A generalized tableau associated with colored convolution trees

A.G. Shannon
University of Technology, Sydney, 2007, Australia

J.C. Turner
University of Waikato, Hamilton, New Zealand

K.T. Atanassov
Institute of Microsystems, Sofia-1184, Bulgaria

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Abstract

This paper considers the tableaux arising from the colors at different levels of arbitrary order convolution trees. The results are related to the elements which constitute the Pascal–Lucas–Turner triangles and to various generalizations of the Fibonacci numbers, some formed by altering the order of the recurrence relations and some by coupling some second order recurrence relations.

1. Introduction

In a previous paper, certain stochastic processes were defined on a sequence of binary trees such that the tree $T_n$ has $F_n$ leaf-nodes where $F_n$ is the $n$th element of the Fibonacci sequence [16]. In a later paper it was shown how to construct trees so that the nodes were weighted with integers from a general sequence $\{C_n\}$ using a sequential method referred to as the ‘dripfeed principle’ [12]. Subsequently it was shown how generalized Fibonacci numbers can be used to color convolution trees so that the shades of the trees establish a generalization of Zeckendorf’s theorem and its dual [14]. There was also a construction which provided an illustration of the original Zeckendorf theorem, which established the completeness of the Fibonacci sequence and generated the Zeckendorf integer representations.
In the sequence of Fibonacci convolution trees \( \{T_n\} \) given in [12], the sum of the weights assigned to the nodes of \( T_n \) is equal to the \( n \)th term of the convolution of \( \{F_n\} \) and \( \{C_n\} \). That is, if \( \Omega \) means the sum of weights, we have

\[
\Omega(T_n) = (F * C)_n = \sum_{i=1}^{n} F_i C_{n-i+1}.
\]

For instance,

\[
(F * F)_5 = F_1 F_5 + F_2 F_4 + F_3 F_3 + F_4 F_2 + F_5 F_1
= 5 + 3 + 4 + 3 + 5 = 20,
\]

to which we shall refer later.

With the same tree construction, and a modified coloring rule, a graphical 'proof' was given of Zeckendorf's theorem, namely that every positive integer can be represented as the sum of distinct Fibonacci numbers, using no two consecutive Fibonacci numbers, and that such a representation is unique [5].

Given a sequence of colors \( C = \{C_1, C_2, C_3, \ldots\} \), we construct \( k \)th order colored, rooted trees, \( T_n \), as follows: The first \( k \) trees:

\[
T_1 = C_1 \bullet; \quad T_n = T_{n-1} \rightarrow \rightarrow \rightarrow C_n, \quad n = 2, 3, \ldots, k,
\]

with the root node being \( C_i \) for each of these; subsequent trees:

\[
T_{n+k} = C_{n+k} \bigvee_{i=0}^{k-1} T_{n+i},
\]

using the 'drip-feed' construction, in which the \( k \)th order fork operation \( \bigvee \) is to mount trees \( T_n, T_{n+1}, \ldots, T_{n+k-1} \) on separate branches of a new tree with root node colored by \( C_{n+k} \). Thus, for example, when \( k = 2 \), and \( C = \{F_n\} \), the sequence of Fibonacci numbers, the first four second-order colored trees are as pictured in Fig. 1.

Now consider the first four trees associated with \( F(a, b) \), pictured in Fig. 2. The coloring sequence is the general Fibonacci one, namely, \( F(a, b) = \{a, b, a + b, a + 2b, \ldots\} \).

Let \( (N_a, N_b) \) represent the number of \( a \)’s and the number of \( b \)’s at a given level of a tree. We may tabulate these pairs as in Table 1.
If we represent the element in the $n$th row and $m$th column of this array by the vector $x_{nm}$, then $x_{nm}$ satisfies the partial recurrence relation

$$x_{nm} = x_{n-1,m-1} + x_{n-2,m-1}, \quad 1 < m < n, \quad n > 2,$$

where the addition of number pairs is elementwise, and the boundary conditions are

$$x_{11} = x_{21} = (1, 0); \quad x_{n1} = (F_{n-2}, F_{n-1}), \quad n > 2;$$

$$x_{22} = (0, 1); \quad x_{nm} = (0, 0), \quad m > n.$$

It is the main purpose of this paper to generalize this result.

2. Generalized tableau

The tableau can be generalized for arbitrary $k$ as follows: Consider $x_{nm}$ as a $k$-component vector, with $x_{nm}$ equal to the null vector when $m > n$, and

$$x_{nm} = \sum_{i=1}^{k} x_{n-i,m-1}, \quad 1 < m < n, \quad n > k,$$

with

$$x_{n1} = (U_{1n}, U_{2n}, \ldots, U_{kn}), \quad n = 1, 2, \ldots, k,$$

where $\{U_{sn}\}, \quad s = 1, 2, \ldots, k$, are the $k$ 'basic' sequences of order $k$ defined by the recurrence relation

$$U_{sn} = \sum_{j=1}^{k} U_{s,n-j}, \quad n > k$$

with initial terms when $n = 1, 2, \ldots, k$, $U_{sn} = \delta_{sn}$, (Shannon and Bernstein [10]), where $\delta_{ij}$ is the Kronecker delta.

When $k = 2$, we have, as before, that if we represent the element in the $n$th row and $m$th column of this array by $x_{nm}$, then $x_{nm}$ satisfies the partial recurrence...
relation

\[ x_{nm} = x_{n-1,m-1} + x_{n-2,m-1}, \quad 1 < m < n, \ n > 2, \]

\[ x_{nm} = (\delta_{1m}, \delta_{2m}), \quad n = 1, 2; \ 1 \leq m \leq n \]

with boundary conditions \( x_{n1} = (F_{n-2}, F_{n-1}) \) and \( x_{nm} = (0, 1) \).

As illustrations of the \( \{U_{nm}\} \) we have Table 2 when \( k = 3 \) and Table 3 when \( k = 4 \).

Various properties of \( \{U_{nm}\} \) have been developed by Shannon [9]. To see more easily what follows, it is useful to continue the tree table of \( (N_n, A_n) \) for \( k = 2 \) (see Table 4).

It can be observed in Tables 4 and 5 that, for \( n > k \) and \( m > 1 \),

\[ x_{nm} = \sum_{i=1}^{k} x_{n-i,m-1}. \quad (2.3) \]

As explained elsewhere [17], the rule of formation comes directly from the construction of the trees. When \( k = 3 \), we have the array as shown in Table 5.

The first main result is that for \( m = 1, 2, \ldots, \lfloor (n-1)/k \rfloor \) (\( \lfloor \rfloor \) is the greatest integer function),

\[ x_{nm} = x_{n1}. \]

**Proof.** The proof follows from induction on \( m \) by utilizing the results

\[ x_{n2} = \sum_{i=1}^{k} x_{n-i,1} = \sum_{i=1}^{k} (U_{1,n-i}, U_{2,n-i}, \ldots, U_{k,n-i}) \]

\[ = \left( \sum_{i=1}^{k} U_{1,n-i}, \sum_{i=1}^{k} U_{2,n-i}, \ldots, \sum_{i=1}^{k} U_{k,n-i} \right) = (U_{1,n}, U_{2,n}, \ldots, U_{k,n}) \]

\[ = x_{n1}, \text{ and so on.} \]
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Table 4

<table>
<thead>
<tr>
<th>m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>(5, 8)</td>
<td>(5, 8)</td>
<td>(5, 8)</td>
<td>(8, 5)</td>
<td>(8, 4)</td>
</tr>
<tr>
<td>$T_2$</td>
<td>(8, 13)</td>
<td>(8, 13)</td>
<td>(8, 13)</td>
<td>(9, 12)</td>
<td>(14, 7)</td>
</tr>
<tr>
<td>$T_3$</td>
<td>(13, 21)</td>
<td>(13, 21)</td>
<td>(13, 21)</td>
<td>(13, 21)</td>
<td>(17, 17)</td>
</tr>
<tr>
<td>$T_4$</td>
<td>(21, 34)</td>
<td>(21, 34)</td>
<td>(21, 34)</td>
<td>(21, 34)</td>
<td>(22, 33)</td>
</tr>
<tr>
<td>$T_5$</td>
<td>(34, 55)</td>
<td>(34, 55)</td>
<td>(34, 55)</td>
<td>(34, 55)</td>
<td>(34, 55)</td>
</tr>
</tbody>
</table>

$m$ 6 7 8 9 10 11

| $T_6$ | (2, 4) | (0, 1) |
| $T_7$ | (10, 7) | (2, 5) | (0, 1) |
| $T_8$ | (22, 11) | (12, 11) | (9, 6) | (0, 1) |
| $T_9$ | (31, 24) | (32, 18) | (14, 16) | (2, 7) | (0, 1) |
| $T_{10}$ | (29, 50) | (53, 35) | (44, 29) | (16, 22) | (2, 8) | (0, 1) |

For instance, when $k = 3$,

$x_{72} = (4, 6, 7) = (U_{17}, U_{27}, U_{37})$,

$x_{10,3} = (24, 37, 44) = (U_{110}, U_{210}, U_{310})$;

when $k = 2$,

$x_{52} = (2, 3) = (U_{15}, U_{25})$,

$x_{73} = (5, 8) = (U_{17}, U_{27})$,

$x_{94} = (13, 21) = (U_{19}, U_{29})$,

$x_{11,3} = (34, 55) = (U_{1,11}, U_{2,11})$,

in which $U_{1n} = U_{2,n-1} = F_{n-2}$ in the conventional Fibonacci notation.

Table 5

<table>
<thead>
<tr>
<th>m</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>(1, 0, 0)</td>
<td>(1, 0, 0)</td>
<td>(1, 0, 0)</td>
<td>(0, 0, 1)</td>
<td>(0, 0, 1)</td>
</tr>
<tr>
<td>$T_2$</td>
<td>(1, 0, 0)</td>
<td>(0, 1, 0)</td>
<td>(0, 1, 0)</td>
<td>(0, 2, 0)</td>
<td>(0, 2, 0)</td>
</tr>
<tr>
<td>$T_3$</td>
<td>(1, 0, 0)</td>
<td>(3, 0, 0)</td>
<td>(3, 0, 0)</td>
<td>(3, 2, 0)</td>
<td>(3, 2, 0)</td>
</tr>
<tr>
<td>$T_4$</td>
<td>(1, 1, 1)</td>
<td>(3, 1, 1)</td>
<td>(6, 2, 1)</td>
<td>(3, 4, 1)</td>
<td>(0, 2, 1)</td>
</tr>
<tr>
<td>$T_5$</td>
<td>(2, 3, 4)</td>
<td>(3, 2, 3)</td>
<td>(6, 2, 1)</td>
<td>(3, 4, 1)</td>
<td>(0, 2, 1)</td>
</tr>
<tr>
<td>$T_6$</td>
<td>(4, 6, 7)</td>
<td>(4, 6, 7)</td>
<td>(9, 4, 4)</td>
<td>(9, 6, 1)</td>
<td>(3, 6, 3)</td>
</tr>
<tr>
<td>$T_7$</td>
<td>(7, 11, 13)</td>
<td>(7, 11, 13)</td>
<td>(10, 10, 11)</td>
<td>(18, 8, 5)</td>
<td>(12, 12, 3)</td>
</tr>
<tr>
<td>$T_8$</td>
<td>(13, 20, 24)</td>
<td>(13, 20, 24)</td>
<td>(14, 20, 23)</td>
<td>(25, 16, 16)</td>
<td>(20, 18, 7)</td>
</tr>
<tr>
<td>$T_9$</td>
<td>(24, 37, 44)</td>
<td>(24, 37, 44)</td>
<td>(24, 37, 44)</td>
<td>(33, 34, 38)</td>
<td>(52, 30, 22)</td>
</tr>
</tbody>
</table>

$m$ 6 7 8 9 10

| $T_6$ | (0, 0, 1) |
| $T_7$ | (0, 0, 1) |
| $T_8$ | (0, 2, 3) | (0, 0, 1) |
| $T_9$ | (3, 8, 6) | (0, 2, 4) | (0, 0, 1) |
| $T_{10}$ | (15, 20, 8) | (3, 10, 10) | (0, 2, 5) | (0, 0, 1) |
| $T_{11}$ | (35, 36, 13) | (18, 30, 17) | (3, 12, 15) | (0, 2, 6) | (0, 0, 1) |
The second result is that for \( m > \lfloor(n - 1)/k \rfloor \), \( x_{nm} \) is formed from the boundary conditions

\[
x_{k+1,m} = (0, 0, \ldots, k - m + 2, \ldots, 0)
\]

in which the nonzero position is the \((m - 1)\)th; thereafter, the elements are generated by the algorithm defined by the vector difference operator \( \Delta \), such that if

\[
\Delta x_{nm} = x_{n+1,m+1} - x_{n,m},
\]

then the \( s \)th order difference is given by

\[
\Delta^s x_{n+s,n} = (0, 0, \ldots, 0, 1), \quad \text{for } n \geq k.
\]

The proof follows from the initial conditions and the ordinary recurrence relation (2.2) for \( \{U_m\} \) to get \( x_{k+1,m} \), and then from the partial recurrence relation (2.3) for \( x_{k+n,k+n-1} \).

As examples, we have when \( k = 2 \),

\[
\begin{align*}
x_{42} &= (2, 1) & \Delta x_{42} &= x_{53} - x_{42} = (2, 0) & \Delta^2 x_{42} &= (0, 1) \\
x_{53} &= (4, 1) & \Delta x_{53} &= x_{64} - x_{53} = (2, 1) & \Delta^2 x_{53} &= (0, 1) \\
x_{64} &= (6, 2) & \Delta x_{64} &= x_{75} - x_{64} = (2, 2) & \Delta^2 x_{64} &= (0, 1) \\
x_{75} &= (8, 4) & \Delta x_{75} &= x_{86} - x_{75} = (2, 3) & \Delta^2 x_{75} &= (0, 1) \\
x_{86} &= (10, 7) & \Delta x_{86} &= x_{97} - x_{86} = (2, 4) & \Delta^2 x_{86} &= (0, 1) \\
x_{97} &= (12, 11) & \Delta x_{97} &= x_{10,8} - x_{97} = (2, 5) & \Delta^2 x_{97} &= (0, 1) \\
x_{10,8} &= (14, 16) & \Delta x_{10,8} &= x_{11,9} - x_{10,8} = (2, 6) \\
x_{11,9} &= (16, 22)
\end{align*}
\]

When \( k = 4 \)

\[
\begin{align*}
x_{52} &= (4, 0, 0, 0) & \Delta x_{52} &= (0, 3, 0, 0) & \Delta^2 x_{52} &= (0, 0, 2, 0) \\
x_{63} &= (4, 3, 0, 0) & \Delta x_{63} &= (0, 3, 2, 0) & \Delta^2 x_{63} &= (0, 0, 2, 1) \\
x_{74} &= (4, 6, 2, 0) & \Delta x_{74} &= (0, 3, 4, 1)
\end{align*}
\]
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\[ x_{85} = (4, 9, 6, 1) \quad \Delta x_{85} = (0, 3, 6, 3) \quad \Delta^2 x_{74} = (0, 0, 2, 2) \]
\[ x_{96} = (4, 12, 12, 4) \quad \Delta x_{96} = (0, 3, 8, 6) \]
\[ x_{10,7} = (4, 15, 20, 10) \]

and
\[ x_{53} = (0, 3, 0, 0) \quad \Delta x_{53} = (0, 0, 2, 0) \]
\[ x_{64} = (0, 3, 2, 0) \quad \Delta x_{64} = (0, 0, 2, 1) \quad \Delta^2 x_{54} = (0, 0, 0, 1) \]
\[ x_{75} = (0, 3, 4, 1) \quad \Delta x_{75} = (0, 0, 2, 2) \quad \Delta^2 x_{65} = (0, 0, 0, 1) \]
\[ x_{86} = (0, 3, 6, 3) \quad \Delta x_{86} = (0, 0, 2, 3) \quad \Delta^2 x_{76} = (0, 0, 0, 1) \]
\[ x_{97} = (0, 3, 8, 6) \]

and
\[ x_{54} = (0, 0, 2, 0) \quad \Delta x_{54} = (0, 0, 0, 1) \]
\[ x_{65} = (0, 0, 2, 1) \quad \Delta x_{65} = (0, 0, 0, 1) \]
\[ x_{76} = (0, 0, 2, 2) \quad \Delta x_{76} = (0, 0, 0, 1) \]
\[ x_{87} = (0, 0, 2, 4) \]

3. Connections with Pascal-T triangles

Turner [14–15] has defined the level counting function

\[ L = \binom{n}{m} \]

as the number of nodes in \( T_n \) which at level \( m \) are colored \( C_i \), where \( T_n \) is the tree colored by integers of the sequence \( C = \{C_1, C_2, C_3, \ldots \} \).

One of the results proved is that

\[ \binom{n}{m} = \sum_{j=1}^{k} \binom{n-j}{m-1} \]  \hspace{1cm} (3.1)

It is also shown in effect that

\[ U_{k,k+n} = \sum_{m=0}^{n} \binom{n}{m} \]
It is also worth noting that (3.1) has the same form as (2.2). Now $U_{k,k+n}$ is, in the terminology of Macmahon [7], the homogeneous product sum of weight $n$ of the zeros $\alpha_j$, $j = 1, 2, \ldots, k$, assumed distinct, of the auxiliary polynomial, $f(x)$, associated with the linear recurrence relation for $\{U_{k,k+n}\}$. Shannon and Horadam [11] have proved that formally

$$
\sum_{n=1}^{\infty} U_{k,k+n} x^n = \left( x f \left( \frac{1}{x} \right) \right)^{-1}.
$$

Thus if we expand the right-hand side of

$$
\sum_{n=1}^{\infty} U_{k,k+n} x^n = 1/(1 - x - x^2 - \cdots - x^k)
$$

by the multinomial theorem and equate corresponding coefficients of powers of $x$ we get

$$
U_{k,k+n} = \frac{(\Sigma \lambda_i)!}{\Sigma \lambda_i! \lambda_1! \lambda_2! \cdots \lambda_k!} \sum_{s+2m+n} \binom{n-m}{m}
$$

which agrees with the analogous result in Macmahon. This is worth noting because Turner [15] has shown that the $\binom{n}{m}$ are multinomial coefficients generated from $x(x + x^2 + x^3 + \cdots + x^k)^m$. For example,

$$
U_{2,2+n} = \sum_{\lambda_1 = s} (\Sigma \lambda_i)! \lambda_1! \lambda_2! \sum_{s+2m+n} \binom{n-m}{m}
$$

where $\lambda_1 = s$ and $\lambda_2 = m$, as in Barakat [4]; and

$$
U_{3,3+n} = \sum_{\lambda_1 = s} (\Sigma \lambda_i)! \lambda_1! \lambda_2! \lambda_3! \sum_{s+2m+3t=n} \frac{(n-m-2t)}{s! m! t!}
$$

$$
= \sum_{s+2m+3t=n} \binom{n-m-2t}{m+t} \binom{m+t}{t}
$$

where $\lambda_1 = s$, $\lambda_2 = m$ and $\lambda_3 = t$, as in Shannon [8].

4. Other generalizations

We can also develop trees for other generalizations. For instance, Atanassov [1, 3] defines 2-F-sequences

$$
\alpha_{n+2} - \beta_{n+1} + \beta_n, \quad \beta_{n+2} = \alpha_{n+1} + \alpha_n, \quad n \geq 0
$$

(4.1)

with $\alpha_0 = a$, $\alpha_1 = b$, $\beta_0 = c$, $\beta_1 = d$ fixed real numbers. The trees for this scheme are shown in Fig. 3.
Similarly, there are 7 basic 3-F-sequences (Atanassov [2]), two of which are shown with their corresponding sets of trees in Fig. 4.

\[
\begin{align*}
\alpha_{n+2} &= \gamma_{n+1} + \gamma_n, \\
\beta_{n+2} &= \alpha_{n+1} + \alpha_n, \\
\gamma_{n+2} &= \beta_{n+1} + \beta_n. \\
\alpha_{n+2} &= \beta_{n+1} + \gamma_n, \\
\beta_{n+2} &= \alpha_{n+1} + \alpha_n, \\
\gamma_{n+2} &= \gamma_{n+1} + \beta_n.
\end{align*}
\]

These trees all have the same structure as the Fibonacci convolution trees, but their node colorings are different since their coloring rules are determined by coupled recurrences such as those of (4.1) and (4.2).

One simple illustration of how studies of the colors arising on the trees lead to interesting tableaux with Fibonacci properties is the following: For the two tree sequences \( S_1 \) and \( S_2 \) (say) from the 2-F scheme, we may compute the total weight (i.e. sum of the node colors) for each tree. For example, the fourth tree in sequence \( S_1 \) has weight \( 4a + 3b + 1c + 2d \). Then we may tabulate the coefficients of \( a, b, c, d \), for each sequence (up to the seventh tree as in Tables 6 and 7.

As we expect from the manner in which the trees were colored (following (4.1)), the table for \( S_2 \) is table \( S_1 \) with its columns permuted thus: \( (a, c)(b, d) \). Note that the sequence of row sums is \( \{1, 2, 5, 10, \ldots \} = (F * F) \), the convolution of the Fibonacci sequence with itself as in (1.1).

If we add Table 6 and Table 7, elementwise, we get Table 8.

We see that the sum of the weights of the \( n \)th trees from the two sequences is:

\[
W^{(1)}_n + W^{(2)}_n = U_n(a + c) + V_n(b + d),
\]

where \( \{U_n\} = 1, 1, 3, 5, 10, 18, 33, \ldots \) and \( \{V_n\} = 0, 1, 2, 5, 10, 20, 38, \ldots \).

Now \( V_n = (F * F)_{n-1} \) (proof given below); and \( U_n + V_n = (F * F)_n \) (since \( \Sigma_n \) in the table is \( 2(F * F)_n \)), therefore

\[
U_n = (F * F)_n - (F * F)_{n-1}.
\]
Fig. 4.
A generalized tableau associated with colored convolution trees

<table>
<thead>
<tr>
<th>Tree</th>
<th>$S_1$ Coefficients of Coefficients</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$\Sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>1 0 0 0 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_2$</td>
<td>1 1 0 0 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_3$</td>
<td>0 0 3 2 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$T_4$</td>
<td>4 3 1 2 10</td>
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<tr>
<td>$T_5$</td>
<td>5 6 5 4 20</td>
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<tr>
<td>$T_6$</td>
<td>7 8 11 12 38</td>
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<tr>
<td>$T_7$</td>
<td>19 20 14 18 71</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4

In [6] the following identity for the Fibonacci convolution term is given

$$5(F \ast F)_{n-1} = (n + 1)F_{n-1} + (n - 1)F_{n+1}.$$  

Using this we obtain

$$V_n = \frac{1}{3}(n + 1)F_{n-1} + (n - 1)F_{n+1};$$

and

$$5U_n = [(n + 2)F_n + nF_{n+2}] - [(n + 1)F_{n-1} + (n - 1)F_{n+1}]
= (n + 1)(F_n + F_{n-2}) + F_{n+1}$$

therefore $U_n = \frac{1}{3}[(n + 1)L_{n-1} + F_{n+1}]$, where $L_{n-1}$ is a Lucas number.

We finally prove the convolution forms given above for $U_n, V_n$.

**Proof.** We showed in [12] that if a single sequence of the convolution trees is colored sequentially, using color $C_n$ of a sequence $\{C_n\}$ to color the root node of $T_n$, and mounting the previously colored $T_{n-1}$ and $T_{n-2}$ on the fork, then the weight of $T_n$ is $(F \ast C)_n$.

Now the general term of $\Sigma_n$ (in the above table) is obtained by setting $a = b = c = d = 1$; in that event, both $S_1$ and $S_2$ are Fibonacci convolution trees (i.e., $C = F$ in both cases), so $\Sigma_n = 2(F \ast C)_n$.

Similarly, if we set $a = 0 = c$ and $b = 1 = d$, we find that $S_1$ and $S_2$ are identical but with color sequences $\{F_i\}$; and then $U_n \cdot 0 + V_n \cdot 2 = 2(F \ast F)_{n-1}$, giving the required form of $V_n$.

<table>
<thead>
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<th>Tree</th>
<th>$S_2$ Coefficients of Coefficients</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$\Sigma$</th>
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<tbody>
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<td>$T_1$</td>
<td>0 0 1 0 1</td>
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<td></td>
<td></td>
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<tr>
<td>$T_2$</td>
<td>0 0 1 1 2</td>
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<td>$T_3$</td>
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<td>$T_4$</td>
<td>4 2 4 3 10</td>
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<td>$T_5$</td>
<td>5 4 5 6 20</td>
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<tr>
<td>$T_6$</td>
<td>11 12 7 8 38</td>
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<tr>
<td>$T_7$</td>
<td>14 18 19 20 71</td>
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Table 8

Total weights $W_n^{(1)} + W_n^{(2)}$, where $W_n^{(i)}$ is the weight of $T_n$ in sequence $S_i$.

<table>
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<th>$n$</th>
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<th>$c$</th>
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Acknowledgement

Gratitude is expressed to the referee for some detailed corrections.

References