Because the approach is fundamentally a state feedback one, an observer will be used to construct an estimate of the unmeasured flux states. A block diagram of the proposed controller is shown in Figure 1. Several techniques have been used for flux, speed and (or) parameter estimation for the induction motor. In References [23, 24], the authors proposed an extended Kalman filter to estimate the rotor flux (or rotor current) together with the rotor speed and the rotor time constant (or rotor resistance). This technique, however, is not robust against external disturbances (for example load torque). In Reference [25], the author used the induction motor equations to estimate the flux. Using two independant subsystems for the rotor flux calculation, an estimation of the rotor speed (considered constant) was given using the model reference adaptive system (MRAS) technique. Under load at low frequency this method gives poor results [25]. A linear observer was proposed in Reference [26] to estimate the rotor flux when the speed is constant. This technique is not robust against motor parameter variations and requires an adaptation mechanism for parameter identification. In Reference [2], the authors propose a sliding mode rotor observer in which the speed is considered variable. Here a modification is proposed in which the sliding surface is time invariant. This observer is more amenable for the stability proof which is presented in this paper.

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3.1 Sliding mode flux observer

This subsection develops a fourth-order observer based on sliding mode ideas, References [18, 19]. Sliding mode approaches generally rely on the specification of a surface (or manifold) \( S \) in the state space such that if the trajectories of the dynamical system are forced to remain on \( S \), the resulting reduced order motion is stable. The reduced order motion is termed the sliding motion and is specified by the choice of \( S \). In terms of observer design, the manifold is usually defined in the error state-space in such a way that when the error states lie on the surface, the observer output is identically equal to the plant output.

Consider only the first four equations of the induction motor model given by Equation (1). In the following exposition, the speed \( x_5 \) will be considered as a varying parameter. The symbol \( \omega \) will be used instead of \( x_2 \) in order to remove any confusion between the states and parameters. The proposed observer aims to estimate the unmeasured flux components \( x_3 \) and \( x_4 \), and is a copy of the original system to which are added switching gains. The observer is given by the following system:

\[
\Sigma_2 = \begin{cases}
\dot{x}_1 &= -a_1 \dot{x}_1 + a_3 \dot{x}_4 \omega + a_2 \dot{x}_3 + bu_a + \Lambda_1 \\
\dot{x}_2 &= -a_1 \dot{x}_2 - a_3 \dot{x}_3 \omega + a_2 \dot{x}_4 + bu_b + \Lambda_2 \\
\dot{x}_3 &= -a_4 x_3 - a_6 \dot{x}_4 \omega + a_4 \dot{x}_1 + \Lambda_3 \\
\dot{x}_4 &= -a_5 \dot{x}_4 + a_6 \dot{x}_3 \omega + a_4 \dot{x}_2 + \Lambda_4
\end{cases}
\]  

(3)
where the \( \Lambda_i \)'s are plant/observer output error dependent gains. If the error components \( e_i = \dot{x}_i - x_i \), then the error dynamics are given as follows:

\[
\Sigma_3 = \begin{cases} 
\dot{e}_1 = -a_1 e_1 + a_5 \omega e_4 + a_2 e_3 + \Lambda_1 \\
\dot{e}_2 = -a_1 e_2 - a_3 \omega e_3 + a_3 e_4 + \Lambda_2 \\
\dot{e}_3 = -a_5 e_3 - a_6 \omega e_4 + a_4 e_1 + \Lambda_3 \\
\dot{e}_4 = -a_5 e_4 + a_6 \omega e_3 + a_4 e_2 + \Lambda_4 
\end{cases}
\]  

(4)

Consider the two first switching gains \( \Lambda_1 \) and \( \Lambda_2 \) with the following structure:

\[
\Lambda_1 = -\rho_1 \text{ sign}(e_1) \\
\Lambda_2 = -\rho_2 \text{ sign}(e_2)
\]

(5)

The observer switching function is given by

\[
S_{\text{obs}} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}
\]

(6)

and the sliding surface is given by \( S_{\text{obs}} = 0 \).

The stability analysis consists of determining \( \Lambda_1 \) and \( \Lambda_2 \) using the so-called reaching condition, which is given by the following equation:

\[
S^{T}_{\text{obs}} \dot{S}_{\text{obs}} < 0
\]

(7)

A sufficient condition for this to be satisfied is that both \( s_1 \dot{s}_1 < 0 \) and \( s_2 \dot{s}_2 < 0 \). This condition guarantees that in finite time \( S_{\text{obs}} = 0 \) and the states remain on the switching surface. Thereafter \( \Lambda_3 \) and \( \Lambda_4 \) are determined such that the reduced-order system obtained when \( S_{\text{obs}} = \dot{S}_{\text{obs}} = 0 \) is stable.

The time derivative of the switching function is

\[
\dot{S}_{\text{obs}} = \begin{bmatrix} \dot{s}_1 \\ \dot{s}_2 \end{bmatrix} = \begin{bmatrix} -a_1 \\ 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} a_2 \\ -a_3 \omega \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} + \begin{bmatrix} -\rho_1 \text{ sign}(e_1) \\ -\rho_2 \text{ sign}(e_2) \end{bmatrix}
\]

(8)

Suppose that the flux components \( x_3 \) and \( x_4 \) are bounded (practical supposition), and consider \( \eta_1 \) and \( \eta_2 \) to be two known positive parameters satisfying \( |x_3| < \eta_1 \) and \( |x_4| < \eta_2 \). Suppose also that there exist two known parameters \( \xi_1 \) and \( \xi_2 \) satisfying the following equations:

\[
\xi_1 > a_1 |e_1| + a_2(\eta_1 + |\dot{x}_3|) + a_3(\omega(\eta_2 + |\dot{x}_4|) \\
\xi_2 > a_1 |e_2| + a_2(\eta_2 + |\dot{x}_4|) + a_3(\omega(\eta_1 + |\dot{x}_3|)
\]

(9)

These equations effectively define a bounded region in the estimation error state space in which the subsequent stability analysis is valid. Implicitly, they impose the restriction that the speed is
bounded. Physically, this is not unrealistic. If

\[ \rho_1 = \dot{c}_1 + c_1 \]
\[ \rho_2 = \dot{c}_2 + c_2 \]  

(10)

where \( c_1 \) and \( c_2 \) are positive design constants, then the reaching conditions are satisfied. Whilst sliding, the trajectories remain on the sliding surfaces and both \( S_{o_1} = 0 \) and \( S_{o_2} = 0 \). From Equation (8), the equivalent output error injection necessary to ensure \( S_{o_2} = 0 \) is given by

\[ \begin{bmatrix} \Lambda_{1eq} \\ \Lambda_{2eq} \end{bmatrix} = - \begin{bmatrix} a_2 & a_3 \omega \\ - a_3 \omega & a_2 \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} + \Gamma \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} \]  

(11)

Suppose

\[ \begin{bmatrix} \Lambda_3 \\ \Lambda_4 \end{bmatrix} = \Lambda \begin{bmatrix} \Lambda_1 \\ \Lambda_2 \end{bmatrix} \]  

(12)

where \( \Lambda \in \mathbb{R}^{2 \times 2} \). When sliding takes place

\[ \begin{bmatrix} \Lambda_3 \\ \Lambda_4 \end{bmatrix} = \Lambda \begin{bmatrix} \Lambda_{1eq} \\ \Lambda_{2eq} \end{bmatrix} = - \Lambda \begin{bmatrix} a_2 & a_3 \omega \\ - a_3 \omega & a_2 \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} = - \Lambda \Gamma \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} \]  

(13)

Choose \( \Lambda = \Delta \Gamma^{-1} \) where

\[ \Delta = \begin{bmatrix} \delta_1 & -a_6 \omega \\ a_6 \omega & \delta_2 \end{bmatrix} \]  

(14)

and \( \delta_1 \) and \( \delta_2 \) are positive design constants. Note that \( \text{det}(\Gamma(\omega)) \neq 0 \) for all \( \omega \) and so the inverse always exists. Consider now the second sub-system concerning the flux error dynamics given by

\[ \begin{align*}
\dot{e}_3 &= - a_5 e_3 - a_6 \omega e_4 + a_4 e_1 + \Lambda_3 \\
\dot{e}_4 &= - a_5 e_4 + a_6 \omega e_3 + a_4 e_2 + \Lambda_4
\end{align*} \]  

(15)

When sliding takes place, substituting from (13) yields

\[ \begin{bmatrix} \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \begin{bmatrix} - a_5 & - a_6 \omega \\ a_6 \omega & - a_5 \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} - \Delta \Gamma^{-1} \Gamma \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} \]  

(16)

and then

\[ \begin{bmatrix} \dot{e}_3 \\ \dot{e}_4 \end{bmatrix} = \begin{bmatrix} - a_5 - \delta_1 & 0 \\ 0 & - a_5 - \delta_2 \end{bmatrix} \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} \]  

(17)

The flux errors \( e_3 \) and \( e_4 \) converge exponentially to zero.
3.2. Nonlinear sliding mode control

This section considers the design of a sliding mode control law. The objective is first to design an equilibrium surface so that the state trajectories of the plant when restricted to $S$ have the desired tracking behaviour. The second objective is to determine a switching control law that is able to drive the state trajectories to and maintain them on the surface $S$, see Reference [19]. The surface $S$ will be designed using the so-called backstepping and non-linear damping techniques, References [16, 17], in such a way that tracking of both the rotor speed and the square of the rotor flux magnitude will be achieved. The control law will be built up in a series of steps.

**Step 1**: Define $Z_{c1} = x_5 - w_{\text{ref}}$, where $w_{\text{ref}}$ is the reference speed. The time derivative of $Z_{c1}$ is given by

$$
\dot{Z}_{c1} = a_7(x_2x_3 - x_1x_4) - a_8 - a_9 x_5 - a_{10} x_2^2 - \dot{w}_{\text{ref}}
$$

(18)

Since $x_3$ and $x_4$ are not measurable, using the equations

$$
x_3 = \dot{x}_3 - e_3
$$

$$
x_4 = \dot{x}_4 - e_4
$$

(19)

results in

$$
\dot{Z}_{c1} = a_7(x_2\dot{x}_3 - x_1\dot{x}_4) - a_8 - a_9 x_5 - a_{10} x_2^2 - \dot{w}_{\text{ref}} - a_7(x_2e_3 - x_1e_4)
$$

(20)

Choose the estimated torque $a_7(x_2\dot{x}_3 - x_1\dot{x}_4)$ as the so-called virtual control treating $e_3, e_4$ as unknown disturbances, which are bounded and decrease to zero exponentially from the observer analysis. To overcome these (unknown) terms, nonlinear damping theory as described in References [16, 17], will be used. This result is applicable to systems with matched uncertainties, i.e. systems in which both the uncertainty and the control appear in the same channel. In this case, Reference [17] has shown that even when no upper bound on the disturbance is known, a nonlinear damping component can be designed to ensure boundedness of the trajectories of the closed loop. For a given Lyapunov function $V_1 = \frac{1}{2} Z_{c1}^2$, to obtain a negative definite time derivative, the stabilization function is chosen as follows:

$$
\phi_{c1} = -z_{c1} Z_{c1} + a_8 + a_9 x_5 + a_{10} x_2^2 + \dot{w}_{\text{ref}} - d_1 a_7^2(x_1^2 + x_2^2) Z_{c1}
$$

(21)

where $d_1 > 0$ and $z_{c1}$ are a scalar design parameters. In Equation (21), $d_1 a_7^2(x_1^2 + x_2^2) Z_{c1}$ is the non-linear damping term. In the case when $a_7(x_2\dot{x}_3 - x_1\dot{x}_4) = \phi_{c1}$, the time derivative of the Lyapunov function $V_1$ is given by

$$
\dot{V}_1 = -z_{c1} Z_{c1}^2 - d_1 a_7^2(x_1^2 + x_2^2) Z_{c1}^2 - a_7 Z_{c1} x_2 e_3 + a_7 Z_{c1} x_1 e_4
$$

(22)

Young's inequality, in Reference [16], is given by

$$
xy \leq \frac{p}{p} |x|^p + \frac{1}{q e^q} |y|^q
$$

(23)
which is satisfied if \( p > 1 \) and \( q > 1 \) are such that \((p - 1)(q - 1) = 1\). Choosing \( p = q = 2 \) and \( e^2 = 2d_1 \) gives

\[
xy \leq d_1 x^2 + \frac{1}{4d_1} y^2
\]  

(24)

Using this inequality

\[
a_7 Z_{e1} x_2 e_3 \leq d_1 a_7^2 Z_{e1}^2 x_2^2 + \frac{e_3^2}{4d_1}
\]  

(25)

\[
a_7 Z_{e1} x_1 e_4 \leq d_1 a_7^2 Z_{e1}^2 x_1^2 + \frac{e_4^2}{4d_1}
\]

and then the time derivative of the Lyapunov function given by Equation (22), becomes

\[
\dot{V}_1 \leq -a_{e1} Z_{e1}^2 + \frac{e_3^2}{4d_1} + \frac{e_4^2}{4d_1}
\]  

(26)

From the inequality above, \( \dot{V}_1 \) is negative whenever \( |Z_{e1}| \geq \sqrt{(e_3^2 + e_4^2)/(4d_1 x_{e1})} \). Since \( e_3 \) and \( e_4 \) are bounded disturbances, \( \dot{V}_1 \) is negative-definite outside the domain

\[
D = \left\{ Z_{e1} : |Z_{e1}| \leq \frac{\sqrt{e_3^2 + e_4^2}}{4d_1 x_{e1}} \right\}
\]  

(27)

Recalling that \( V_1 = \frac{1}{2} Z_{e1}^2 \), \( |Z_{e1}| \) decreases whenever \( Z_{e1} \) is outside the set \( D \), and hence \( Z_{e1} \) is bounded. From the observer analysis, \( e_3 \) and \( e_4 \) are bounded and converge to zero, then \( Z_{e1} \) converges to zero in addition to being bounded.

**Step 2:** Define the first switching function variable as

\[
s_{e1} = a_7 (x_2 \dot{x}_3 - x_1 \dot{x}_4) - \phi_{e1}
\]  

(28)

The time derivative of the switching function \( s_{e1} \) is

\[
\dot{s}_{e1} = a_7 (\dot{x}_2 \dot{x}_3 + x_2 \dot{x}_3 - \dot{x}_1 \dot{x}_4 - x_1 \dot{x}_4) - \dot{\phi}_{e1}
\]  

(29)

where the time derivative of the first stabilization function is given by

\[
\dot{\phi}_{e1} = -a_{e1} \dot{Z}_{e1} + a_9 \dot{x}_5 + 2a_{10} x_5 \dot{x}_5 + \ddot{w}_{ref} - d_1 a_7^2 (x_1^2 + x_2^2) \dot{Z}_{e1} - d_1 a_7^2 Z_{e1} (2x_1 \dot{x}_1 + 2x_2 \dot{x}_2)
\]  

(30)

Finally the time derivative of the first sliding surface is obtained as

\[
\dot{s}_{e1} = A_1 u_a + A_2 u_b + G_1 e_3 + G_2 e_4 + F_1
\]  

(31)
with

\[ A_1 = -a_7 b f_2 \]
\[ A_2 = a_7 b f_1 \]
\[ G_1 = a_7 (a_3 x_5 f_1 + a_2 f_2 - x_2 f_4) \]
\[ G_2 = a_7 (a_3 x_3 f_2 - a_2 f_1 + x_1 f_4) \]
\[ F_1 = a_7 f_1 (-a_1 x_2 - a_3 \hat{x}_3 x_5 + a_2 \hat{x}_4) - a_7 f_2 (-a_1 x_1 + a_3 \hat{x}_4 x_5 + a_2 \hat{x}_3) \]
\[ + f_4 (a_7 (x_2 \hat{x}_3 - x_1 \hat{x}_4) - a_8 - a_9 x_5 - a_{10} x_5^2) - \hat{w}_{\text{ref}} - f_3 \hat{w}_{\text{ref}} \]
\[ - a_7 x_1 (-a_5 \hat{x}_4 + a_6 \hat{x}_3 x_5 + a_4 \hat{x}_2 + \Lambda_4) + a_7 x_2 (-a_5 \hat{x}_3 - a_6 \hat{x}_4 x_5 + a_4 \hat{x}_1 + \Lambda_3) \]

where

\[ f_1 = \hat{x}_3 + 2d_1 a_7 Z_{c1} x_2 \]
\[ f_2 = \hat{x}_4 - 2d_1 a_7 Z_{c1} x_1 \]
\[ f_3 = \alpha_{c1} + d_1 a_7^2 (x_1^2 + x_2^2) \]
\[ f_4 = f_3 - a_9 - 2a_{10} x_5 \]

In Equation (31) both control components $u_a$ and $u_b$ appear. In order to use this additional degree of freedom, flux tracking will be considered. In this case, it is assumed that sliding is taking place in the observer. This presents no theoretical difficulties since the results of Section 3.1 show that sliding takes place in finite time. The equivalent output error injection vector will replace the observer gains.

**Step 3:** Consider the following flux tracking error term given by

\[ Z_{c2} = x_3^2 + x_4^2 - \Phi_{\text{ref}} \]  

(32)

where $\Phi_{\text{ref}}$ is the square of the rotor flux reference.

Since $x_3$ and $x_4$ are not measurable, consider their estimates, to obtain

\[ Z_{c2} = \hat{x}_3^2 + \hat{x}_4^2 - \Phi_{\text{ref}} \]  

(33)

Its time derivative is given by

\[ \dot{Z}_{c2} = 2\hat{x}_3 (-a_5 \hat{x}_3 + a_4 \hat{x}_1 - a_6 \hat{x}_3 x_5 + \Lambda_{3\text{eq}}) + 2\hat{x}_4 (-a_5 \hat{x}_4 + a_6 \hat{x}_2 + a_4 \hat{x}_3 x_5 + \Lambda_{4\text{eq}}) - \Phi_{\text{ref}} \]  

(34)

Since

\[ \begin{bmatrix} \Lambda_{3\text{eq}} \\ \Lambda_{4\text{eq}} \end{bmatrix} = \Lambda \begin{bmatrix} \Lambda_{1\text{eq}} \\ \Lambda_{2\text{eq}} \end{bmatrix} = \Delta \Gamma^{-1} (- \Gamma) \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} = - \Delta \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} \]  

(35)
it follows that once sliding takes place

\[
\Lambda_{3\text{eq}} = -\delta_1 e_3 + a_6 x_5 e_4
\]

\[
\Lambda_{4\text{eq}} = -a_6 x_5 e_3 - \delta_2 e_4
\]

(36)

and the time derivative of the square rotor flux tracking error can be rewritten as

\[
\dot{Z}_{e^2} = 2a_4 (\dot{x}_1 \dot{x}_3 + \dot{x}_2 \dot{x}_4) - 2a_5 (\dot{x}_3^2 + \dot{x}_4^2) - 2(\delta_1 \dot{x}_3 + a_6 x_5 \dot{x}_4) e_3 + 2(a_6 x_5 \dot{x}_3 - \delta_2 \dot{x}_4) e_4 - \dot{\Phi}_{\text{ref}}
\]

(37)

Choosing the second virtual control as \(2a_4 (\dot{x}_1 \dot{x}_3 + \dot{x}_2 \dot{x}_4)\) and using nonlinear damping theory (\(e_3\) and \(e_4\) are unknown disturbances which converge exponentially to zero), the second stabilisation function, associated with the Lyapunov function \(V_3 = \frac{1}{2} Z_{e^2}^2\), is given by the following equation:

\[
\phi_{e^2} = -Z_{e^2} Z_{e^2} + 2a_5 (\dot{x}_3^2 + \dot{x}_4^2) + \dot{\Phi}_{\text{ref}} - 4d_2 [(\delta_1 \dot{x}_3 + a_6 x_5 \dot{x}_4)^2 + (a_6 x_5 \dot{x}_3 - \delta_2 \dot{x}_4)^2] Z_{e^2}
\]

(38)
Following the same procedure in Step 1, it can easily be shown that

$$\dot{V}_3 = -\alpha Z_c^2 + \frac{e_3^2}{4d_z} + \frac{e_4^2}{4d_z}$$

(39)

is negative-definite outside the region \( \{Z_c: |Z_c| < \sqrt{(e_3^2 + e_4^2)/4d_z} \} \). Since \( e_3 \) and \( e_4 \) are bounded and converge to zero, \( Z_c \) converges to zero in addition to being bounded.

**Step 4:** The error between the virtual control and the stabilisation function is taken as the second switching function, so that

$$s_{c2} = 2a_4(\hat{x}_1\hat{x}_3 + \hat{x}_2\hat{x}_4) - \phi_{c2}$$

(40)

The time derivative of this second switching function \( s_{c2} \) is given by the following equation

$$\dot{s}_{c2} = 2a_4(\hat{x}_1\dot{x}_3 + \hat{x}_2\dot{x}_4 + \dot{x}_1\hat{x}_3 + \dot{x}_2\hat{x}_4) - \dot{\phi}_{c2}$$

(41)
where the time derivative of the second stabilization function is

\[
\dot{\phi}_{c2} = -\alpha_{c2}Z_{c2} + 4a_5(\dot{x}_3\dot{x}_3 + \dot{x}_4\dot{x}_4) + \Phi_{ref} - 4d_2[[\delta_1\dot{x}_3 + a_6x_5\dot{x}_4]^2 + (a_6x_5\dot{x}_3 - \delta_2\dot{x}_4)^2]Z_{c2}
\]

\[
- 4d_2[2(\delta_1\dot{x}_3 + a_6x_5\dot{x}_4)(\delta_1\dot{x}_3 + a_6x_5\dot{x}_4 + a_6x_5\dot{x}_4) + 2(a_6x_5\dot{x}_3 - \delta_2\dot{x}_4)(a_6x_5\dot{x}_3 + a_6x_5\dot{x}_3 - \delta_2\dot{x}_4)]Z_{c2}
\]

(42)

For notational simplicity make the following definitions:

\[
f_5 = \alpha_{c2} + 4d_2(f_2^2 + f_2^2)
\]
\[
f_6 = \delta_1\dot{x}_3 + a_6x_5\dot{x}_4
\]
\[
f_7 = a_6x_5\dot{x}_3 - \delta_2\dot{x}_4
\]
\[
f_8 = 2a_4(\dot{x}_1\dot{x}_3 + \dot{x}_2\dot{x}_4) - 2a_5(\dot{x}_3^2 + \dot{x}_4^2) - \Phi_{ref}
\]
\[
f_9 = 4d_2(2\delta_1f_6 + 2a_6x_5f_7)Z_{c2}
\]
\[
f_{10} = 4d_2(2a_6x_5f_6 - 2\delta_2f_7)
\]
\[
f_{11} = 4d_2(2a_6\dot{x}_4f_6 + 2a_6\dot{x}_3f_7)Z_{c2}
\]
Figure 5. Stator current modulus and motor torque.

\[ f_{12} = -a_1 \dot{x}_1 + a_3 \dot{x}_4 x_5 + a_2 \dot{x}_3 \]
\[ f_{13} = -a_1 \dot{x}_2 - a_3 \dot{x}_3 x_5 + a_2 \dot{x}_4 \]
\[ f_{14} = -a_5 \dot{x}_3 - a_6 \dot{x}_4 x_5 + a_4 \dot{x}_1 \]
\[ f_{15} = -a_5 \dot{x}_4 + a_6 \dot{x}_3 x_5 + a_4 \dot{x}_2 \]
\[ f_{16} = a_7 (x_2 \dot{x}_3 - x_1 \dot{x}_4) - a_8 - a_9 x_5 - a_{10} x_5^2 \]
\[ f_{17} = f_{10} + 2a_4 \dot{x}_1 - 4a_5 \dot{x}_3 \]
\[ f_{18} = f_{10} + 2a_4 \dot{x}_2 - 4a_5 \dot{x}_4 \]

Then the speed dynamic becomes

\[ \dot{x}_5 = \dot{\omega} = f_{16} - a_7 (x_2 \dot{e}_3 - x_1 \dot{e}_4) \quad (43) \]

Finally, the time derivative of the second sliding surface \( s_{c2} \) can be shown to be

\[ \dot{s}_{c2} = A_3 u_a + A_4 u_b + G_3 e_3 + G_4 e_4 + F_2 \quad (44) \]
Consider the two Equations (31) and (44), with the following control law:

\[
\begin{bmatrix}
    u_a \\
    u_b
\end{bmatrix} = \left[ \begin{array}{cc}
    A_1 & A_2 \\
    A_3 & A_4
\end{array} \right]^{-1} \left[ \begin{array}{c}
    F_1 \\
    F_2
\end{array} \right] - \left[ \begin{array}{cc}
    q_1 & 0 \\
    0 & q_2
\end{array} \right] \left[ \begin{array}{c}
    \text{sign}(s_{c_1}) \\
    \text{sign}(s_{c_2})
\end{array} \right] - \left[ \begin{array}{cc}
    q_3 + d_3(G_1^2 + G_2^2) & 0 \\
    0 & q_3 + d_4(G_3^2 + G_4^2)
\end{array} \right] \left[ \begin{array}{c}
    s_{c_1} \\
    s_{c_2}
\end{array} \right]
\]

(45)
The expression above is valid only when \[
\begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix}
\] is invertible, i.e., if, \(A_1A_4 - A_2A_3 \neq 0\). It can be shown

\[
A_1A_4 - A_2A_3 = -2a_4a_7b^2[\dot{x}_3^2 + \dot{x}_4^2 + 2d_1a_7Z_{c1}(x_2\dot{x}_3 - x_1\dot{x}_4)]
\] (46)

and since the term \(\dot{x}_3^2 + \dot{x}_4^2\), is the square of the rotor flux which is always positive and the term \(x_2\dot{x}_3 - x_1\dot{x}_4\), is the motor torque, the above equation becomes

\[
A_1A_4 - A_2A_3 = -2a_4a_7b^2[\text{flux}^2 + 2d_1a_7Z_{c1} \text{torque}]
\] (47)

If the quantity \(Z_{c1}\) is negative, it means that the value of the rotor speed is less than its reference and then more (positive) motor torque is needed in order to compensate this difference.
$Z_{c1}$ is positive, the motor tends to decelerate and the motor torque is negative. Consequently $Z_{c1}$ torque $\leq 0$. As a result, $A_1A_4 - A_2A_3 \neq 0$, is equivalent to the requirement that

$$\text{flux}^2 \neq \frac{1}{2d_1a_{c1}|Z_{c1}||\text{torque}|} \quad (48)$$

Since $d_1$ is a design parameter, it is easy to overcome this problem.

Substituting Equation (45) into Equations (31) and (44) gives

$$\dot{s}_{c1} = -q_1 \text{sign}(s_{c1}) - (q_3 + d_3(G_1^2 + G_2^2))s_{c1} + G_1e_3 + G_2e_4 \quad (49)$$

$$\dot{s}_{c2} = -q_2 \text{sign}(s_{c2}) - (q_4 + d_4(G_3^2 + G_4^2))s_{c2} + G_3e_3 + G_4e_4 \quad (50)$$

Following the same procedure in Steps 1 and 2 and using Young’s inequality

$$s_{c1}\dot{s}_{c1} \leq -q_1s_{c1}\text{sign}(s_{c1}) - q_3s_{c1}^2 + \frac{e_3^2 + e_4^2}{4d_3} \quad (51)$$

$$s_{c2}\dot{s}_{c2} \leq -q_2s_{c2}\text{sign}(s_{c2}) - q_4s_{c2}^2 + \frac{e_3^2 + e_4^2}{4d_4} \quad (52)$$
Consequently $s_{c1}\dot{s}_{c1}$ and $s_{c2}\dot{s}_{c2}$ are negative if

$$|s_{c1}| \geq -\frac{q_1 + \sqrt{q_1^2 + 4q_3(e_{23}^2 + e_{24}^2)/4d_3}}{2q_3}$$

(53)

and

$$|s_{c2}| \geq -\frac{q_2 + \sqrt{q_2^2 + 4q_4(e_{23}^2 + e_{24}^2)/4d_4}}{2q_4}$$

(54)

Consequently, in finite time, the controller switching functions are forced into a (decreasing) boundary layer about the ideal sliding surface $s_{c1} = s_{c2} = 0$.

3.3. Summary

In terms of the observer, the four parameters $\delta_1$, $\delta_2$, $\rho_1$ and $\rho_2$ are available for design purposes. The scalars $\delta_1$ and $\delta_2$ define the flux estimation error decay rates. Once these have been specified, $\Delta$ from Equation (14) is determined and so is the output error injection gain matrix $\Lambda = \Delta \Gamma^{-1}$. 

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The scalars $\rho_1$ and $\rho_2$ premultiply and determine the magnitude of the discontinuous output error injection signal. These scalars will determine the rate at which the sliding motion is attained.

In terms of the controller, the parameters $a_1$, $a_2$, $d_1$, $d_2$, $d_3$, $d_4$, $q_1$, $q_2$, $q_3$, $q_4$ are available to tune the closed-loop response. The pair $x_{c1}$ and $x_{c2}$ represent the gains for the linear error feedback terms in the stabilization functions (21) and (38) and define the rate at which the tracking errors $Z_{c1}$ and $Z_{c2}$ tend to zero. The scalars $d_1$ and $d_2$ are the corresponding non-linear damping gains. The parameters $q_3$ and $q_4$ are linear gains associated with the switching functions in the expression for the stator voltages (45). Again $d_3$ and $d_4$ are the gains associated with the non-linear damping terms. Finally, $q_1$ and $q_2$ represent the gains which scale the relay components in (45).

4. SIMULATION RESULTS

This section considers the induction motor corresponding to an experimental set-up in the CNRS Laboratory of Electrical Engineering of PARIS (LGEP). The nominal parameters are given as follows:

\[ \text{Stator inductance (} L_s \text{) = 0.47 H} \]
\[ \text{Rotor inductance (} L_r \text{) = 0.47 H} \]
Mutual inductance ($L_m$) = 0.44 H
Total leakage factor ($\sigma$) = 0.12
Stator resistance ($R_s$) = 8.0 $\Omega$
Rotor resistance ($R_r$) = 3.6 $\Omega$
Moment of inertia ($J$) = 0.06 Kg m$^2$
Mechanical viscous damping ($f$) = 0.04 N m s
Number of pole pairs ($p$) = 2

In the benchmark experiment, it is assumed that the only measurable signals are the stator currents ($x_1, x_2$) and the rotor speed $\omega$. It is assumed that the load torque is unknown, but all the parameters are known and constant except for the rotor time constant which will change during the experiments. The parameter change will be introduced only in the plant, and the controller will continue to work with the nominal values.
In order to reduce the stator current demand, the step reference for speed and flux has been replaced by an exponential one.

\[ w_{\text{ref}} \rightarrow w_{\text{ref}}(1 - e^{-k_1 t}) \]  \hspace{1cm} (55)

\[ \phi_{\text{ref}} \rightarrow \phi_{\text{ref}}(1 - e^{-k_2 t}) \]  \hspace{1cm} (56)

The transient behaviour of the stator current for the various references is fixed by the value of the two parameters \( k_1 \) and \( k_2 \). In order to reduce the chattering phenomenon, a continuous function interpolation is added in the neighbourhood of \( S = 0 \). This continuous function is given by

\[ \text{sign}(S) \rightarrow \frac{S}{|S| + \kappa}, \quad \kappa > 0 \]  \hspace{1cm} (57)

\( \kappa \) is chosen small as possible. The first control objective is to follow the speed and the flux reference in spite of disturbances in the load torque and the rotor time constant. The control parameters for this first simulation are chosen as follows: \( z_{c_1} = 250, \ z_{c_2} = 400, \ q_1 = 1000, \ q_2 = 1000, \ q_3 = 1000, \ q_4 = 1000, \) and \( \delta_1 = \delta_2 = 100, \ \eta_1 = \eta_2 = 1.5 \). In Figure 2 the motor speed and the demand profile are shown together with the square of the tracking error. It can be seen that the speed tracking is quite good. The flux norm tracking and the corresponding stator voltage modulus are shown in Figure 3. In Figure 4, the control inputs \( u_a \) and \( u_b \) are shown. The stator current modulus and the motor torque are given in Figure 5. It can be seen that the torque is over the nominal value (7 N m) when the speed is higher than the nominal speed. Finally, the flux observer sliding surfaces are shown in Figure 6. It can be seen that in the presence of load and rotor resistance variations, the controller is robust. The value of the stator voltage modulus is sometimes higher than the nominal value 210 V. In the simulations however, there is a hard limit of 210 V on each component. The flux observer sliding surfaces display some oscillations when the rotor resistance varies. These oscillations are proportional to the rotor speed. However, good flux estimation is still obtained together with quite good tracking performance.

The second control objective concerns position tracking. Because the controller in Section 3 is designed for speed tracking, the desired position trajectory will be replaced by the corresponding speed reference. In this case, in order to reduce the stator current demand, the same exponential reference will be introduced as in the first part. The control parameters for this second objective are chosen as follows: \( z_{c_1} = 600, \ z_{c_2} = 800, \ q_1 = 2000, \ q_2 = 1200, \ q_3 = 2000, \ q_4 = 1000 \) and \( \delta_1 = \delta_2 = 100, \ \eta_1 = \eta_2 = 1.5 \). In Figure 7, the speed tracking together with the rotor flux tracking are shown. The stator inputs \( u_a, u_b \) and the stator voltage modulus are given in Figure 8. The stator current modulus and the motor torque are given in Figure 9. The corresponding flux observer sliding surfaces are given in Figure 10. Finally, the rotor position corresponding to the speed tracking is shown in Figure 11. It can be seen that the results are quite satisfactory even if our reference is the speed trajectory.

5. CONCLUSION

This paper has presented a non-linear sliding mode controller for an induction machine. Assuming that only the stator current and the rotor speed are available, a sliding mode observer
was designed to estimate the rotor flux. A non-linear sliding mode controller was derived taking into account the estimated values of the flux and uses a systematic method for the design of the controller sliding manifold. The closed-loop stability of the algorithm was shown. The results for the benchmark problem were quite satisfactory.

REFERENCES