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Continuous and Discrete Frames of Subspaces in Hilbert Spaces

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Abstract. In this paper, we introduce the concept of continuous frame of subspaces which is the generalization of discrete frame of subspaces and we also develop the frame theory of subspaces for separable Hilbert spaces. Since the discrete frames are a special case of continuous frames, we expect that some results that occur in frame theory will be generalized to these frames.

Keywords: Frame; Continuous frame; Frame of subspaces; Continuous frame of subspaces; Measure space.

1. Introduction

Throughout this paper, $\mathcal{H}$ is a Hilbert space, (Ω, μ) is a measure space with a positive measure μ and $\mathcal{H}_C$ is the collection of all non zero closed subspaces of $\mathcal{H}$.

Definition 1.1. *Let* $\upsilon : \Omega \to (0, +\infty)$ *be a measurable function, a mapping* $F : \Omega \to \mathcal{H}_C$ *is called a* continuous frame of subspaces *with respect to* υ *for* $\mathcal{H}$ *if* (*i*) *$\omega \to \pi_{F(\omega)}$ is a measurable function from Ω to $B(\mathcal{H})$;*

(*ii*) *there exist constants $0 < A \leq B < \infty$ such that*

$$A\|f\|^2 \leq \int_\Omega \upsilon^2(\omega)\|\pi_{F(\omega)}(f)\|^2 \, d\mu(\omega) \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1.1}$$

A continuous frame F is called *tight* if $A = B$ and *Parseval* if $A = B = 1$. A frame F is called *Bessel* if the second inequality in (1.1) holds. In this case, B is called the *Bessel bound*.

Lemma 1.2. *Let $F : \Omega \to \mathcal{H}_C$ be a Bessel mapping with respect to υ for $\mathcal{H}$ with bound B. Then the mapping*

$$\sigma : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

$$\sigma(f, g) = \int_\Omega \upsilon^2(\omega)\langle f, \pi_{F(\omega)}(g)\rangle \, d\mu(\omega)$$

is a sesquilinear form and $\|\sigma\| \leq B$.

Proof. Let $f, g \in \mathcal{H}$. It is clear that the mapping $\omega \to \langle f, \pi_{F(\omega)}\rangle$ is a measurable function from Ω to $\mathbb{C}$. Also it is obvious that σ is a sesquilinear form and by Cauchy-Schwarz' inequality we have

$$\begin{aligned} |\sigma(f,g)| &\leq \left(\int_\Omega \upsilon^2(\omega)\|\pi_{F(\omega)}(f)\|^2 \, d\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega \upsilon^2(\omega)\|\pi_{F(\omega)}(g)\|^2 \, d\mu(\omega)\right)^{\frac{1}{2}} \\ &\leq B\|f\|\|g\|. \end{aligned}$$

Therefore $\|\sigma\| \leq B$. ∎

In the discrete frame of subspaces, we have a frame operator. The following Proposition will provide us to obtain a frame operator for a continuous frame of subspaces.

Proposition 1.3. *Let $F : \Omega \to \mathcal{H}_C$ be a continuous frame of subspaces with respect to υ for $\mathcal{H}$ with frame bounds A, B. Then there exists a unique positive and invertible operator $S : \mathcal{H} \to \mathcal{H}$ such that for each $f, g \in \mathcal{H}$*

$$\langle Sf, g\rangle = \int_\Omega \upsilon^2(\omega)\langle f, \pi_{F(\omega)}(g)\rangle \, d\mu(\omega).$$

Furthermore, with the notation of Lemma 1.2, $\|S\| = \|\sigma\|$ and $AI \leq S \leq BI$.

Proof. The existence and uniqueness of operator S follows from Lemma 1.2 and Theorem 2.3.6 of [8]. Let $f \in \mathcal{H}$. Then

$$\langle Sf, f\rangle = \int_\Omega \upsilon^2(\omega)\|\pi_{F(\omega)}(f)\|^2 \, d\mu(\omega).$$

This shows that S is a positive operator and $AI \leq S \leq BI$. Since $\|I - B^{-1}S\| \leq 1 - AB^{-1} < 1$, so S is an invertible operator. ∎

Corollary 1.4. *Let $F : \Omega \to \mathcal{H}_C$ be a Bessel mapping with respect to υ for $\mathcal{H}$ with bound B. Then there exists an unique positive operator $S : \mathcal{H} \to \mathcal{H}$ such that for each $f, g \in \mathcal{H}$*

$$\langle Sf, g\rangle = \int_\Omega \upsilon^2(\omega)\langle f, \pi_{F(\omega)}(g)\rangle \, d\mu(\omega).$$

For each measure space (Ω, μ) and a mapping $F : \Omega \to \mathcal{H}_C$, we define the space of measurable functions

$$L^2(\Omega, F) = \Big\{ f : \Omega \to \mathcal{H} \mid f(\omega) \in F(\omega) \text{ and } \int_\Omega \|f(\omega)\|^2 \, d\mu(\omega) < \infty \Big\}$$

with inner product given by

$$\langle f, g\rangle = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega).$$

It is clear that $L^2(\Omega, F)$ is a Hilbert space with pointwise operations. We will denote the norm of $f \in L^2(\Omega, F)$ by $\|f\|_2$.

Proposition 1.5. *Let $F : \Omega \to \mathcal{H}_C$ be a Bessel mapping with respect to υ for $\mathcal{H}$ with bound B. Then the mapping $T : L^2(\Omega, F) \to \mathcal{H}$ defined by*

$$\langle Tf, x\rangle = \int_\Omega \upsilon(\omega)\langle f(\omega), x\rangle \, d\mu(\omega), \quad (x \in \mathcal{H})$$

is linear and bounded with $\|T\| \leq \sqrt{B}$. Furthermore for each $x \in \mathcal{H}$ and $\omega \in \Omega$

$$T^*(x)(\omega) = \upsilon(\omega)\pi_{F(\omega)}(x).$$

Proof. It is clear that T is linear and

$$\|Tf\| = \sup_{\|x\|=1} |\langle Tf, x\rangle| \leq \sup_{\|x\|=1} \left(\int_\Omega \upsilon^2(\omega)\|\pi_{F(\omega)}(x)\|^2 \, d\mu(\omega)\right)^{\frac{1}{2}} \|f\|_2.$$

Hence $\|T\| \leq \sqrt{B}$. Also for each $f \in L^2(\Omega, F)$ and $x \in \mathcal{H}$, we have

$$\langle T^*x, f\rangle = \langle x, Tf\rangle = \int_\Omega \upsilon(\omega)\langle \pi_{F(\omega)}(x), f(\omega)\rangle \, d\mu(\omega) = \langle g, f\rangle$$

where $g \in L^2(\Omega, F)$ defined by $g(\omega) = \upsilon(\omega)\pi_{F(\omega)}(x)$. ∎

The operators T and T^* in Proposition 1.5 are called *synthesis* and *analysis* operators, respectively. By the assumptions of Proposition 1.5, by letting $x, y \in \mathcal{H}$ and $T^*(x) = f$, $T^*(y) = g$, we obtain

$$\begin{aligned}\langle TT^*x, y\rangle &= \int_\Omega \upsilon(\omega)\langle f(\omega), y\rangle \, d\mu(\omega)\\ &= \int_\Omega \langle f(\omega), \upsilon(\omega)\pi_{F(\omega)}(y)\rangle \, d\mu(\omega)\\ &= \langle f, g\rangle\\ &= \langle x, Tg\rangle\\ &= \int_\Omega \upsilon^2(\omega)\langle x, \pi_{F(\omega)}(y)\rangle \, d\mu(\omega)\\ &= \langle Sx, y\rangle.\end{aligned}$$

Hence $TT^* = S$, where S is a positive operator which is described in Corollary 1.4. If $F : \Omega \to \mathcal{H}_C$ is a continuous frame of subspaces with respect to υ for $\mathcal{H}$, then TT^* will be the frame operator S.

The converse of Proposition 1.5 holds where μ is σ-finite. At the first we need the following lemma.

Lemma 1.6. *Let (Ω, μ) be a measure space, where μ is σ-finite. Suppose that $L^2(\Omega, \mathcal{H})$ is the collection of all measurable functions $f : \Omega \to \mathcal{H}$ such that*

$$\int_\Omega \|f(\omega)\|^2 \, d(\mu(\omega) < \infty.$$

If $f : \Omega \to \mathcal{H}$ is a measurable function and for each $g \in L^2(\Omega, \mathcal{H})$,

$$\Big|\int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu(\omega)\Big| < \infty,$$

then $f \in L^2(\Omega, \mathcal{H})$.

Proof. We may suppose that

$$\int_\Omega \|f(\omega)\|.\|g(\omega)\| \, d\mu(\omega) < \infty, \quad \text{for all} \quad g \in L^2(\Omega, \mathcal{H})$$

since we can replace $g \in L^2(\Omega, \mathcal{H})$ by $h \in L^2(\Omega, \mathcal{H})$ which is defined by

$$h(\omega) := \begin{cases} \frac{\|g(\omega)\|}{\|f(\omega)\|} f(\omega) & \text{if } f(\omega) \neq 0 \\ 0 & \text{if } f(\omega) = 0 \end{cases}.$$

Let $\{\Omega_n\}_{n=1}^\infty$ be a family of disjoint measurable subsets of Ω such that $\Omega = \bigcup_{n=1}^\infty \Omega_n$ with $\mu(\Omega_n) < \infty$ for each $n \geq 1$. For each integer $m \geq 0$, let

$$\Delta_m = \{\, \omega \in \Omega : m \leq \|f(\omega)\| < m + 1 \,\}.$$

It is clear that $\Delta_m \subseteq \Omega$ is measurable and $\Omega = \bigcup_{m=0}^{\infty} \bigcup_{n=1}^{\infty} \left(\Omega_n \cap \Delta_m\right)$ where $\{\Omega_n \cap \Delta_m\}_{n=1}^{\infty}{}_{m=0}^{\infty}$ is a family of disjoint and measurable subsets of Ω. Hence

$$\int_{\Omega} \|f(\omega)\|^2 \, d\mu(\omega) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \int_{\Omega_n \cap \Delta_m} \|f(\omega)\|^2 \, d\mu(\omega)$$

and

$$\int_{\Omega_n \cap \Delta_m} \|f(\omega)\|^2 \, d\mu(\omega) \leq (m+1)^2 \mu(\Omega_n) < \infty.$$

Suppose that $\int_{\Omega} \|f(\omega)\|^2 \, d\mu(\omega) = +\infty$. Then there exists a family $\{E_k\}_{k=1}^{\infty}$ of disjoint and finite subsets of $\mathbb{N}_0 \times \mathbb{N}$ such that

$$\sum_{(m,n) \in E_k} \int_{\Omega_n \cap \Delta_m} \|f(\omega)\|^2 \, d\mu(\omega) > 1,$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let

$$\Gamma_k = \bigcup_{(m,n) \in E_k} \left(\Omega_n \cap \Delta_m\right), \, \Gamma = \bigcup_{k=1}^{\infty} \bigcup_{(m,n) \in E_k} \left(\Omega_n \cap \Delta_m\right)$$

and let $g : \Omega \to \mathcal{H}$ defined by

$$g(\omega) := \begin{cases} c_k f(\omega) & \text{if } \omega \in \Gamma_k \\ 0 & \text{if } \omega \in \Omega \setminus \Gamma \end{cases}$$

where

$$c_k := \frac{1}{k. \left(\int_{\Gamma_k} \|f(\omega)\|^2 \, d\mu(\omega) \right)^{\frac{1}{2}}}.$$

Since

$$\begin{aligned} \int_{\Omega} \|g(\omega)\|^2 \, d\mu(\omega) &= \int_{\Gamma} \|g(\omega)\|^2 \, d\mu(\omega) \\ &= \sum_{k=1}^{\infty} \sum_{(m,n) \in E_k} \int_{\Omega_n \cap \Delta_m} \|g(\omega)\|^2 \, d\mu(\omega) \\ &= \sum_{k=1}^{\infty} \int_{\Gamma_k} c_k^2 \|f(\omega)\|^2 \, d\mu(\omega) \\ &= \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty \end{aligned}$$

$g \in L^2(\Omega, \mathcal{H})$. But on the other hand, we have

$$\begin{aligned}
&\int_{\Omega} \|f(\omega)\|.\|g(\omega)\| \; d\mu(\omega) \\
&= \sum_{k=1}^{\infty} \frac{1}{k} \left(\int_{\Gamma_k} \|f(\omega)\|^2 \; d\mu(\omega) \right)^{\frac{1}{2}} \\
&\geq \sum_{k=1}^{\infty} \frac{1}{k} = +\infty.
\end{aligned}$$

This contradiction proves the Lemma. ■

Proposition 1.7. *Let (Ω, μ) be a measure space where μ is σ-finite and let $F : \Omega \to \mathcal{H}_C$. Suppose $\omega \to \pi_{F(\omega)}$ and $\upsilon : \Omega \to (0, \infty)$ are measurable functions. Then $F : \Omega \to \mathcal{H}_C$ is a Bessel mapping with respect to υ for $\mathcal{H}$ with bound B if and only if the mapping $T : L^2(\Omega, F) \to \mathcal{H}$ defined by*

$$\langle Tf, x \rangle = \int_{\Omega} \upsilon(\omega) \langle f(\omega), x \rangle \, d\mu(\omega), \quad (x \in \mathcal{H})$$

is linear and bounded with $\|T\| \leq \sqrt{B}$.

Proof. Let T be a bounded operator and let $x \in \mathcal{H}$ and $f \in L^2(\Omega, F)$. Then

$$\langle T^* x, f \rangle = \langle x, Tf \rangle = \int_{\Omega} \Big\langle \upsilon(\omega) \pi_{F(\omega)}(x), f(\omega) \Big\rangle \, d\mu(\omega).$$

Hence by Lemma 1.6

$$T^* x(\omega) = \upsilon(\omega) \pi_{F(\omega)}(x), \quad (\omega \in \Omega).$$

Therefore $\|T^* x\|^2 = \int_{\Omega} \upsilon^2(\omega) \|\pi_{F(\omega)}(x)\|^2 \, d\mu(\omega)$, which it implies that $F : \Omega \to \mathcal{H}_C$ is a Bessel mapping with respect to υ for $\mathcal{H}$ with bound B. The converse is given in Proposition 1.5. ■

The following lemma which was proved in [5] is useful.

Lemma 1.8. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and suppose that $U : \mathcal{H} \to \mathcal{K}$ is a bounded operator with closed rang $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger : \mathcal{K} \to \mathcal{H}$ for which*

$$UU^\dagger f = f, \quad (f \in \mathcal{R}_U).$$

We now give a characterization of continuous frames of subspaces in terms of synthesis operator. It is analogous to characterization of frames in terms of synthesis operator, which was proved by Christensen [3].

Theorem 1.9. *Let (Ω, μ) be a measure space where μ is σ-finite and let $F : \Omega \to \mathcal{H}_C$. Suppose $\omega \to \pi_{F(\omega)}$ and $\upsilon : \Omega \to (0, \infty)$ are measurable functions. Then $F : \Omega \to \mathcal{H}_C$ is a continuous frame of subspaces with respect to υ for $\mathcal{H}$ if and only if the synthesis operator T is a bounded mapping from $L^2(\Omega, F)$ onto $\mathcal{H}$.*

Proof. Let $F : \Omega \to \mathcal{H}_C$ be a continuous frame with respect to υ for $\mathcal{H}$. Then $S = TT^*$ is invertible and hence T is onto. Conversely, let T be a bounded and onto operator, then by Lemma 1.8, there exists a bounded operator $T^\dagger : \mathcal{H} \to L^2(\Omega, F)$ such that $TT^\dagger = I_{\mathcal{H}}$, so $(T^\dagger)^* T^* = I_{\mathcal{H}}$. By Propositions 1.5 and 1.7, we have

$$\|T^* x\|^2 = \int_\Omega \upsilon^2(\omega) \|\pi_{F(\omega)} x\|^2 \, d\mu(\omega)$$

and

$$\|T^\dagger\|^{-1} \|x\| \leq \|T^* x\| \leq \|T\| \|x\|$$

for each $x \in \mathcal{H}$. Therefore $F : \Omega \to \mathcal{H}_C$ is a continuous frame with respect to υ for $\mathcal{H}$. ∎

Proposition 1.10. *Let $F : \Omega \to \mathcal{H}_C$ be a continuous frame with respect to υ for $\mathcal{H}$ with bounds A, B. Then*

$$A \dim \mathcal{H} \leq \int_\Omega \upsilon^2(\omega) \dim F(\omega) \, d\mu(\omega) \leq B \dim \mathcal{H}.$$

Furthermore, if $\mathcal{H}$ is a finite dimensional Hilbert space, then

$$A \leq \int_\Omega \upsilon^2(\omega) \, d\mu(\omega) \leq B \dim \mathcal{H}.$$

Proof. For each $\omega \in \Omega$, let $\{e_{\omega j}\}_{j \in J_\omega}$ be an orthonormal basis for $F(\omega)$ and let $\{e_j\}_{j \in J}$ be an orthonormal basis for $\mathcal{H}$. Thus

$$\begin{aligned}
&\sum_{j \in J} \int_\Omega \upsilon^2(\omega) \|\pi_{F(\omega)}(e_j)\|^2 \, d\mu(\omega) \\
&= \sum_{j \in J} \int_\Omega \sum_{i \in J_\omega} \Big| \langle e_j, \upsilon(\omega) e_{\omega i} \rangle \Big|^2 \, d\mu(\omega) \\
&= \int_\Omega \sum_{i \in J_\omega} \sum_{j \in J} \Big| \langle e_j, \upsilon(\omega) e_{\omega i} \rangle \Big|^2 \, d\mu(\omega) \\
&= \int_\Omega \sum_{i \in J_\omega} \|\upsilon(\omega) e_{\omega i}\|^2 \, d\mu(\omega) = \int_\Omega \upsilon^2(\omega) \dim F(\omega) \, d\mu(\omega).
\end{aligned}$$

Since for each $j \in J$

$$A \leq \int_\Omega \upsilon^2(\omega) \|\pi_{F(\omega)}(e_j)\|^2 \, d\mu(\omega) \leq B$$

$$A \dim \mathcal{H} \leq \int_{\Omega} \upsilon^2(\omega) \dim F(\omega)\, d\mu(\omega) \leq B \dim \mathcal{H}.$$

Now let $\dim \mathcal{H} < \infty$, then

$$A \dim \mathcal{H} \leq \int_{\Omega} \upsilon^2(\omega) \dim F(\omega)\, d\mu(\omega) \leq \dim \mathcal{H} \int_{\Omega} \upsilon^2(\omega)\, d\mu(\omega)$$

and

$$\int_{\Omega} \upsilon^2(\omega)\, d\mu(\omega) \leq \int_{\Omega} \upsilon^2(\omega) \dim F(\omega)\, d\mu(\omega) \leq B \dim \mathcal{H}.$$

■

Corollary 1.11. *Let $F : \Omega \to \mathcal{H}_C$ be a Parseval continuous frame with respect to υ for $\mathcal{H}$. Then*

$$\dim \mathcal{H} = \int_{\Omega} \upsilon^2(\omega) \dim F(\omega)\, d\mu(\omega).$$

If $\mathcal{H}$ is a finite dimensional Hilbert space, then

$$1 \leq \int_{\Omega} \upsilon^2(\omega)\, d\mu(\omega) \leq \dim \mathcal{H}.$$

If $F : \Omega \to \mathcal{H}_C$ is an orthogonal mapping, (i.e., for every $\omega, \gamma \in \Omega$, if $\omega \neq \gamma$ then $F(\omega) \perp F(\gamma)$) we have the following proposition.

Proposition 1.12. *Let (Ω, μ) be a measure space. If an orthogonal mapping $F : \Omega \to \mathcal{H}_C$ is a continuous frame with respect to υ for $\mathcal{H}$ with bounds A, B, then each finite subset $F \subseteq \Omega$ is measurable and if $F \neq \emptyset$, then*

$$A \sum_{\omega \in F} \upsilon^{-2}(\omega) \leq \mu(F) \leq B \sum_{\omega \in F} \upsilon^{-2}(\omega). \tag{1.2}$$

Proof. For each $f \in \mathcal{H}$, let $\Omega_f = \{\, \omega \in \Omega : \pi_{F(\omega)}(f) \neq 0 \,\}$. Since $\omega \to \pi_{F(\omega)}$ is a measurable function, Ω_f is measurable. Let $\omega_0 \in \Omega$ and $0 \neq f_0 \in F(\omega_0)$, then $\{\omega_0\} = \Omega_{f_0}$ is measurable. Hence every finite subset of Ω is measurable and

$$A \|f_0\|^2 \leq \int_{\Omega} \upsilon^2(\omega) \|\pi_{F(\omega)}(f_0)\|^2\, d\mu(\omega) = \upsilon^2(\omega_0) \|f_0\|^2 \mu(\{\omega_0\}) \leq B \|f_0\|^2.$$

Therefore

$$A \leq \upsilon^2(\omega) \mu(\{\omega\}) \leq B, \quad (\omega \in \Omega).$$

Hence if F is a nonempty finite subset of Ω, then $\mu(F) = \sum_{\omega \in F} \mu(\{\omega\})$ and so (1.2) holds. ■

Theorem 1.13. *Let (Ω, μ) be a measure space and let an orthogonal mapping $F : \Omega \to \mathcal{H}_C$ be a continuous frame with respect to υ for $\mathcal{H}$ with bounds A, B. Then for each $f \in \mathcal{H}$*

$$f = \sum_{\omega \in \Omega} \pi_{F(\omega)}(f).$$

Proof. Let

$$K = \Big\{ f \in \mathcal{H} : f = \sum_{\omega \in \Omega} \pi_{F(\omega)}(f) \Big\}.$$

It is clear that K is a subspace of $\mathcal{H}$. We prove that K is closed. Let $\{f_n\}$ be a sequence in K that convergence to $f \in \mathcal{H}$. For each given $\epsilon > 0$, we can find a positive integer number N such that $\|f_N - f\| < \epsilon$. Since $f_N = \sum_{\omega \in \Omega} \pi_{F(\omega)}(f_N)$, there exists a finite subset $\Gamma \subseteq \Omega$ such that for each finite subset $\Gamma \subseteq E \subseteq \Omega$

$$\Big\| f_N - \sum_{\omega \in E} \pi_{F(\omega)}(f_N) \Big\| < \epsilon.$$

For each finite subset $\Gamma \subseteq E \subseteq \Omega$, let $S_E(f) = \sum_{\omega \in E} \pi_{F(\omega)}(f)$. Then

$$\begin{aligned} &\|f - S_E(f)\| \\ &\leq \|f - f_N\| + \Big\| f_N - \sum_{\omega \in E} \pi_{F(\omega)}(f_N) \Big\| + \Big\| S_E(f) - \sum_{\omega \in E} \pi_{F(\omega)}(f_N) \Big\|. \end{aligned} \tag{1.3}$$

Since

$$\begin{aligned} A \Big\| S_E(f) - \sum_{\omega \in E} \pi_{F(\omega)}(f_N) \Big\|^2 &= A \|S_E(f_N - f)\|^2 \\ &\leq \int_\Omega \upsilon^2(\omega) \|\pi_{F(\omega)} S_E(f_N - f)\|^2 \, d\mu(\omega) \\ &= \int_E \upsilon^2(\omega) \|\pi_{F(\omega)} S_E(f_N - f)\|^2 \, d\mu(\omega) \\ &= \int_E \upsilon^2(\omega) \|\pi_{F(\omega)}(f_N - f)\|^2 \, d\mu(\omega) \\ &\leq \int_\Omega \upsilon^2(\omega) \|\pi_{F(\omega)}(f_N - f)\|^2 \, d\mu(\omega) \\ &\leq B \|f_N - f\|^2, \end{aligned}$$

by using (1.3), we obtain $f = \sum_{\omega \in \Omega} \pi_{F(\omega)}(f)$. Hence K is a closed subspace of $\mathcal{H}$. To complete the proof, we prove that $K^\perp = 0$. Let $g \in K^\perp$, since for each $\gamma \in \Omega$, $\pi_{F(\gamma)}(g) = \sum_{\omega \in \Omega} \pi_{F(\omega)} \pi_{F(\gamma)}(g)$ then $\pi_{F(\gamma)}(g) \in K$ and

$$\|\pi_{F(\gamma)}(g)\|^2 = \langle g, \pi_{F(\gamma)}(g) \rangle = 0, \quad (\gamma \in \Omega).$$

Since $F : \Omega \to \mathcal{H}_C$ is a continuous frame, $g = 0$. ∎

Proposition 1.14. *Let (Ω, μ) be a measure space and let an orthogonal mapping $F : \Omega \to \mathcal{H}_C$ be a continuous frame with respect to υ for a (separable) Hilbert space $\mathcal{H}$ with bounds A, B. Then there exists a countable and measurable subset $\Gamma \subseteq \Omega$ such that $\mathcal{F} = \{F(\omega)\}_{\omega\in\Gamma}$ is a frame of subspaces with respect to weights $\{\upsilon(\omega)\sqrt{\mu(\omega)}\}_{\omega\in\Gamma}$ for $\mathcal{H}$ with bounds A, B.*

Proof. Let $\{e_n\}_{n\in I}$ be an orthonormal basis for $\mathcal{H}$ and let $f \in \mathcal{H}$. By Theorem 1.13, we have

$$\|f\|^2 = \sum_{\omega\in\Omega} \|\pi_{F(\omega)}(f)\|^2.$$

Therefore $\Omega_f = \{\omega \in \Omega : \pi_{F(\omega)}(f) \neq 0\}$ is a countable and measurable subset of Ω. Let $\Gamma = \bigcup_{n\in I} \Omega_{e_n}$, then Γ is a countable and measurable subset of Ω. If $\omega \in \Omega \setminus \Gamma$, then $\pi_{F(\omega)}(e_n) = 0$ for each $n \in I$. Therefore for each $f \in \mathcal{H}$

$$\pi_{F(\omega)}(f) = 0, \quad (\omega \in \Omega \setminus \Gamma).$$

Hence by Proposition 1.12, for each $f \in \mathcal{H}$, we have

$$\begin{aligned}
&\sum_{\omega\in\Gamma} \upsilon^2(\omega)\mu(\{\omega\})\|\pi_{F(\omega)}(f)\|^2 \\
&= \int_\Gamma \upsilon^2(\omega)\|\pi_{F(\omega)}(f)\|^2\, d\mu(\omega) \\
&= \int_\Omega \upsilon^2(\omega)\|\pi_{F(\omega)}(f)\|^2\, d\mu(\omega).
\end{aligned}$$

■

In the next example, we construct a continuous frame of subspaces for a finite dimensional Hilbert space.

Example 1.15. Let $\Omega = [-1, 1]$ with the Lebesgue measure μ and let $\mathcal{H}_n$ be a n-dimensional Hilbert space with orthonormal basis $\{e_i\}_{i=1}^n$. For each $\omega \in \Omega$, let

$$F(\omega) = \Big\{\lambda \sum_{i=0}^{n-1} \omega^i e_{i+1} : \lambda \in \mathbb{C}\Big\} \quad \text{and} \quad \upsilon(\omega) = \sqrt{\sum_{i=1}^{n} \omega^{2i-2}}.$$

Then $F : \Omega \to \mathcal{H}_C$ is a continuous frame of subspaces with respect to υ for $\mathcal{H}_n$.

Proof. **Case 1.** Let n be odd and let $S : \mathcal{H}_n \to \mathcal{H}_n$ be an operator defined by

$$\langle Sf, g\rangle = \int_{-1}^{1} \upsilon^2(\omega)\langle f, \pi_{F(\omega)}(g)\rangle d\omega, \quad (f, g \in \mathcal{H}_n).$$

Let $g = (\xi_1, \xi_2, ..., \xi_n) = \sum_{i=1}^n \xi_i e_i$. Then

$$\langle Se_i, g\rangle = \int_{-1}^{1} \omega^{i-1}. \sum_{j=1}^{n} \overline{\xi_j}\omega^{j-1}\ d\omega$$

for each $1 \leq i \leq n$.

We can select two cases.

Case I. Let i be odd. Then

$$Se_i = (\frac{2}{i}, 0, \frac{2}{i+2}, 0, ..., 0, \frac{2}{i+n-1}).$$

Case II. Let i be even. Then

$$Se_i = (0, \frac{2}{i+1}, 0, \frac{2}{i+3}, 0, ..., \frac{2}{i+n-2}, 0).$$

Therefore the matrix representation of S with respect to orthonormal basis $\{e_i\}_{i=1}^n$ is

$$[S] = \begin{pmatrix} 2 & 0 & \frac{2}{3} & \cdots & \frac{2}{n} \\ 0 & \frac{2}{3} & 0 & \cdots & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} & \cdots & \frac{2}{n+2} \\ 0 & \frac{2}{5} & 0 & \cdots & 0 \\ \frac{2}{5} & 0 & \frac{2}{7} & \cdots & \frac{2}{n+4} \\ 0 & \frac{2}{7} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \frac{2}{n} & 0 & \cdots & 0 \\ \frac{2}{n} & 0 & \frac{2}{n+2} & \cdots & \frac{2}{2n-1} \end{pmatrix}.$$

Case 2. Let n be even. By similar operations the matrix representation of S with respect to orthonormal basis $\{e_i\}_{i=1}^n$ is

$$[S] = \begin{pmatrix} 2 & 0 & \frac{2}{3} & \cdots & 0 \\ 0 & \frac{2}{3} & 0 & \cdots & \frac{2}{n+1} \\ \frac{2}{3} & 0 & \frac{2}{5} & \cdots & 0 \\ 0 & \frac{2}{5} & 0 & \cdots & \frac{2}{n+3} \\ \frac{2}{5} & 0 & \frac{2}{7} & \cdots & 0 \\ 0 & \frac{2}{7} & 0 & \cdots & \frac{2}{n+5} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{2}{n-1} & 0 & \frac{2}{n+1} & \cdots & 0 \\ 0 & \frac{2}{n+1} & 0 & \cdots & \frac{2}{2n-1} \end{pmatrix}.$$

One can see that $[S]$ is a invertible matrix, so S is a invertible operator. Since

$$\langle Sf, f\rangle = \int_{-1}^{1} v^2(\omega)\|\pi_{F(\omega)}(f)\|^2 d\omega, \quad (f \in \mathcal{H}_n)$$

S is a positive operator and all eigenvalues of S are positive. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of S. Then for each $f \in \mathcal{H}_n$

$$\|f\|^2 \min_{1\leq i\leq n} \lambda_i \leq \langle Sf, f\rangle = \int_{-1}^{1} v^2(\omega)\|\pi_{F(\omega)}(f)\|^2 d\omega \leq \|f\|^2 \max_{1\leq i\leq n} \lambda_i.$$

■

2. Discrete Frames of Subspaces

Let $\Omega = I$ be a countable set and let μ be the counting measure on Ω. Then the continuous frame of subspaces $F : I \to \mathcal{H}_C$ with respect to $\upsilon = \{\upsilon_i\}_{i\in I}$ ($\upsilon_i > 0$ for all $i \in I$) for $\mathcal{H}$, is called *(discrete) frame of subspaces.* In this case, we called $\{\upsilon_i\}_{i\in I}$ weights, i.e., $\upsilon_i > 0$ for all $i \in I$.

Casazza and Kutyniok in [2] introduced the concept of frames of subspaces.

Definition 2.1. *A family $\mathcal{F} = \{W_i\}_{i\in I}$ of closed subspaces of $\mathcal{H}$ is called* frame of subspaces *with respect to weights $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$ if there exist constants $0 < A \le B < \infty$ such that for all $f \in \mathcal{H}$*

$$A\|f\|^2 \le \sum_{i\in I} \upsilon_i^2 \|\pi_{W_i}(f)\|^2 \le B\|f\|^2.$$

The numbers A, B are called frame bounds*. The frame $\mathcal{F} = \{W_i\}_{i\in I}$ is called a* tight frame of subspaces *with respect to weights $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$, if $A = B$ and is called a* Parseval frame of subspaces *with respect to weights $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$, if $A = B = 1$. We call a frame of subspaces with respect to $\{\upsilon_i\}_{i\in I}$,* υ-uniform *if $\upsilon := \upsilon_i = \upsilon_j$ for all $i, j \in I$. A sequence $\mathcal{F} = \{W_i\}_{i\in I}$ of closed subspaces of $\mathcal{H}$ is called an* orthonormal basis of subspaces *if $\mathcal{H} = \bigoplus_{i\in I} W_i$.*

In specially case, suppose that $\{f_i\}_{i\in I}$ is a sequence of non zero vectors in $\mathcal{H}$. Let $W_i = \operatorname{span}\{f_i\}$ and $\upsilon_i = \|f_i\|^2$ for each $i \in I$. We say that $\{f_i\}_{i\in I}$ is a *frame* for $\mathcal{H}$ with bounds $0 < A \le B < \infty$ if $\mathcal{F} = \{W_i\}_{i\in I}$ is a discrete frame of subspaces with respect to $\upsilon = \{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$ with bounds $0 < A \le B < \infty$. In this case, for each $f \in \mathcal{H}$

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

For a family of closed subspaces $\mathcal{F} = \{W_i\}_{i\in I}$ of $\mathcal{H}$, we define the Hilbert space $L^2(I, \mathcal{F})$ by

$$L^2(I, \mathcal{F}) = \Big\{ \{f_i\}_{i\in I} : f_i \in W_i \text{ and } \sum_{i\in I} \|f_i\|^2 < \infty \Big\}$$

with pointwise operations and inner product given by

$$\Big\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I} \Big\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

In [2], Casazza and Kutyniok have showed that if $\mathcal{F} = \{W_i\}_{i\in I}$ is a discrete frame of subspaces with respect to weights $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$, then $\sum_{i\in I} \upsilon_i f_i$ converges (unconditionally) for each $\{f_i\}_{i\in I} \in L^2(I, \mathcal{F})$. For converse, let $\{f_i\}_{i\in I}$

be a sequence in $\mathcal{H}$ and let $\{\upsilon_i\}_{i\in I}$ be a sequence of weights where $\inf_{i\in I} \upsilon_i > 0$ and suppose that $\sum_{i\in I} \upsilon_i f_i$ converges unconditionally. Then by Orlicz' theorem, $\sum_{i\in I} \upsilon_i^2 \|f_i\|^2 < \infty$ and hence $\sum_{i\in I} \|f_i\|^2 < \infty$.

Definition 2.2. *Let $\mathcal{F} = \{W_i\}_{i\in I}$ be a discrete frame of subspaces with respect to weights $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$, the* synthesis operator *for discrete frame of subspaces $\mathcal{F} = \{W_i\}_{i\in I}$ with respect to $\{\upsilon_i\}_{i\in I}$ is the operator*

$$T : L^2(I, \mathcal{F}) \to \mathcal{H}$$

defined by

$$T(\{f_i\}_{i\in I}) = \sum_{i\in I} \upsilon_i f_i.$$

We call the adjoint T^* of synthesis operator the analysis operator.

It is clear that for each $f \in \mathcal{H}$

$$T^*(f) = \{\upsilon_i \pi_{W_i}(f)\}_{i\in I}.$$

Definition 2.3. *Let $\mathcal{F} = \{W_i\}_{i\in I}$ be a discrete frame of subspaces with respect to weights $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$. The* frame operator *$S_{\mathcal{F},\upsilon} : \mathcal{H} \to \mathcal{H}$ for $\mathcal{F} = \{W_i\}_{i\in I}$ and $\{\upsilon_i\}_{i\in I}$ is defined by*

$$S_{\mathcal{F},\upsilon}(f) = TT^*(f) = \sum_{i\in I} \upsilon_i^2 \pi_{W_i}(f).$$

In the following theorem, Casazza and Kutyniok have showed the well-known relations between a discrete frame of subspaces and the associated analysis and synthesis operator (see [2]).

Theorem 2.4. *Let $\mathcal{F} = \{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$, and let $\{\upsilon_i\}_{i\in I}$ be a family of weights. Then the following conditions are equivalent.*

- *(i) $\{\upsilon_i e_{ij}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$; where $\{e_{ij}\}_{j\in J_i}$ is an orthonormal basis for W_i for each $i \in I$.*
- *(ii) $\mathcal{F} = \{W_i\}_{i\in I}$ is a discrete frame of subspaces with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$.*
- *(iii) The synthesis operator T is bounded, linear and onto.*
- *(iv) The analysis operator T^* is a (possibly into) isomorphism.*

The next theorem generalizes a result of Holub [7] to situation of discrete frames of subspaces.

Theorem 2.5. *Let $\mathcal{F} = \{W_i\}_{i\in I}$ be a discrete frame of subspaces with respect to $\{v_i\}_{i\in I}$ for $\mathcal{H}$. Then $\mathcal{F} = \{W_i\}_{i\in I}$ is equivalent to one of the form $\{PZ_i\}_{i\in I}$, where P is some orthogonal projection on $L^2(I,\mathcal{F})$ and $\{Z_i\}_{i\in I}$ is an orthonormal basis for $L^2(I,\mathcal{F})$.*

Proof. Let T be the synthesis operator for $\mathcal{F} = \{W_i\}_{i\in I}$. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i and for each $i \in I, j \in J_i$, let E_{ij} be an element of $L^2(I,\mathcal{F})$ defined by

$$(E_{ij})_k := \begin{cases} 0 & \text{if } k \neq i \\ e_{ij} & \text{if } k = i \end{cases}$$

for all $k \in I$. It is clear that $\{E_{ij}\}_{i\in I, j\in J_i}$ is an orthonormal basis for $L^2(I,\mathcal{F})$. For each $i \in I$, let $Z_i = \overline{\text{span}}\{E_{ij}\}_{j\in J_i}$. Then $\{Z_i\}_{i\in I}$ is an orthonormal basis for $L^2(I,\mathcal{F})$. Let $P : L^2(I,\mathcal{F}) \to (\ker T)^{\perp}$ be the orthogonal projection. Then $TP(E_{ij}) = T(E_{ij}) = v_i e_{ij}$ for all $i \in I, j \in J_i$. Therefore T maps $(\ker T)^{\perp}$ isomorphically onto $\mathcal{H}$. That is $\{PZ_i\}_{i\in I}$ and $\{W_i\}_{i\in I}$ are equivalent, since $TPZ_i = W_i$ for all $i \in I$. ∎

3. Riesz Bases, Besselian Frames and Near-Riesz Bases of Subspaces

A family of vectors $\{f_i\}_{i\in I}$ of $\mathcal{H}$ is called a *Riesz basis*, if $\overline{\text{span}}\{f_i\}_{i\in I} = \mathcal{H}$ and there exist constants $0 < C \leq D < \infty$ such that

$$C\Big(\sum_{i=1}^{n} |c_i|^2\Big)^{\frac{1}{2}} \leq \Big\|\sum_{i=1}^{n} c_i f_i\Big\| \leq D\Big(\sum_{i=1}^{n} |c_i|^2\Big)^{\frac{1}{2}}$$

for every finite scalar sequence $c_1, c_2, ..., c_n$.

The numbers C, D in the above definition are called *Riesz basis constants.* Holub in [7], by analogy to the case of a Schauder basis [9] introduced Besselian frames. As usual, we denote by l^2 the Hilbert space of all square-summable sequences of scalars.

Definition 3.1. *We say that a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$ is:*

(*i*) Besselian, *if whenever $\sum_{i\in I} c_i f_i$ converges, then $\{c_i\}_{i\in I} \in l^2$.*

(*ii*) unconditional, *if whenever $\sum_{i\in I} c_i f_i$ converges, it converges unconditionally.*

(*iii*) *a* near-Riesz basis, *if there is a finite set σ for which $\{f_i\}_{i\in I\setminus\sigma}$ is a Riesz basis for $\mathcal{H}$.*

The definition of Riesz basis of a sequence gives rise to a definition of Riesz basis for a sequence of subspaces.

Definition 3.2. *Let $\{\upsilon_i\}_{i\in I}$ be a family of weights. A family of closed subspaces $\mathcal{F} = \{W_i\}_{i\in I}$ of $\mathcal{H}$ is called a* Riesz basis of subspaces *with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$, if $\overline{span}\{W_i\}_{i\in I} = \mathcal{H}$ and there exist constants $0 < C \le D < \infty$ such that for each finite subset $J \subseteq I$*

$$C\Big(\sum_{j\in J}\|f_j\|^2\Big)^{\frac{1}{2}} \le \Big\|\sum_{j\in J}\upsilon_j f_j\Big\| \le D\Big(\sum_{j\in J}\|f_j\|^2\Big)^{\frac{1}{2}}, \quad (f_j \in W_j).$$

We say that a discrete frame of subspaces $\mathcal{F} = \{W_i\}_{i\in I}$ with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$ is:

(i) Besselian, *if whenever $\sum_{i\in I}\upsilon_i f_i$ converges ($f_i \in W_i$ for each $i \in I$) then $\{f_i\}_{i\in I} \in L^2(I,\mathcal{F})$.*

(ii) unconditional, *if whenever $\sum_{i\in I}\upsilon_i f_i$ converges ($f_i \in W_i$ for each $i \in I$), it converges unconditionally.*

(iii) *a* near-Riesz basis, *if there is a finite set σ for which $\{W_i\}_{i\in I\setminus\sigma}$ is a Riesz basis for $\mathcal{H}$.*

For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. If $\{W_i\}_{i\in I}$ is a Besselian frame of subspaces, then $\{\upsilon_i e_{ij}\}_{i\in I, j\in J_i}$ is a Besselian frame.

The next theorem gives both necessary and sufficient condition for us to be able to string together Riesz basis for each of the subspaces W_i (with uniformly bounded Riesz basis constants) to get a Riesz basis for $\mathcal{H}$.

Theorem 3.3. *Suppose that $\{\upsilon_i\}_{i\in I}$ is a family of weights and let $\mathcal{F} = \{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$. For each $i \in I$, let $\{f_{ij}\}_{j\in J_i} \subseteq \mathcal{H}$ be a Riesz basis for W_i with Riesz basis bounds C_i, D_i. Suppose that $0 < C = \inf_{i\in I} C_i \le D = \sup_{i\in I} D_i < \infty$. Then the following conditions are equivalent:*

(i) *$\{\upsilon_i f_{ij}\}_{i\in I, j\in J_i}$ is a Riesz basis for $\mathcal{H}$.*

(ii) *$\mathcal{F} = \{W_i\}_{i\in I}$ is a Riesz basis of subspaces with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$.*

Proof. To prove $(i) \Rightarrow (ii)$. Fix $F \subseteq I$ with $|F| < \infty$ and let $g_i \in W_i$ for each $i \in F$. Hence for each $i \in F$, we have

$$g_i = \sum_{j\in J_i}\lambda_{ij} f_{ij} \quad \text{and} \quad \sum_{j\in J_i}|\lambda_{ij}|^2 < \infty.$$

Therefore for each $i \in F$

$$\begin{aligned} C^2\sum_{j\in J_i}|\lambda_{ij}|^2 &\le C_i^2\sum_{j\in J_i}|\lambda_{ij}|^2 \le \|g_i\|^2 \\ &= \Big\|\sum_{j\in J_i}\lambda_{ij} f_{ij}\Big\|^2 \le D_i^2\sum_{j\in J_i}|\lambda_{ij}|^2 \le D^2\sum_{j\in J_i}|\lambda_{ij}|^2 \end{aligned}$$

and

$$\Big\|\sum_{i\in F}\upsilon_i g_i\Big\| = \Big\|\sum_{i\in F}\sum_{j\in J_i}\lambda_{ij}\upsilon_i f_{ij}\Big\|.$$

Let A, B be the Riesz basis bounds for $\{\upsilon_i f_{ij}\}_{i\in I, j\in J_i}$, then

$$\begin{aligned}\frac{A^2}{D^2}\sum_{i\in F}\|g_i\|^2 \le A^2\sum_{i\in F}\sum_{j\in J_i}|\lambda_{ij}|^2 \le \Big\|\sum_{i\in F}\upsilon_i g_i\Big\|^2\\ \le B^2\sum_{i\in F}\sum_{j\in J_i}|\lambda_{ij}|^2 \le \frac{B^2}{C^2}\sum_{i\in F}\|g_i\|^2.\end{aligned}$$

Since $\{\upsilon_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame (see [5], Theorem 5.4.1), then by Theorem 2.4, $\overline{\text{span}}\{W_i\}_{i\in I} = \mathcal{H}$. Hence $\mathcal{F} = \{W_i\}_{i\in I}$ is a Riesz basis of subspaces with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$ with Riesz basis bounds $\frac{A}{D}, \frac{B}{C}$.

To prove $(ii) \Rightarrow (i)$. Let $\{c_{ij}\}$ be a finite scalar sequence and let A_0, B_0 be the Riesz basis bounds for $\mathcal{F} = \{W_i\}_{i\in I}$. Then

$$A_0^2\sum_i\Big\|\sum_j c_{ij}f_{ij}\Big\|^2 \le \Big\|\sum_i\upsilon_i\sum_j c_{ij}f_{ij}\Big\|^2 \le B_0^2\sum_i\Big\|\sum_j c_{ij}f_{ij}\Big\|^2$$

and

$$C^2\sum_j|c_{ij}|^2 \le C_i^2\sum_j|c_{ij}|^2 \le \Big\|\sum_j c_{ij}f_{ij}\Big\|^2 \le D_i^2\sum_j|c_{ij}|^2 \le D^2\sum_j|c_{ij}|^2.$$

Therefore

$$A_0^2C^2\sum_{ij}|c_{ij}|^2 \le \Big\|\sum_{ij}c_{ij}\upsilon_i f_{ij}\Big\|^2 \le B_0^2D^2\sum_{ij}|c_{ij}|^2.$$

Since $\mathcal{H} = \overline{\text{span}}\{W_i\}_{i\in I} = \overline{\text{span}}\{\upsilon_i f_{ij}\}_{i\in I, j\in J_i}$, then $\{\upsilon_i f_{ij}\}_{i\in I, j\in J_i}$ is a Riesz basis for $\mathcal{H}$ with bounds A_0C, B_0D. ∎

Corollary 3.4. *Suppose that $\{\upsilon_i\}_{i\in I}$ is a family of weights and let $\mathcal{F} = \{W_i\}_{i\in I}$ be a family of closed subspaces in $\mathcal{H}$. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. Then the following conditions are equivalent:*

- *(i) $\{\upsilon_i e_{ij}\}_{i\in I, j\in J_i}$ is a Riesz basis for $\mathcal{H}$ with Riesz basis bounds A, B.*
- *(ii) $\mathcal{F} = \{W_i\}_{i\in I}$ is a Riesz basis of subspaces with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$ with Riesz basis bounds A, B.*

The following theorem gives a relationship between Besselian frame and the kernel of synthesis operator.

Theorem 3.5. *[7] Let $\{x_n\}_{n=1}^{\infty}$ be a Besselian frame in $\mathcal{H}$. If $T : l^2 \to \mathcal{H}$ is the synthesis operator associated with $\{x_n\}_{n=1}^{\infty}$. Then T has finite-dimensional kernel.*

Also in [7], J. Holub has proved that the deletion of a finite set of vectors from a frame $\{x_n\}_{n=1}^{\infty}$ leaves a Riesz basis if and only if the frame is Besselian.

Theorem 3.6.*[7] Let $\{x_n\}_{n=1}^{\infty}$ be a frame in $\mathcal{H}$ and $T : l^2 \to \mathcal{H}$ the associated synthesis operator. Then the following are equivalent:*

- *(i)* $\ker T$ *is finite dimensional.*
- *(ii)* $\{x_n\}_{n=1}^{\infty}$ *is a near-Riesz basis for* $\mathcal{H}$.
- *(iii)* $\{x_n\}_{n=1}^{\infty}$ *is Besselian.*
- *(iv)* $\sum_{n=1}^{\infty} a_n x_n$ *converges in* $\mathcal{H} \Leftrightarrow \{a_n\}_{n=1}^{\infty} \in l^2$.

The next theorems, give us similar results for discrete frames of subspaces.

Theorem 3.7. *Let $\mathcal{F} = \{W_i\}_{i\in I}$ be a Besselian discrete frame of subspaces with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$ and let T be the associated synthesis operator for $\mathcal{F} = \{W_i\}_{i\in I}$. Then* $\ker T$ *is finite-dimensional.*

Proof. For each $i \in I$, let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i. Since $\mathcal{F} = \{W_i\}_{i\in I}$ is a Besselian discrete frame, $\{\upsilon_i e_{ij}\}_{i\in I, j\in J_i}$ is a Besselian frame for $\mathcal{H}$. Let Q be the associated synthesis operator for $\{\upsilon_i e_{ij}\}_{i\in I, j\in J_i}$. Then

$$Q\big(\{c_{ij}\}_{i\in I, j\in J_i}\big) = \sum_{i\in I}\sum_{j\in J_i} c_{ij}\upsilon_i e_{ij} = T\left(\sum_{i\in I}\sum_{j\in J_i} c_{ij}E_{ij}\right),$$

where $E_{ij} \in L^2(I, \mathcal{F})$ defined by

$$(E_{ij})_k = \begin{cases} e_{ij} & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

for each $k \in I$. Let $\Psi : \ker Q \to \ker T$ be a mapping defined by

$$\Psi\big(\{c_{ij}\}_{i\in I, j\in J_i}\big) = \sum_{i\in I}\sum_{j\in J_i} c_{ij}E_{ij}.$$

It is clear that Ψ is well defined, linear and injective. Let $f = \{f_i\}_{i\in I} \in \ker T$. Then $f_i = \sum_{j\in J_i} \lambda_{ij} e_{ij}$ for each $i \in I$. Since $\sum_{i\in I, j\in J_i} |\lambda_{ij}|^2 = \sum_{i\in I} \|f_i\|^2 < \infty$, $\{\lambda_{ij}\}_{i\in I, j\in J_i} \in l^2$ and

$$\Psi\big(\{\lambda_{ij}\}_{i\in I, j\in J_i}\big) = \sum_{i\in I, j\in J_i} \lambda_{ij}E_{ij} = f.$$

Therefore Ψ is surjective and by Theorem 3.5, $\dim\ker T = \dim\ker Q < \infty$. ■

4. Riesz Frames of Subspaces

Riesz frames of subspaces have been introduced in [2]. In this section we show that every Riesz frame of subspaces contains a minimal frame of subspaces. We start by giving the definitions of Riesz frame of subspaces and minimal subspaces.

Definition 4.1. *We call a discrete frame of subspaces* $\{W_i\}_{i\in I}$ *a* Riesz frame of subspaces with respect to $\{\upsilon_i\}_{i\in I}$, *if there exist constants* $0 < A \leq B < \infty$ *so that every subfamily* $\{W_i\}_{i\in J}$ *with* $J \subseteq I$ *is a discrete frame of subspaces with respect to* $\{\upsilon_i\}_{i\in J}$ *for its closed linear span with frame bounds* A, B.

The numbers A, B *are called the* Riesz frame bounds.

Definition 4.2. *A family of subspaces* $\{W_i\}_{i\in I}$ *of* $\mathcal{H}$ *is called* minimal, *if for each* $i \in I$

$$W_i \cap \overline{span}\{W_j\}_{j\in I, j\neq i} = \{0\}.$$

The following theorem gives us a reconstruction formula when we have a special case of frames of subspaces.

Proposition 4.3. *Let* $\{W_i\}_{i\in I}$ *be a frame of pairwise orthogonal nonzero closed subspaces of* $\mathcal{H}$ *with respect to* $\{\upsilon_i\}_{i\in I}$ *for* $\mathcal{H}$ *with bounds* A, B. *For each* $E \subseteq I$, *let* $W = \overline{span}\{W_i\}_{i\in E}$. *Then*

$$f = \sum_{i\in E} \pi_{W_i}(f), \quad (f \in W).$$

Proof. It is clear that for each $f \in \text{span}\{W_i\}_{i\in E}$, we have

$$f = \sum_{i\in E} \pi_{W_i}(f). \tag{4.1}$$

Let $f \in W$, then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subseteq \text{span}\{W_i\}_{i\in E}$ such that $f_n \to f$ as $n \to \infty$. Since (4.1) holds for each f_n,

$$\begin{aligned} A\Big\|\sum_{i\in E}\pi_{W_i}(f_n) - \sum_{i\in E}\pi_{W_i}(f)\Big\|^2 &= A\sum_{i\in E}\|\pi_{W_i}(f_n - f)\|^2 \\ &\leq \sum_{i\in I}\upsilon_i^2\|\pi_{W_i}(f_n - f)\|^2 \\ &\leq B\|f_n - f\|^2. \end{aligned}$$

Therefore the result is proved. ∎

Corollary 4.4. *Let $\mathcal{F} = \{W_i\}_{i\in I}$ be a υ-uniform frame of pairwise orthogonal nonzero closed subspaces of $\mathcal{H}$ for $\mathcal{H}$ with bounds A, B. Then $S_{\mathcal{F},\upsilon} = \upsilon^2 I$.*

The following proposition gives us to construct a Riesz frame of subspaces:

Proposition 4.5. *Let $\{W_i\}_{i\in I}$ be a discrete frame of pairwise orthogonal non zero closed subspaces with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$. Then $\{W_i\}_{i\in I}$ is a Riesz frame of subspaces with respect to $\{\upsilon_i\}_{i\in I}$ with bounds A, B.*

Proof. Let $J \subseteq I$ and let $W = \overline{\text{span}}\{W_i\}_{i\in J}$. By Proposition 4.3, we have

$$A\|f\|^2 \leq \sum_{i\in J} \upsilon_i^2 \|\pi_{W_i}(f)\|^2 \leq B\|f\|^2$$

for all $f \in W$. ∎

In [4] Christensen has showed that every Riesz frame contains a Riesz basis. We show that every Riesz frame of subspaces contains a minimal frame of subspaces:

Theorem 4.6. *Let $\{W_i\}_{i\in I}$ be a Riesz frame of subspaces with respect to $\{\upsilon_i\}_{i\in I}$ for $\mathcal{H}$. Then there exists $K \subseteq I$ such that $\{W_i^{'}\}_{i\in K}$ is a minimal discrete frame of subspaces with respect to $\{\upsilon_i\}_{i\in K}$ for $\mathcal{H}$, where $W_i^{'} \subseteq W_i$ for all $i \in K$.*

Proof. Let A and B be the Riesz frame of subspaces bounds for $\{W_i\}_{i\in I}$ and let $\{e_{ij}\}_{j\in J_i}$ be an orthonormal basis for W_i for each $i \in I$. By Proposition 5.8 of [2], $\{\upsilon_i e_{ij}\}_{i\in I, j\in J_i}$ is a Riesz frame for $\mathcal{H}$. Therefore $\{e_{ij}\}_{i\in I, j\in J_i}$ is a Riesz frame for $\mathcal{H}$ since $A \leq \upsilon_i^2 \leq B$ for all $i \in I$. Hence by Theorem 6.3.3 in [5], there exists a subfamily $\{f_{ij}\}_{i,j}$ of $\{e_{ij}\}_{i\in I, j\in J_i}$ such that $\{f_{ij}\}_{ij}$ is a Riesz basis for $\mathcal{H}$, where $i \in K \subseteq I$. So $\{f_{ij}\}_{i,j}$ is minimal. Let $W_i^{'} = \overline{\text{span}}\{f_{ij}\}_j$ for each $i \in K$. Then by Corollary 3.4 and Lemma 4.2 in [2], $\{W_i^{'}\}_{i\in K}$ is a Riesz basis of subspaces for $\mathcal{H}$ and minimal. ∎

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