The use of Sinc-collocation method for solving multi-point boundary value problems

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ABS TRACT

Multi-point boundary value problems have received considerable interest in the mathematical applications in different areas of science and engineering. In this work, our goal is to obtain numerically the approximate solution of these problems by using the Sinc-collocation method. Some properties of the Sinc-collocation method required for our subsequent development are given and are utilized to reduce the computation of solution of multi-point boundary value problems to some algebraic equations. It is well known that the Sinc procedure converges to the solution at an exponential rate. Numerical examples are included to demonstrate the validity and applicability of the new technique.

1. Introduction

In the last three decades a variety of numerical methods based on the Sinc approximation have been developed. Sinc methods developed by Frank Stenger, the pioneer of this field, people in his school and others [1,2] and it is widely used for solving a wide range of linear and nonlinear problems arising from scientific and engineering applications including oceanographic problems with boundary layers [3], two-point boundary value problems [4], astrophysics equations [5], Blasius equation [6], Volterra’s population model [7], Hallen’s integral equation [8], third-order boundary value problems [9], system of second-order boundary value problems [10], fourth-order boundary value problems [11], heat distribution [12], elasto-plastic problem [13], inverse problem [14,15], integro-differential equation [16], optimal control [1]. Very recently authors of [17] used the Sinc procedure to solve linear and nonlinear Volterra integral and integro-differential equations. We also refer the interested reader to [18–22] for more research works on Sinc methods.

There are several advantages to using approximations based on Sinc numerical methods. First, unlike most numerical techniques, it is now well-established that they are characterized by exponentially decaying errors [1]. Second, approximations by Sinc functions handles singularities in the problem. The effect of any such singularities will appear in some form in any scheme of the numerical solution, and it is well known that polynomial methods do not perform well near singularities. Finally, due to their rapid convergence, Sinc numerical methods do not suffer from the common instability problems associated with other numerical methods [23].

As said in [24], although many physical problems are modeled by a single or a system of differential equations over a finite domain, there are also applications where the system is modeled by different differential equations over subdomains

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of the overall domain and require the solution to satisfy certain conditions across the subdomains boundaries, in addition to conditions at the overall domain boundaries. This is what constitutes a multi-point boundary value problem (MBVP). In this paper, we consider MBVP in the following form

\[ y^{(m)} = g(x, y, y', \ldots, y^{(n-1)}), \quad x \in [0, 1], \]  

for which the boundary conditions are given as the value of the function \( y \) or its derivatives or combination of them at \( m \) points \( (m \leq n) \) in the domain \([0, 1]\). Also, it is assumed that \( g \) has properties which guarantee the existence and uniqueness of the solution of the problem. The study of MBVP’s for the linear second order ordinary differential equations was initiated by Il’in and Moiseev in \([25,26]\). In 1992, Gupta \([27]\) firstly studied the existence and uniqueness of solutions to the certain nonlinear three-point boundary value problems. Since then, the existence and uniqueness of solutions of the more general nonlinear MBVPs have been investigated by many authors (see \([28]\) and references therein).

Multi-point boundary value problems arise in variety of different areas of applied mathematics and physics. For instance, Hajji in \([24]\) considered MBVP which occurs in many areas of engineering applications such as in modelling the flow of fluid such as water, oil and gas through ground layers, where each layer constitutes a subdomain. The vibrations of a guy wire of a uniform cross-section and composed of such as water, oil and gas through ground layers, where each layer constitutes a subdomain. The vibrations of a guy wire of a uniform cross-section and composed of different densities can be formulated as a MBVP \([29]\). Many problems in the theory of elastic stability can be handled by multi-point problems \([30]\). Also as said in \([31]\) large size bridges are sometimes contrived with multi-point supports which correspond to a multi-point boundary value condition.

Despite of the large amount of works which are done on the theoretical aspects of these kind of equations (see \([32]\) and references therein), few works are available on the numerical analysis of MBVPs. Authors of \([31]\) introduced a method for solving a class of nonlinear second-order MBVPs in bridge design by combining He’s homotopy perturbation and variational iteration methods. In \([33]\) an adaptive finite difference method \([34]\) for the first order nonlinear systems of ordinary differential equations subject to multi-point nonlinear boundary conditions is presented. In \([35]\), based upon the shooting method, a numerical method for approximating solutions and fold bifurcation solutions of second order MBVPs is set up. In \([36]\) Adomian decomposition method and in \([37]\) homotopy analysis method employed for solving MBVPs. Also the optimal homotopy asymptotic method \([38]\) and He’s variational iteration method \([39]\) are employed to solve MBVPs.

In this paper, the solution of MBVPs is presented by means of Sinc-collocation method. Our method consists of reducing the solution of MBVP to a set of algebraic equations by expanding \( y(x) \) as Sinc functions with unknown coefficients. The properties of Sinc function are then utilized to evaluate the unknown coefficients. This paper is organized in the following way:

Section 2 is devoted to the basic formulation of the Sinc functions required for our subsequent development. Section 3 applies the Sinc-collocation method to MBVPs. In Section 4, we present some numerical examples to show the efficiently and applicability of this method for MBVPs. An application of the model is described in Section 5 and a conclusion is drawn in Section 6. Note that we have computed the numerical results by Maple programming.

2. Sinc function properties

In this section we only state the main properties which will be useful in the following. The reader interested in more detailed properties of the Sinc function should refer to \([1,2]\).

The Sinc function is defined on the whole real line, \(-\infty < x < \infty\), by

\[ \text{Sinc}(x) = \begin{cases} \sin(\pi x) / \pi x, & x \neq 0, \\ 1, & x = 0. \end{cases} \]

For \( h > 0 \), and \( k = 0, \pm 1, \pm 2, \ldots \), the translated Sinc functions with evenly spaced nodes are given by

\[ S(k, h)(x) = \text{Sinc}\left(\frac{x - kh}{h}\right) = \begin{cases} \sin\left[\pi\left(\frac{x - kh}{h}\right)\right] / \pi\left(\frac{x - kh}{h}\right), & x \neq kh, \\ 1, & x = kh. \end{cases} \]

The Sinc functions form an interpolatory set of functions, i.e.,

\[ S(k, h)(j h) = \delta_{kj}, \quad k = j, \]

\[ 0, \quad k \neq j. \]

If a function \( f(x) \) is defined on the real axis, then for \( h > 0 \) the series

\[ C(f, h)(x) = \sum_{k=\infty}^{\infty} f(kh) \text{Sinc}\left(\frac{x - kh}{h}\right), \]

is called the Whittaker cardinal expansion of \( f \) whenever this series converges. The properties of Whittaker cardinal expansion have been extensively studied in \([2]\).

As mentioned in \([2]\), a class of functions where the Whittaker cardinal expansion of \( f \) converges to \( f \) is characterized in the following definition.
**Definition 1.** Let $h$ be a positive constant. The Paley–Wiener class of functions $B(h)$ is the family of entire functions $f$ such that on the real line $f \in L^2(\mathbb{R})$ and in the complex plane $f$ is of exponential type $\pi/h$, i.e.,

$$|f(z)| \leq K \exp(\pi|z|/h),$$

for some $K > 0$.

But there are functions $f \notin B(h)$ where the approximation of $f$ by its cardinal series has errors that decrease exponentially. One is led to seek a less restrictive class of functions than $B(h)$ where the exponential decay of the error can be maintained.

The properties of Whittaker cardinal expansion are derived in the infinite strip $D_s$ of the complex $w$-plane, where for $d > 0$,

$$D_s = \{ w = t + is : |s| < d \leq \frac{\pi}{2} \}.$$  

Approximations can be constructed for infinite, semi-infinite and finite intervals. To construct approximations on the interval $(0,1)$, which are used in this paper, the eye-shaped domain in the $z$-plane,

$$D_E = \{ z = x + iy : \arg\left(\frac{z}{1-z}\right) < d \leq \frac{\pi}{2}\},$$

is mapped conformally onto the infinite strip $D_s$ via

$$w = \phi(z) = \ln\left(\frac{z}{1-z}\right).$$

The basis functions on $(0,1)$ are taken to be the composite translated Sinc functions

$$S_k(x) = S(k, h) \circ \phi(x) = \text{Sinc}\left(\frac{\phi(x) - kh}{h}\right),$$

where $S(k, h) \circ \phi(x)$ is defined by $S(k, h)(\phi(x))$. The inverse map of $w = \phi(z)$ is

$$z = \phi^{-1}(w) = \frac{\exp(w)}{1 + \exp(w)}.$$  

Thus we may define the inverse images of the real line and of the evenly spaced nodes $(kh)_{k=-\infty}^{\infty}$ as

$$\Gamma = \{ \phi^{-1}(x) \in D_E : -\infty < x < \infty \} = (0,1),$$

and

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \ldots$$

respectively. The class of functions such that the known exponential error estimates exist for Sinc interpolation is denoted by $B(D_E)$ and is defined in the following.

**Definition 2.** Let $B(D_E)$ be the class of functions $F$ which are analytic in $D_E$, satisfy

$$\int_{\phi^{-1}(L)} |F(z)|dz \to 0, \quad t \to \pm\infty,$$

where $L = \{ iv : |v| < d \leq \frac{\pi}{2}\}$, and on the boundary of $D_E$, (denoted $\partial D_E$), satisfy

$$N(F) = \int_{\partial D_E} |F(z)|dz < \infty.$$  

Interpolation for function in $B(D_E)$ is defined in the following theorem whose proof can be found in [1].

**Theorem 1.** If $\phi' F \in B(D_E)$ then for all $x \in \Gamma$

$$\left| F(x) - \sum_{k=-\infty}^{\infty} F(x_k) S(k, h) \circ \phi(x) \right| \leq \frac{N(F\phi')}{2\pi d \sinh(\pi d/h)} \leq \frac{2N(F\phi')}{\pi d} e^{-nd/h}.$$  

Moreover, if $|F(x)| \leq Ce^{-\pi |x|^2}$, $x \in \Gamma$, for some positive constants $C$ and $\pi$, and if the selection $h = \sqrt{\pi d/2N} \leq 2\pi d/\ln 2$, then

$$\left| F(x) - \sum_{k=-N}^{N} F(x_k) S(k, h) \circ \phi(x) \right| \leq C_2 \sqrt{N} \exp\left(-\sqrt{\pi d} x\right), \quad x \in \Gamma,$$

where $C_2$ depends only on $F$, $d$ and $\pi$.

The above expressions show Sinc interpolation on $B(D_E)$ converges exponentially [3]. We also require derivatives of composite Sinc functions evaluated at the nodes. The expressions required for the present discussion are [11].
\[ \delta_{kj}^{(0)} = |S(k, h) \circ \phi(x)|_{x \to x_j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \]  
\[(5)\]

\[ \delta_{kj}^{(1)} = \frac{d}{d\phi} |S(k, h) \circ \phi(x)|_{x \to x_j} = \frac{1}{H} \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \]  
\[(6)\]

\[ \delta_{kj}^{(2)} = \frac{d^2}{d\phi^2} |S(k, h) \circ \phi(x)|_{x \to x_j} = \frac{1}{H^4} \begin{cases} \frac{\pi^2}{(j-k)^2}, & k = j, \\ -\frac{2(1-j)^{-1}}{(j-k)^2}, & k \neq j. \end{cases} \]  
\[(7)\]

\[ \delta_{kj}^{(3)} = \frac{d^3}{d\phi^3} |S(k, h) \circ \phi(x)|_{x \to x_j} = \frac{1}{H^6} \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k}6 - \pi^2(j-k)^2}{(j-k)^2}, & k \neq j. \end{cases} \]  
\[(8)\]

and

\[ \delta_{kj}^{(4)} = \frac{d^4}{d\phi^4} |S(k, h) \circ \phi(x)|_{x \to x_j} = \frac{1}{H^8} \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k}6 - \pi^2(j-k)^2}{(j-k)^2}, & k \neq j. \end{cases} \]  
\[(9)\]

General expressions for higher order derivatives are also available [19]

\[ \delta_{kj}^{(2r)} = \frac{1}{H^{2r}} \begin{cases} \frac{(-1)^{j-k} \pi^{2r}}{(j-k)^{2r}}, & k = j, \\ \frac{(-1)^{j-k} \pi^{2r}}{(j-k)^{2r}} \sum_{s=0}^{r-1} \frac{(-1)^{j-k} \pi^{2s}}{(j-k)^{2s}} (j-k)^{2s}, & k \neq j. \end{cases} \]  
\[(10)\]

\[ \delta_{kj}^{(2r+1)} = \frac{1}{H^{2r+1}} \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k} \pi^{2r}}{(j-k)^{2r}} \sum_{s=0}^{r-1} \frac{(-1)^{j-k} \pi^{2s}}{(j-k)^{2s}} (j-k)^{2s}, & k \neq j. \end{cases} \]  
\[(11)\]

with \( r = 1, 2, 3, \ldots \)

3. Applying the Sinc-collocation method to MBVP

Consider Eq. (1) for which the boundary conditions are given as the value of the function \( y \) or its derivatives or combination of them at \( m \) points \( (m \leq n) \) in the domain \([0,1]\). Also suppose we need \( p \)th derivative of \( y \) in points 0 and 1.

The Sinc basis functions in Eq. (3) do not have a derivative when \( x \) tends to 0 or 1. Thus we modify the Sinc basis functions as

\[ w(x)S_h(x), \]  
\[(12)\]

where \( w(x) = x^p(x - 1)^p \). Now the first, second, \ldots, and \( p \)th derivatives of the modified Sinc basis functions are defined as \( x \) approaches 0 or 1, and are equal to 0. In order to discretize Eq. (1) by using Sinc-collocation, we approximate solution for \( y(x) \) in Eq. (1) as

\[ y_N(x) = u_N(x) + p(x), \]  
\[(13)\]

where

\[ u_N(x) = \sum_{k=-N}^{N} c_k w(x)S_h(x), \]  
\[(14)\]

and \( p(x) \) is a polynomial given by

\[ p(x) = a_0 + a_1x + \cdots + a_nx^n. \]  
\[(15)\]

In Eq. (15), \( a_0, a_1, \ldots, a_n \) are constants to be determined.

The \( 2N + 1 \), coefficients \( \{c_j\}_{j=-N}^{N} \) and the unknown coefficients \( a_0, a_1, \ldots, a_n \), are determined by substituting \( y_N(x) \) into Eq. (1) and evaluating the result at the Sinc points

\[ x_j = \frac{e^{i\theta}}{1 + e^{i\theta}} \quad j = -N - 1, \ldots, N. \]  
\[(16)\]

It is worth pointing out that \( u_N(x) = u^1_N(x) = \cdots = u^{(p)}_N(x) = 0 \) when \( x \) tends to 0 or 1.

Setting
\[
\frac{d^r}{dx^r} [S_k(x)] = S_k^{(r)}(x), \quad r = 1, \ldots, n.
\]

we have
\[
\begin{align*}
(w(x)S_k(x))' &= w'S_k + w\phi S_k^{(1)} \\
(w(x)S_k(x))'' &= w''S_k + (2w'\phi' + w\phi'')S_k^{(1)} + w\phi'^2S_k^{(2)} \\
(w(x)S_k(x))''' &= w'''S_k + (3w''\phi' + 3w'\phi'' + w\phi'''')S_k^{(1)} + (3w'\phi'^2 + 3w\phi''\phi')S_k^{(2)} + w\phi'^3S_k^{(3)}, \\
(w(x)S_k(x))^{(4)} &= w^{(4)}S_k + (4w'''\phi' + 6w''\phi'' + 4w'\phi''' + w\phi''''')S_k^{(1)} + (6w''\phi'^2 + 12w'\phi''\phi' + 4w\phi'''\phi')S_k^{(2)} + (6w'\phi'^3 + 12w\phi''\phi''\phi' + 3w\phi''\phi''' + 3w\phi''''\phi' + 3w\phi''''')S_k^{(3)} + w\phi'^4S_k^{(4)}.
\end{align*}
\]

Similarly by taking the rth derivative from \((w(x)S_k(x))\) and using Eqs. (10), (11) and (14) we obtain \(u_N^{(r)}(x_j)\). Substituting \(u_N^{(r)}(x_j)\) in (13) we get
\[
\begin{align*}
y_N^{(r)}(x_j) &= u_N^{(r)}(x_j) + p^{(r)}(x_j), \quad r = 0, 1, \ldots, n, \quad j = -N, -N + 1, \ldots, N.
\end{align*}
\]

Substituting Eq. (25) in Eq. (1) we obtain
\[
\begin{align*}
y_N^{(j)}(x) &= g(x, y_N(x), y_N^{(1)}(x), \ldots, y_N^{(j-1)}(x_j), \quad j = -N, -N + 1, \ldots, N.
\end{align*}
\]

The number of the unknown coefficients \(c_l\) and \(a_l\) is equal to \(2N + n + 2\). Eq. (26) gives \(2N + n + 2\) nonlinear algebraic equations and also \(n\) equations can be found from the boundary conditions. Therefore these \(2N + n + 2\) algebraic equations can be solved for the unknown coefficients \(c_l\) and \(a_l\) by using Newton's method. Consequently \(y_N(x)\) given in Eq. (13) can be calculated.

4. Numerical Examples

In this section, we present some examples to show the efficiency of the Sinc-collocation method for solving MBVP's. In all examples we choose \(a = 1/2\) and \(d = \pi/2\) which leads to \(h = \pi/\sqrt{N}\).

Example 1. As the first example consider the four-point second-order nonlinear ordinary differential equation [31]
\[
y''(x) + (x^2 + x + 1)y'(x) = f(x), \quad 0 \leq x \leq 1,
\]
with the boundary conditions
\[
\begin{align*}
y(0) &= \frac{1}{5}y(\frac{1}{2}) + \frac{1}{2}y'(\frac{1}{2}) - 0.0286634, \\
y(1) &= \frac{1}{5}y(\frac{1}{2}) + \frac{1}{2}y'(\frac{1}{2}) - 0.0401287.
\end{align*}
\]

where
\[
f(x) = \frac{1}{5}([-6 \cos(x - x^2) + \sin(x - x^2)(-3(1 - 2x)^2 + (1 + x + x^2) \sin(x - x^2))].
\]

The exact solution is given by \(y(x) = \frac{1}{2} \sin(x - x^2)\). Using the Sinc-collocation method with \(N = 10, 20\), the approximate solution is calculated and the absolute errors \(|y - y_N|\) are plotted in Fig. 1.
Example 2. In this example, we consider the following three-point second-order nonlinear ordinary differential equation

$$y''(x) + \frac{3}{8}y'(x) + \frac{2}{1089}y^2(x) + 1 = 0, \quad 0 \leq x \leq 1,$$

with the boundary conditions

$$y(0) = 0, \quad y\left(\frac{1}{3}\right) = y(1).$$

In Table 1, the results of the present method with $N = 10$ are compared with the Adomian decomposition method [36], optimal homotopy asymptotic method [38] and the results of the successive iteration method introduced in [40]. Also the residual

$$Res = y''_N(x) + \frac{3}{8}y'_N(x) + \frac{2}{1089}y^2_N(x) + 1,$$

with $N = 20, 25$ are plotted in Fig. 2. The given comparison in Table 1 indicates that Sinc-collocation method is in good agreement with other methods.

Example 3. Consider the following third-order linear differential equation [36,38]

$$y'''(x) - k^2y' + a = 0, \quad 0 \leq x \leq 1,$$

with the boundary conditions at three points

$$y'(0) = y'(1) = 0, \quad y\left(\frac{1}{2}\right) = 0,$$

where the physical constants are $k = 5$ and $a = 1$. The function $y(x)$ shows the shear deformation of sandwich beams. The analytic solution of this problem is written as

![Fig. 1. Plot of the absolute error with $N = 10$ (left) and $N = 20$ (right) for example 1.](image)

### Table 1
Comparison of the values of $y(x)$ by different methods for example 2.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Successive iteration method</th>
<th>Adomian decomposition method</th>
<th>Optimal homotopy asymptotic method</th>
<th>Sinc-collocation method</th>
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</table>
\[ y(x) = \frac{a}{k^3} \left( \sinh \frac{k^2}{2} - \sinh kx \right) + \frac{a}{k^2} \left( x - \frac{1}{2} \right) + \frac{a}{k} \left( \cosh kx - \cosh \frac{k}{2} \right) \tanh \frac{k}{2}. \]

Fig. 3 shows the plot of absolute error with \( N = 20, 25 \) using Sinc-collocation method. Of course the accuracy of our method can be improved by increasing \( N \).

**Example 4.** Consider the following four-point fourth-order nonlinear ordinary differential equation [36,40]

\[ y^{(4)}(x) + y(x)y'(x) - 4x^2 - 24 = 0, \quad 0 \leq x \leq 1, \tag{33} \]

with boundary conditions given at four different points,

\[ y(0) = 0, \quad y'(\frac{1}{2}) = 3, \quad y'''(\frac{1}{4}) = 6, \quad y(1) = 1. \tag{34} \]

Using the Sinc-collocation method with \( N = 2 \) we obtain \( y_2(x) = x^4 \), which is the exact solution of this problem.

**5. An application**

Multi-point boundary value problems for ordinary differential equations are applied to model many problems in different areas of science and engineering. The following, expresses a case in which the second order differential equation with multi-point boundary conditions is used as a model for designing bridges [31].

Consider a second-order ordinary differential equation in bridge design
$y''(x) + f(x) + g(x, y(x)) = 0.$

where $y(x)$ denotes the displacement of the bridge from the unloaded position. As said in [31] small size bridges are often designed with two supported points (cf. the left-hand side of Fig. 4), which leads to the standard two-point boundary value conditions

$y(0) = 0, \quad y(1) = 0.$

Large size bridges are sometimes contrived with multi-point supports (cf. the right-hand side of Fig. 4), which correspond to a multi-point boundary value condition. Near each endpoint of the bridge, we can set up two different type of boundary conditions. If we emphasize the position of the bridge at supporting points near $x = 0$, we propose the following boundary value condition:

$y(0) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) + \lambda_1,$

where $\xi_i \in (0,1), \ i = 1, 2, \ldots, m - 2,$ and $\lambda_1$ is a parameter. If we are interested in controlling the angles of the bridge at supporting points near $x = 0$, we submit the following boundary value condition

$y'(0) = \sum_{i=1}^{m-2} \alpha_i y'(\xi_i) + \lambda_2.$

Similar situation holds near $x = 1$ and the multi-point boundary value conditions can be formulated [31].

6. Conclusion

In this work we employed the Sinc-collocation method for solving the multi-point boundary value problems. Properties of the Sinc function are utilized to reduce the computation of this problem to some algebraic equations. The method is computationally attractive and applications are demonstrated through illustrative examples. The obtained results showed that this approach can solve the problem effectively and the comparison shows that the proposed technique is in good agreement with the existing results in the literature.

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