Conditional symmetries for ordinary differential equations and applications

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We refine the definition of conditional symmetries of ordinary differential equations and provide an algorithm to compute such symmetries. A proposition is proved which provides criteria as to when the symmetries of the root system of ODEs are inherited by the derived higher-order system. We provide examples and then investigate the conditional symmetry properties of linear nth-order equations subject to root linear second-order equations. First this is considered for simple linear equations and then for arbitrary linear systems. We prove criteria when the symmetries of the root linear ODEs are inherited by the derived scalar linear ODEs and even order linear system of ODEs. Furthermore, we show that if a system of ODEs has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves. Moreover, we give examples of the conditional symmetries of non-linear third-order equations which are linearizable by second-order Lie linearizable equations. Applications to classical and fluid mechanics are presented.

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1. Introduction

One of the principal stages in the development of the solution of differential equations was the algebraic approach due to Lie [1]. After the advent of calculus, there arose a number of disparate methods for the solution of ordinary differential equations (ODEs) such as linear, homogeneous, and Bernoulli. This was the state of affairs some 200 years after Newton and Leibnitz had introduced the derivative and integral of a function. Guided by Galois theory, Lie introduced the notion of a continuous one-parameter group of transformations that leaves a differential equation form invariant. Lie showed how the group properties of a differential equation enable its reduction or solution and significantly unify the unraveling of solutions to differential equations. However, the Lie approach is not complete for the integrability of differential equations. In the case of ODEs, on the one hand an equation may have sufficient number of symmetries but may not yet be solvable by quadratures via the Lie approach. An example of this is a variable coefficient linear second-order ODE which has eight point symmetries but is not integrable by quadratures. On the other hand, an ODE or a system of ODEs has general solution in terms of quadratures without admitting any non-trivial symmetry group [2,3]. These insights have given rise to extensions to different types of symmetries such as non-local, λ as well as other types of symmetries [4–8].

Scalar linear second-order ODEs are similar to each other. Characteristically, they can be converted to the simplest equation by invertible maps. In the earlier literature, much focus has been on the study of non-linear differential equations and the procedures to reduce them to linear form. Reduction and exact solutions of ODEs were investigated by Lie in a systematic manner by use of his continuous groups of point transformations. He, in [1], obtained a complete classification in the complex domain of such second-order ODEs in terms of their groups. In particular, Lie determined algebraic and invariant criteria of second-order ODEs that could be linearized. Furthermore, he discovered the class of scalar second-order ODEs that transform to linear form under point transformations. These are called Lie linearizable ODEs. Lie found criteria for linearizability and obtained the most general form being that of the cubically non-linear second-order ODE in its first derivative which was linearizable via point transformations.

The linearization of ODEs of the second-order was extended to the third order in [9,10] and [11,12]. It was shown in the latter works that there are two classes of third-order ODEs that are linearizable by point transformations. It is possible to extend the...
linearization procedure for any order \( n \) with available algebraic computation packages. For \( n = 4 \) this was achieved in [13].

Lie algebraic properties of scalar linear nth-order \((n \geq 3)\) ODEs were investigated in [14] as well as [15]. It was shown in [15] that there are three classes of such higher order ODEs, viz. \( n + 1, n + 2 \) and \( n + 4 \). This is in contrast to scalar linear second-order ODEs.

A significant extension of classical Lie symmetries for partial differential equations (PDEs) is that of non-classical or what is imposed on the PDEs [18,19]. Olver and Rosenau [18] showed that there are three classes of such higher order ODEs, viz. systems in the work of Noether [23]. Conditional symmetries of ODEs which has not been considered amount of work on conditional symmetries of PDEs which pervade the literature. Our interest here is that of conditional symmetry of ODEs which has not enjoyed much interest.

One of the initial investigations of the non-classical method to ODEs is due to Gaeta [20]. In this study [20], he discussed, inter alia, conditional constants of the motion as well as their relation to conditional symmetries for first-order dynamical systems of ODEs. In the definition of conditional symmetries proposed in [20], the invariant curve condition is appended as a side condition. We in our study consider higher-order ODE systems and provide a definition in this context (see the next section). First integrals or constants of the motion are naturally studied in the formalsm of Hamiltonian and Lagrangian systems. The direct relationship between symmetries and first integrals for canonical Hamilton’s equations first appeared in the work of Levi-Civita [21] (see also the interesting translation of the work [21] and its historical perspective in [22]) and for symmetries related to Lagrangian systems in the work of Noether [23]. Configurational invariants and second integrals were studied by Darboux, Poincaré, Painlevé, Hess amongst others (see, e.g. [24] for a review). In the paper by Sarlet et al. [25], configurational invariants were investigated in the context of Hamiltonian systems and a weak form of complete integrability. This notion is utilized as conditional constants of the motion in [20] and in the present treatment we formulate the definition of conditional first integrals in the context of higher-order ODEs under discussion.

Conditional classification of ODEs was proposed in [26] in which the authors provide definitions of conditional symmetry and linearizability of scalar ODEs subject to lower order root ODEs. They, in [27], discussed the invariant criteria for conditional linearizability of third-order equations subject to root second-order Lie linearizable ODEs. They have shown that certain gaps may be filled by the recent development of conditional linearizability for the deficiency of certain cases when the ODEs are not Lie linearizable. Differentiating the quadratically and cubically scalar semi-linear second order ODE relative to the independent variable gives scalar third-order ODEs. By investigating the general class of scalar third-order ODEs that are conditionally linearizable one finds conditional linearizability criteria for scalar third-order ODEs subject to root second-order linearizable ODEs [27]. These classes were not subsumed in [11] or in [12]. This in effect was a new type of linearizability. Further work on conditional linearization of fourth-order ODEs was performed in [28] as well as the conditional linearization of third-order systems in [29]. These were subject to Lie linearizable lower second-order ODEs.

We further study conditional symmetries of ODEs from the algorithmic viewpoint. The previous work by Mahomed and Qadir [26] provided basic definitions and invariant criteria without the computation of symmetries. They used invariant criteria which were in terms of the coefficients of the root second-order ODE. Herein we use and extend the initial basic notions of [26] and investigate, inter alia, the computational aspects of conditional symmetries. Moreover, we provide a conditional classification of scalar linear nth-order \((n \geq 2)\) ODEs subject to root linear \( m < n \) order ODEs. Further, we extend these to systems of linear nth-order \((n \geq 2)\) ODEs when the derived system is even order. Moreover, we show that if a system of ODEs has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which results from the known solution curves. We also deal with conditionally linearizable ODEs subject to Lie linearizable root second-order ODEs by computation of their conditional symmetries.

2. Conditional symmetries: definitions and algorithm

We firstly consider some preliminaries before proceeding to the notion of conditional symmetries of ODEs.

Consider the system of nth-order ODEs of the form

\[ y^{(n)}_a = f_a(x, y, y^{(1)}, \ldots, y^{(n-1)}), \quad n \geq 1, \quad a = 1, \ldots, \bar{p}, \quad (1) \]

where \( x \) is the independent variable, \( y = (y_1, \ldots, y_p) \) is the dependent variable and \( y^{(1)}, y^{(2)}, \ldots, y^{(n-1)} \) denote the derivatives of \( y \) with respect to \( x \) up to order \( n \). We may also say that the system (1) is of the order \( np \) which then directly gives the number of essential constants that arises if (1) has general solution.

We now have the following notion of compatibility which is much more familiar in the setting for PDEs as in [18,19].

If every solution of the nth-order ODE

\[ y^{(m)}_a = g_a(x, y, y^{(1)}, \ldots, y^{(m-1)}), \quad m \leq n, \quad \beta = 1, \ldots, q \leq \bar{p}. \quad (2) \]

is also a solution of the nth-order ODE (1), then the nth-order ODE (2) is said to be compatible to the nth-order ODE (1).

For compatible systems of ODEs, the number of essential constants for the appended side condition system is \( mq \) which is in general lower than that of the original ODE system of order \( n \) which is \( np \). This reduction in the number of essential constants will be seen in the examples considered in the sequel.

Furthermore, it is well-known that a first integral of the system (1) is a function \( l(x, y, y^{(1)}, \ldots, y^{(n)}), k \leq n \), which satisfies the relation

\[ D_x l = 0, \quad (3) \]

where \( D_x = d/dx \) is the total derivative operator with respect to \( x \), on the solutions of Eqs. (1).

Following Sarlet et al. [25], who stated this definition for a system of two second-order ODEs, we have the notion of conditional first integrals for an arbitrary system of ODEs. Note that in [25] and before, the terminology configurational invariant was in vogue (presumably due to the advent of conditional symmetries appearing later) and in [20], Gaeta used the notion of conditional constants of the motion for investigating dynamical systems. Here the context is higher-order ODEs. We therefore have the definition below.

**Definition 1.** A function \( l(x, y, y^{(1)}, \ldots, y^{(n)}), k \leq n \), is called a conditional first integral of the system of ODEs (1) with respect to the first integral \( l(x, y, y^{(1)}, \ldots, y^{(n)}) = C, k \leq n \), \( C \) a constant, if the restriction of \( J \) to a certain fixed surface obtained by setting \( C = C_0, \text{i.e.} \int_C = C_0 \) is a first integral of the corresponding reduced system.
It is important to remark that the number of essential constants for our system reduces by one if we have a proper conditional first integral of the system. The examples that ensue will amply illustrate this fact.

We start here by giving a precise definition of conditional symmetries. This is a natural extension of that given in the work [26]. Also this in general differs from the work [20] which requires only invariant curve conditions.

**Definition 2.** An nth-order system of ODEs \( n \geq 1 \) (1) is conditionally classifiable by a symmetry algebra \( \mathcal{A} \) with respect to an nth-order system of ODEs \( m \leq n \) (2) called the root ODEs if and only if the nth-order system of ODEs jointly with the nth order system of ODEs forms an overdetermined compatible system (so the solutions of the nth-order system of ODEs reduce to the solutions of the nth-order system of ODEs) and the nth-order system of ODEs has symmetry algebra \( \mathcal{A} \) which is the conditional symmetry algebra of the nth-order system.

Now we present an algorithm for computing the conditional point symmetries of the system (1) subject to the root system (2).

Let \( X \) be the vector field of dependent and independent variables given by

\[
X = \xi(x,y) \frac{\partial}{\partial x} + \eta(x,y) \frac{\partial}{\partial y},
\]

where \( \xi \) and \( \eta \) are the coefficient functions of the vector field \( X \) and the summation convention applies on \( \alpha \) from 1 to \( p \).

Suppose that the vector field \( X \) is a conditional symmetry generator of system (1) subject to an nth-order root system of ODEs (2). Then the conditional symmetry condition

\[
X^{[n]} \left| _{y_{\alpha} = \eta(x,y)} = 0 \right., \quad j, \ldots, \eta = 0 = 0,
\]

holds, where \( X^{[n]} \) denotes the \( n \)th prolongation of the generator \( X \)

\[
X^{[n]} = X + \sum_{j=1}^{n} c_j \frac{\partial}{\partial y^j}
\]

in which

\[
c_j = D_x(c_j) - \eta^{(j)}D_{x}(\xi) \quad j = 1, \ldots, n, \quad c_0 = \eta
\]

and \( D_x \) is again the usual total differentiation operator.

The conditional symmetry approach proposed for a given system of ODEs is a technique that allows us to construct particular compatible side conditions to the system to enable one to obtain the construction of exact solutions of the system if there is a sufficient number of conditional symmetries. The number of conditional symmetries required for a solution of a non-linear ODE system is the same as for the Lie approach as we seek reduction and solution to the compatible root equation (see also [26]). The exact solutions that arise in this approach have in general fewer or for many equations that occur in applications no essential constants when compared to the original system. This is seen in the examples presented below and in the section on physical applications. In relation to this, we mention that the conditional linearization of classes of third-order ODEs subject to Lie linearizable second-order ODEs gives rise to two essential constants for the original classes [27] as the root ODEs are of order two for which we know the general solutions. If the side condition is of one order (order in the sense of essential constants) less than the original system, then first or conditional first integrals are the appropriate side conditions. We mention these when they arise in the examples considered.

It is now opportune to provide examples of conditional symmetries for second- and third-order ODEs.

(1) Consider firstly the scalar second-order ODE:

\[
y'' = B + 3Ax^2y^{-1} - A^2Bx^3y^{-3},
\]

where \( A \) and \( B \) are constants. By using the Lie symmetry condition

\[
X^{[2]}(y'' - B - 3Ax^2y^{-1} - A^2Bx^3y^{-3}) = 0
\]

whenever (8) holds, we obtain the following determining equations:

\[
\eta_{yy} + (2\xi_{xy} - \xi_{xx})y'' + (\eta_{yy} - 2\xi_{xy})y' - \xi_{yy}y^2 + \eta_{yy}(B + 3Ax^2y^{-1} - A^2Bx^3y^{-3})
\]

\[
- ABx^2y' - A^2Bx^3y^{-3} - 3\xi_{yy}(B + 3Ax^2y^{-1} - A^2Bx^3y^{-3}) - A^2Bx^3y^{-3} - 6Ax^2y' - 4A^2Bx^3y^{-2} + 6\xi^2x^3y^{-3}
\]

\[
+ 3y^2x^3y^{-2} - 4\eta_{yy}x^3y^{-3} - 3\xi^2x^3y^{-4} = 0.
\]

Separation of (10) by powers of \( y' \), as \( \xi \) and \( \eta \) are independent of the derivatives of \( y \), leads to the overdetermined system of linear homogeneous partial differential equations:

\[
\xi_{yy} = 0,
\]

\[
\eta_{yy} - 2\xi_{xy} = 0,
\]

\[
2\eta_{xy} - 3\xi_{yy}B - 9A^2y^2 - 3Ax^2y^{-2} \xi_{yy} + 3A^2x^3y^{-3} = 0,
\]

\[
\eta_{yy} + (B + 3Ax^2y^{-1} - A^2Bx^3y^{-3})B + 3Ax^2y^{-1} - ABx^3y^{-2} - A^2Bx^3y^{-3} - 3\xi_{yy}(B + 3Ax^2y^{-1} - A^2Bx^3y^{-3}) - A^2Bx^3y^{-3} - 6Ax^2y' - 4A^2Bx^3y^{-2} - 6\xi^2x^3y^{-3}
\]

\[
+ 3y^2x^3y^{-2} - 4\eta_{yy}x^3y^{-3} - 3\xi^2x^3y^{-4} = 0.
\]

By solving the above system (11)–(14), we find the trivial solution:

\[
\xi = 0, \quad \eta = 0.
\]

Hence the ODE (8) has no Lie point symmetry and we consequently cannot use the Lie reduction technique to solve it.

We now compute the conditional symmetries of the second-order ODE (8) subject to the non-linear Abel first-order ODE:

\[
y' = Bx + Ax^3y^{-1}.
\]

That is the conditional symmetry condition is

\[
X^{[2]}(y'' - B - 3Ax^2y^{-1} + A^2Bx^3y^{-3}) = 0
\]

If we assume that

\[
\eta = \alpha(x)y = \beta(x)y,
\]

since a symmetry of (16) is of this form, then the determining equations (12)–(15) after the split in powers of \( y \) result in

\[
-2\beta + 2\alpha x + 6\alpha = 0,
\]

\[
-3\beta x + 2\alpha x^2 + 4\alpha = 0,
\]

\[
x^3(2\beta - \alpha') + 6\beta x - 6\alpha x - 6\alpha = 0,
\]

\[
x(2\beta - \alpha') + \beta - 2\alpha = 0.
\]

After we solve the system (19)–(22) for \( \alpha \) and \( \beta \), we obtain

\[
X = \xi(x) + 2\eta(x)y,
\]

which is now the conditional symmetry of our non-linear ODE (8) subject to the root equation (16). By using this symmetry (23) we can construct a one-parameter family of exact solutions of (8). Here we see that the root ODE (16) is a conditional first integral of our equation (8). Thus a reduction occurs in the number of essential constants from two to one therefore yielding a one-parameter family of exact solutions.

We proceed further to a simpler equation by considering a linear equation in order to understand the behavior of such symmetries.

(2) We work out the conditional symmetries of the simplest third-order ODE, viz.

\[
y'' = 0
\]
subject to the free particle equation
\[ y'' = 0. \tag{25} \]

Here the determining equation is
\[ \xi y'' = 0. \tag{26} \]

The expansion of (26) gives rise to
\[ a_{xxx} + 3a_{xxy} y'' + 3a_{xyy} y'' + 2a_{yyy} y'' = 0. \tag{27} \]

Separation of (27) and then solving the resultant linear homogeneous system yield 15 conditional symmetries which are
\[ X_1 = \delta_t, \quad X_2 = \delta_y, \quad X_3 = \theta_{x}, \quad X_4 = \phi_{y}, \quad X_5 = \phi_{x}, \quad X_6 = \phi_{\theta}, \quad X_7 = \phi_{\delta}, \quad X_8 = \phi_{\eta}, \quad X_9 = \phi_{\lambda}, \quad X_{10} = \phi_{\xi}, \quad X_{11} = \phi_{\eta}, \quad X_{12} = \phi_{\xi}, \quad X_{13} = \phi_{\xi}, \quad X_{14} = \phi_{\xi}, \quad X_{15} = \phi_{\xi}. \tag{28} \]

We note that these conditional symmetries contain both the symmetries of the free particle equation (25) (these are \( X_1 \) to \( X_6 \), \( X_7 \) and \( X_8 \)) as well as those of the simplest third-order ODE (24) (these are \( X_1 \) to \( X_5 \), \( X_6 \) and \( X_7 \) and \( 2X_8 \)). This should be the case as conditional symmetries are more general than Lie point symmetries as in the case of PDEs. Moreover, it should be seen that \( X_{12}, X_{14} \) and \( X_{15} \) are neither symmetries of (25) nor of (24). These are in addition. Further, the symmetries (28) do not close under the Lie bracket and therefore do not form a Lie algebra. One can observe that by merely computing \([X_2, X_{12}] = 2X_1^3\delta_y\) which is not a conditional symmetry of (24). Also of interest is that \( X_{13}, X_{14} \) and \( X_{15} \) form an Abelian three-dimensional Lie algebra.

In general we have that the free particle symmetries are conditional symmetries of the third-order ODE (24). The question arises as to which symmetries of the free particle equation are inherited by the derived third-order ODE as point symmetries and which remain as proper conditional symmetries.

We focus now on the symmetries of the free particle ODE (25) that are inherited by the third-order ODE (24). It is seen that the generators \( X_1, X_2, X_3, X_4 \) and \( X_5 \) given in (28) are inherited by (24). The remaining generators \( X_6, X_7 + X_8 \) and \( X_9 + X_{10} \) of the free particle equation (25) thus become the proper conditional symmetries of the third-order ODE (24). Also note that the solution symmetry \( X_{12} \) and the symmetry \( X_8 + 2X_9 \) of equation \( y'' = 0 \) given in (28) do not come from the symmetries of the equation \( y'' = 0 \). In this example the conditional first integral is \( y' = 0 \) which restricts the number of essential constants to two of the original third-order ODE \( y'' = 0 \) for solution purposes.

(3) Next we consider the non-linear third-order ODE
\[ y'' + y' + 3y^2y'' - \frac{2}{X} y' + \frac{2}{X^2} y' = 0, \tag{29} \]

which is of the class considered in [26]. This ODE (29) is also conditionally linearizable subject to the root Lie linearizable ODE \( y'' + y^3 + 2y' = 0 \) [27]. The determining equations for the symmetries of the ODE (29) are
\[ \xi y'' = 0, \quad \eta_{yy} = 0, \quad \eta_x = 0, \quad a_{xxx} - 4a_{xx} x - 2a_x x^2 + a_{xxx} = 0, \quad -2\eta_y + 3a_{xx} y - 3a_{xxx} = 0, \quad \xi - a_{xx} x + 2a_y y = 0. \tag{30} \]

The solution of the determining system (30) gives
\[ Y_1 = x\phi_y, \quad Y_2 = \delta_y. \tag{31} \]

One can at most perform a double reduction of the ODE (29) by using the Lie reduction approach. However, we cannot solve (29) completely by using the Lie approach.

The conditional symmetries of (29) subject to the second-order ODE
\[ y'' + y^3 - \frac{2}{X} y' = 0, \tag{32} \]

are
\[ X_1 = x^3 \cos y\phi_y, \quad X_2 = x^3 \sin y\phi_y, \quad X_3 = \sin y\phi_y + \frac{1}{x} \cos y\phi_y, \quad X_4 = -\cos y\phi_y + \frac{1}{x} \sin y\phi_y, \quad X_5 = x \cos 2y\phi_y - \sin 2y\phi_y, \quad X_6 = x \sin 2y\phi_y + \cos 2y\phi_y, \quad X_7 = x\phi_y, \quad X_8 = \delta_y. \tag{33} \]

We see that only \( X_7 = Y_1 \) and \( X_8 = Y_2 \) are inherited symmetries of (29).

In order to understand when the symmetry of the base scalar root equation is inherited by the derived equation, we therefore prove the following proposition.

**Proposition 1.** Consider the system of \( n \)th ODEs
\[ E_n(x, y, y', y'', \ldots, y^{(n)}) = 0, \quad \alpha = 1, \ldots, \tilde{p} \tag{34} \]

where \( x \) is the independent variable, \( y = (y_1, \ldots, y_p) \) is the dependent variable and \( y', y'', \ldots, y^{(n)} \) denote respectively the first, second and up to \( n \)th order derivatives of \( y \) with respect to \( x \). If the generator (4) is a Lie point symmetry generator of the ODE (34), then
\[ X^{(n+1)}D_x E_n = D_x (x_i \partial_i E_n) - D_x (\partial_x E_n)X_{i} \tag{35} \]

holds. Thus \( X \) is a symmetry of the derived ODE \( D_x E_n = 0 \) for each \( \alpha \) if \( X_{i} \) are constants. Moreover if \( X_{i} \) are constants, then one has the following relation for the \( k \)th derived ODEs \( E_n = 0 \), \( \alpha = 1, \ldots, \tilde{p} \):
\[ X^{(n)}D_x E_n = -\sum_{i=0}^{k-1} (x_i \partial_i E_n)D_x E_n = 0 \tag{36} \]

Furthermore, for \( X_{i} \) constants and \( D_x \xi = 0 \), we obtain
\[ X^{(n)}D_x E_n = -\sum_{i=0}^{k-1} (x_i \partial_i E_n)D_x E_n = 0 \tag{37} \]

This relation (37) implies that for the derived \( k \)-th order ODEs \( D_x E_n = 0 \), for each \( \alpha \), the symmetry \( X \) is inherited from \( E_n = 0 \) provided that \( X_{i} \) are constants and \( \xi \) is linear in \( x \).

The proof of the first part follows from the identity \( X^{(n+1)}D_x E_n = D_x (x_i \partial_i E_n) - D_x (\partial_x E_n)X_{i} \) which acts on a differential function of order \( n \). It is the case that
\[ X^{(n)}D_x E_n|_{D_x E_n = 0} = D_x (\xi_j \partial_j E_n|_{D_x E_n = 0} = 0 + (\xi_j \partial_x E_n)|_{D_x E_n = 0} = 0 - D_x (\xi_j \partial_x E_n|_{D_x E_n = 0} = 0) \tag{38} \]

Thus \( D_x \xi = 0 \) implies that \( X_{i} \) are constants, meaning that \( X \) is an inherited symmetry of the derived ODEs \( D_x E_n = 0 \) for each \( \alpha \). Moreover, the second part follows from induction. Certainly, it applies for \( k = 1 \) as
\[ X^{(n+1)}D_x E_n = x_0 D_x E_n - D_x \xi \partial_x E_n \tag{39} \]

For \( k = 2 \), we have
\[ X^{(n+2)}D_x E_n = - (D_x \xi \partial_x E_n) + D_x (\xi_0 \partial_0 E_n) \tag{40} \]

The rest follows by induction that if it is true for \( k = m \), then for \( k = m + 1 \) one has
\[ X^{(n+m)}D_x E_n = D_x (x_i \partial_i E_n) - D_x \xi_j \partial_x E_n \]

which results in
\[ X^{(n+m+1)}D_x E_n = x_0 D_x (\xi_j \partial_j E_n) - \sum_{i=0}^{m-1} D_x \xi_j \partial_0 E_n = \frac{m!}{(m-1)!} \partial_d^{m+1} E_n \]
Operating the ODE (42) with $X$ since $X^{1/2}X^{1/2}X^{1/2}X^{1/2}$, the final part is a consequence of the fact that if $\xi$ is linear in $x$, then $D_x^k=0$ for $k \geq 2$ in (36). This directly gives (37).

We now provide applications of Proposition 1.

(4) The non-linear equation which is of the class in [26]

$$yy''+3yy'=0$$  

subject to the Lie linearizable ODE

$$yy''+y^2=0$$

has conditional symmetries

$$X_1 = \partial_x, \quad X_2 = y \partial_y, \quad X_3 = x \partial_x,$$

$$X_4 = y \partial_y, \quad X_5 = y^2 \partial_y, \quad X_6 = x \partial_y,$$

$$X_7 = x^2 \partial_x + \frac{3}{4} xy \partial_y, \quad X_8 = x^2 \partial_x + \frac{1}{2} xy \partial_y.$$  

Clearly $X_1$ is a inherited symmetry of our ODE (42).

Now consider the operator $X_2 = (1/y) \partial_y$ with its second prolongation

$$X_2^{[2]} = \frac{1}{y} \partial_y - \frac{1}{y^2} y' \partial y' + \left( - \frac{1}{y^2} y'' + \frac{2}{y^3} y'' ight) \partial y'',$$

Operating $X_2^{[2]}$ on ODE (42), we find

$$X_2^{[2]}(yy''+y^2)=0,$$

on the solution; hence $X_2$ by Proposition 1 is also a symmetry of the ODE (42) with $\lambda=0$. Consider the generator $X_3 = x \partial_x$ with its second-prolongation:

$$X_3^{[2]} = x \partial_x - y \partial y' - 2y \partial y''.$$  

Applying $X_3^{[2]}$ on ODE (42), results in

$$X_3^{[2]}(yy''+y^2) = -2yy''(yy''+y^2),$$

where we have $\lambda = -2$. Thus $X_3$ is an inherited symmetry by Proposition 1.

Now consider $X_4 = y \partial_y$, with

$$X_4^{[2]} = y \partial_y + y \partial y' + y \partial y''.$$  

Making use of $X_4^{[2]}$ on ODE (42), we deduce

$$X_4^{[2]}(yy''+y^2) = 2yy''(yy''+y^2),$$

with $\lambda=2$. Hence $X_4$ is a inherited symmetry by Proposition 1.

Now for operator $X_5 = y^2 \partial_y$, we have

$$X_5^{[2]} = y^2 \partial_y - 2yy'' \partial y' + ( -2y'' - 4yy'' - 2yy' ) \partial y''.$$  

Operating $X_5^{[2]}$ on ODE (42), we get

$$X_5^{[2]}(yy''+y^2) = -6yy''(yy''+y^2),$$

where $\lambda = -6yy'$, so that $X_5$ is a proper conditional symmetry.

Consider the symmetry $X_6 = (x/y) \partial_y$ with second prolongation given by

$$X_6^{[2]} = \frac{x}{y} \partial_y - \left( \frac{1}{y} \partial y - \frac{x}{y^2} y'' \right) y' + \left( \frac{x}{y} \partial y ight) y''.$$  

Applying $X_6^{[2]}$ on ODE (42), we find

$$X_6^{[2]}(yy''+y^2) = 0,$$

with $\lambda=0$. This symmetry is inherited as well.

Now consider $X_7 = x^2 \partial_x + \frac{1}{2} xy \partial_y$ with second prolongation:

$$X_7^{[2]} = 2x^2 \partial_x + xy \partial_y + (y - 3xy' \partial y' + ( -2y' - 7xy'' \partial y''.$$  

We have

$$X_7^{[2]}(yy''+y^2) = -6xy(yy''+y^2)=0,$$

where $\lambda = -6x$; this implies that $X_7$ is a proper conditional symmetry.

Finally, we consider $X_8 = x^2 \partial_x + \frac{1}{2} xy \partial_y$ with

$$X_8^{[2]} = x^2 \partial_x + \frac{1}{2} xy \partial_y + \left( \frac{x^2 y'}{2} - 2xyy'' + 2xy'' + 6xyy''y' \right) \partial y'',$$

Now with the use of $X_8^{[2]}$ on ODE (42), we determine

$$X_8^{[2]}(yy''+y^2) = -6x(2yy''+y^2)=0,$$

which in this case yields $\lambda = -6xy'$. Thus $X_8$ is a proper conditional symmetry.

Therefore the inherited symmetries of [42] are $X_1$ to $X_4$ and $X_8$. Here five symmetries are inherited whereas in Example 3 we had only two inherited symmetries. These symmetries are also the inherited symmetries for any derived ODE of (42) as $\xi$ is linear in $x$.

A few examples on systems of two ODEs are now given.

(5) The variable coefficient linear system

$$E_1 = x' = tx = 0,$$

$$E_2 = y' = ty = 0,$$

obviously admits the scaling symmetries $X_1 = x \partial_x$ and $X_2 = y \partial_y$. According to Proposition 1 these are inherited symmetries of any derived even order system $D_x^kE_0 = 0$, $\alpha = 1, 2$, $k \geq 1$. In fact we see that for $X_1$, the $\lambda$ are $x_1^1 = 1$ and $\lambda_2 = 0$ for the other values. Similarly one can obtain $x_1^2 = 1$ for $X_2$ with the rest zero.

(6) The non-linear system (see [30])

$$E_1 = x' = -y^2 = 0,$$

$$E_2 = y' = ty = 0,$$

admits the symmetries

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = y \partial_y, \quad X_4 = y \partial_y,$$

By Proposition 1, the symmetry $X_1$, $X_2$ and $X_3$ are inherited by any derived system $D_x^kE_0 = 0$, $\alpha = 1, 2$, $k \geq 1$. For the first two it is evident and for $X_3$ we have that the non-zero $\lambda$ is $x_1^1 = 1$ which is constant. In the case of $X_4$ we have that $x_1^1 = -2y'$, $x_1^2 = -x'$, $x_2^1 = 0$ and $x_2^2 = -3y'$. Hence not all $x_i^j$ s are constant and $X_4$ is a condition symmetry of system (60).

(6) Finally for the non-linear Newtonian system (see [30])

$$E_1 = x' = -x^2 - y^2 = 0,$$

$$E_2 = y' = -2xy = 0,$$

the admitted symmetries are

$$X_1 = \partial_t, \quad X_2 = \exp (-x+y) \partial_t, \quad X_3 = \exp (-x-y) \partial_t, \quad X_4 = t \partial_t.$$  

The symmetry $X_1$ is clearly inherited and so is $X_4$ since $x_1^1 = -2$ and $x_1^2 = -2$ with the rest zero. Any derived system of (62) also has inherited symmetries $X_1$ and $X_4$. These follow from Proposition 1. However, $X_2$ and $X_3$ are conditional symmetries.

### 3. Integrable equations admitting conditional symmetries

It is almost a folklore even nowadays that any ODE integrable by quadratures has non-trivial symmetries. This was dispelled in the two telling examples provided in [2,3]. The second paper was on a system of two second-order ODEs which we focus on later. The example we first draw attention to is from [2]. In [2] the scalar...
second-order ODE
\[ y'' = y^{-1/2} + pg(x)y^p + g'(x)y^{p+1}, \quad y > 0, \]  
where \( p \neq 0 \) is a constant and \( g \neq 0 \) an arbitrary function of \( x \), shown to be completely integrable by quadratures but devoid of point symmetry.

Here we demonstrate that (64) has a conditional symmetry subject to the Bernoulli equation:
\[ y' - Cy = g(x)y^{p+1}, \]  
where \( C \) is an arbitrary constant.

We assume that \( \xi = 0, \eta = \alpha(x)y^{p+1} \) since a symmetry of (65) is of this form. The substitution of this form into the determining equation (we invoke (51) on the solutions of (64) and (65) gives after separation with respect to separate powers of \( y \) that
\[ \alpha(x) = \exp(-Cpx). \]  
Thus a conditional symmetry of (64) subject to (65) is
\[ X = \exp(-Cpy^{p+1})y, \]  
In fact the conditional symmetry (66) results in the complete integrability by quadratures of the ODE (64) for arbitrary \( g(x) \) and \( p \neq 0 \). Here the Bernoulli equation (65) is a first integral of the original second-order ODE (64). Thus the number of essential constants in the solution is two which results in the general solution.

In fact we can go a step further and construct what appears to be the first higher-order ODE which is integrable by quadratures but has no point symmetry.

We utilize (65) in the form
\[ y^{-1}y' - Kg(x)y^p = C, \]  
where \( C \) and \( K \) are arbitrary constants. Then the corresponding second-order ODE after differentiation is
\[ y^{-1}y'' = y^{-2}y'' + pg(x)K^{-1}y^{-1} + g'(x)Ky^p, \quad y > 0, \]  
Utilizing (68) as a first integral for the third-order ODE, we determine
\[ pg(x)y^{-2}y'' + g'(x)y^{-1}y'' - 3g(x)y^{-2}y'' - 2pg(x)y^{-2}y'' + pg(x)y^{-1}y'' + pg(x)y^{-1}y'' + 2pg(x)y^{-4} + 2pg(x)y^{-4} + 2pg(x)y^{-1}y'' - 2pg(x)y^{-2}y'' - 2pg(x)y^{-1}y'' + pg(x)y^{-1}y''. \]  
This ODE (69) subject to (67) admits the conditional symmetry (66). Here the compatible side condition (67) has two constants \( C \) and \( K \), and its integrability then provides a further constant yielding in all three required for complete integrability.

We are in a position to prove the following proposition.

**Proposition 2.** If a scalar nth-order, \( n \geq 2 \), ODE of the form (34) is completely integrable by quadratures, then it admits a conditional symmetry subject to the first-order ODE related to the invariant curve condition which arises from the known solution curves.

The proof of this follows at once. Indeed, since the nth-order ODE of the form (34) is completely integrable, it has integral curves \( \phi(x, y, C_1, \ldots, C_{n-1}) = 0 \) in which \( C_i, i = 1, \ldots, n-1 \) are arbitrary constants (these \( n \) constants result in \( n \) first integrals). We solve for one of the \( C_{i} \), say \( C_i \), to deduce \( C_i = \phi(x, y, C_1, \ldots, C_{i-1}) \). The total differentiation of this, viz., \( D_y \phi(x, y, C_1, \ldots, C_{n-1}) = 0 \) yields
\[ \phi_x + y\phi_y = 0, \quad \phi_y \neq 0 \]  
which is the invariant curve condition. The condition (70) then gives the conditional symmetry
\[ X = \partial_x - \frac{\phi_x}{\phi_y} y \]  
of our nth-order ODE (34) subject to (70).

In the light of Proposition 2, we can now re-look at the example with which we started. For the second-order ODE (64), a conditional symmetry is
\[ X = \partial_x - (Cy + g(x)y^{p+1})y \]  
which is subject to the invariant curve condition (65). Similar considerations apply to the third-order ODE (69).

An ODE may not be completely integrable such as the one given in Example 1 of the previous section in which the second-order ODE has a one-parameter family of exact solutions. In this event, one can weaken Proposition 2 and write the integral curves as \( \phi(x, y) = 0 \) or \( \phi(x, y, C_1, \ldots, C_n) = 0 \), where \( r \) ranges from 1 to \( r < n \). We can state a weaken form of Proposition 2 as follows.

**Proposition 3.** If a scalar nth-order, \( n \geq 2 \), ODE of the form (34) has exact solutions \( \phi(x, y) = 0 \) or \( \phi(x, y, C_1, \ldots, C_n) = 0 \), where \( r \) ranges from 1 to \( r < n \), then it admits a conditional symmetry subject to the first-order ODE related to the invariant curve condition which arises from the known solution curves.

The proof of this is analogous to that of Proposition 2. In this case there are \( r \) first integrals resulting in \( r \) essential constants or no first integral thereby giving an exact solution with no arbitrary constants – the side condition is the invariant condition which is compatible.

4. **Conditional symmetries of scalar linear ODEs**

Now we generalize the above analysis of the free particle ODE to scalar nth-order linear ODEs. We begin our discussion with simple linear ODEs.

Consider the simplest \((n-1)\)th order ODE:
\[ y^{(n-1)} = 0, \quad n \geq 4. \]  
The above equation (73), as is well-known, has \((n-1)+4 = n+3\) point symmetries for \( n \geq 4 \) which are (see, e.g. [15])
\[ X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x, \quad X_4 = y\partial_y, \quad X_5 = x\partial_y, \quad X_6 = x^4\partial_y, \quad X_{n+3} = x^2\partial_x + (n-2)xy\partial_y. \]  
The derived ODE
\[ y^{(n)} = 0 \]  
has \(n+4\) symmetries. The operators \( X_1, \ldots, X_{n+2} \) of (74) are the inherited symmetries of the nth-order ODE (75) and \( X_{n+3} \) of (74) is proper conditional. This can be verified by Proposition 1 as only the translations, uniform scalings as well as solution symmetries of (73) have \( i \) constant. Thus these symmetries are inherited by the simplest derived higher-order linear ODE. If one looks for the inherited symmetries that arise from the free particle equation to any subsequent order equation \( y^{(n)} = 0, \quad s \geq 3 \), then these are \( X_1, \ldots, X_5 \). However, if we desire the inherited symmetries from \( y^{(n)} = 0, \quad p \geq 3 \to y^{(n)} = 0, \quad q \geq 4 \), then these are transformations, uniform scalings and solution symmetries up to \( X_{p+4} = x^{(p-1)}\partial_x \). The symmetry \( X_{p+4} \) is a proper conditional symmetry given as
\[ X_{p+4} = x^2\partial_x + (p-1)xy\partial_y. \]  

We now consider how the symmetries of scalar linear second-order ODEs with variable coefficients are inherited by the derived compatible higher-order linear ODEs. The linear second-order ODE
\[ y'' + a(x)y' + b(x)y = 0 \]  
has the point symmetries
\[ X_1 = \alpha_1\partial_x + (\alpha_1' - \alpha_1\alpha_2)y\partial_y, \quad X_2 = \alpha_2\partial_x + (\alpha_2' - \alpha_2\alpha_2)y^2\partial_y. \]
X_3 = \beta_1 \phi_k + \frac{(\beta_1 - \beta_2)(y)}{2},
X_4 = \beta_2 \phi_k + \frac{(\beta_2 - \beta_1)(y)}{2},
X_5 = \beta_3 \phi_k + \frac{(\beta_3 - \beta_2)(y)}{2},
X_6 = \delta \phi_k, \quad X_7 = \delta \phi_k, \quad X_8 = y \phi_k.

where the \alpha, \beta, and \delta are independent solutions of
\alpha - (ax) + bu = 0,
\beta - (ax^2 + ax^2 - 4b)\phi - \frac{1}{2}(2ax^2 + ax^2 - 4b)\phi = 0,
\delta + ax + bx = 0.

It transpires by Proposition 1 that any derived ODE from (77) of third- or higher-order inherits in general the symmetries \( X_8, X_7, X_5 \) for non-constant coefficients. If the coefficients of (77) are constants then a further symmetry \( X = \delta \phi \) (which comes from one of \( X_1 \) to \( X_5 \)) is inherited.

We consider a particular example by putting \( a = 0 \) and \( b = -1 \) in ODE (77). Then the ODE is

\[ y'' - y = 0. \]

Thus the symmetries \( X_3, X_4 \) and \( X_5 \) of ODE (80) are (these come from the corresponding symmetries in (78))
\[ X_3 = e^t y \phi_k + e^t \phi_k, \quad X_4 = e^t \phi_k + e^t \phi_k, \quad X_5 = e^t \phi_k - e^{-t} \phi_k. \]

The only inherited symmetries to the derived ODE are \( X_5, X_6, \ldots, X_8 \). Those of \( X_1, X_2, X_3, X_4 \) and \( X_5 \) are proper conditional symmetries of the derived equation. In fact if one computes \( \lambda \) for \( X_4 \) and \( X_5 \), one gets \( \lambda = -3e^{2t} \) and \( \lambda = 3e^{2t} \), respectively. Hence these are not inherited symmetries by Proposition 1.

For the simple harmonic oscillator equation
\[ y'' + y = 0, \]

again the inherited symmetries to any higher-order derived ODE are translations in \( x \), uniform scalings in \( y \), and the two solution symmetries \( \cos \phi_k \) and \( \sin \phi_k \).

We may as well investigate what happens to the inherited symmetries if we substitute the highest derivative of the root third-order ODE in the derived ODE. Consider the first derived ODE of (77), viz.

\[ y'' + a y' + (a' + b)y' + b' y = 0. \]

Now we substitute the value for \( y' \) from (77) into this ODE (83).

This results in
\[ y'' + (a'' + b) y' + (b' - ab) y = 0. \]

Eq. (84) inherits the solution and homogeneity symmetries \( X_6, X_7 \) and \( X_8 \) as given in (78) of the second-order ODE (77) although Proposition 1 does not apply. The reason being that \( \delta \) and \( \delta \) of \( X_6 \) and \( X_7 \) solve (84) as well as \( X_8 \) is clearly a symmetry of (84).

We now consider higher-order variable coefficient linear nth-order \((n \geq 2)\) ODEs of the homogeneous type

\[ E \equiv y'' + \sum_{i=0}^{n} a_i(x)y^{(i)} = 0. \]

We content ourselves with this form as the treatment of the linear ODE without the second highest derivative coefficient annulled is the same. We thus use the canonical form (85) as it has received full treatment for its symmetry properties in Mahomed and Leach [15]. Equation has point symmetries [15]

\[ X_1 = y \phi_k, \quad X_{i+1} = \alpha(x) \phi_k, \quad i = 1, \ldots, n. \]

Any derived linear equation of (85), viz. \( \mathcal{D}_n^2 E = 0 \) inherits the symmetries (86) as Proposition 1 applies. The symmetries (87) become proper conditional symmetries of \( \mathcal{D}_n^2 E = 0 \).

Now the differentiation of (85) twice, \( \mathcal{D}_n^2 E = 0 \), and then the substitution of the value of \( y'' \) from (85) into \( \mathcal{D}_n^2 E = 0 \) give

\[ \mathcal{D}_n^2 E = 0. \]

The linear ODE (89) inherits the symmetries (86) of the root equation (85) which comprise the scaling and solution symmetries. Here Proposition 1 does not apply. This result though follows from the fact that the ODE (85) is compatible with the ODE (89) and the solutions of (85) are the solutions of (89) meaning that the solution symmetries in (86) are inherited by the ODE (89). Also clearly (89) admits the uniform dilation symmetry \( X_1 \) of (86).

The above argument is valid and can be extended to an arbitrary order derived ODE subject to appropriate substitutions of (85) at each stage in the derived ODE.

In the case of constant coefficient second- and higher-order linear ODEs (77) and (85), the derived ODE and derived ODE after substitution of the root equation inherit in addition to the solution and homogeneity symmetries, the translation symmetry viz. \( \delta \).

One thus has the following proposition.

**Proposition 4.** Any derived scalar linear ODE \( \mathcal{D}_n^2 E = 0 \), \( m \geq 1 \), with or without substitution of the root linear nth-order \((n \geq 2)\) ODE of the homogeneous type \( E \equiv y'' + \sum_{i=0}^{n} a_i(x)y^{(i)} = 0 \) inherits the homogeneity symmetry \( \phi_k \) and \( n \) solution symmetries \( \alpha_i(x) \phi_k \), \( i = 1, \ldots, n \), where \( \alpha_i \) solves \( E = 0 \). Further if the coefficients \( \alpha_i \) are constants, then in addition the translation symmetry \( \delta \) is inherited by the derived ODE. The remaining symmetries are proper conditional symmetries of the derived ODE.

**5. Conditional symmetries of systems**

Symmetries and integrability for systems of second-order ODEs were considered recently in [31]. In general we have that the symmetries of the root system are conditional symmetries of the derived higher-order system of ODEs. The question arises as to which symmetries of the root system are inherited by the derived system of ODEs as point symmetries and which remain as proper conditional symmetries.

We commence our initial investigation by considering the simplest system of two second-order ODEs, viz.

\[ x' = 0, \]
\[ y'' = 0, \]

where the prime denotes differentiation with respect to \( t \). This system (90) has the sl(4, R) symmetry algebra with well-known point symmetries

\[ X_1 = \phi_k, \quad X_2 = \phi_k, \quad X_3 = \phi_k, \quad X_4 = \phi_k, \quad X_5 = \phi_k, \quad X_6 = \phi_k, \quad X_7 = \phi_k, \quad X_8 = \phi_k, \quad X_9 = \phi_k, \quad X_{10} = \phi_k, \quad X_{11} = \phi_k, \quad X_{12} = \phi_k, \quad X_{13} = \phi_k. \]

It is easy to deduce that the derived system

\[ x'' = 0, \]
\[ y'' = 0, \]

has the inherited symmetries \( X_1 \) to \( X_6 \) as well as \( X_{11} \) and \( X_{12} \). This
means that 10 symmetries are inherited. These are composed of 3 translations, 3 homogeneous, 2 solution and two symmetries of mixed type in the space variables, not time t, viz, \( X_{101} \) and \( X_{102} \). We remind the reader that upper bounds for symmetry algebras for higher-order systems were given in [32].

One can extend the above argument to a system of more than two second-order ODEs. To begin with we consider a system of three equations. In this case we have 4 translations, 4 homogeneous, 3 solution and 6 symmetries of mixed type in the space variables. One can continue in this vein and we have the following proposition.

Proposition 5. For a system of \( n, n \geq 2 \), free particle equations which admits the symmetry algebra \( sl(n+2, R) \), the derived even order system inherits \( n^2 + 2n + 2 \) of the point symmetries of the free particle system comprising \( n+1 \) translations, \( n+1 \) homogeneous, \( n \) solution and \( n(n-1) \) symmetries of mixed type in the space variables. The remaining \( 2n+1 \) symmetries are proper conditional symmetries of the derived system.

The proof is constructive and follows easily. The free particle system has \( n^2 + 4n + 3 \) point symmetries. Of these, \( n \) are the symmetries \( y_i \), \( i = 1 \ldots n \), if we label the space variables as \( y_i \), and \( n+1 \) are the true projective symmetries which involve all the variables. These constitute \( 2n+1 \) symmetries which one can check are not inherited by the derived system and are conditional symmetries. Therefore \( n^2 + 2n + 2 \) are the inherited symmetries.

Consider now the \( nth \)-order system of ODEs:

\[
y^{(n)} + \sum_{i=0}^{n-1} A_i(t)y^{(i)} = 0,
\]

where the \( A_i \)s are \( m \times m \) time-dependent matrices and \( y = (y_1, \ldots, y_m) \).

We now have the following result which is a natural extension of Proposition 4.

Proposition 6. Any derived linear even order system of ODEs \( D_nE = 0 \), \( p \geq 1 \), with or without substitution of the root linear \( nth \)-order \( (n \geq 2) \) ODE of the homogeneous type \( E = y^{(n)} + \sum_{i=0}^{n-1} A_i(t)y^{(i)} = 0 \) inherits the \( m \) homogeneous symmetries \( y_i \partial y_i \), \( i = 1 \ldots m \), and \( nm \) solution symmetries \( a_i(x) \partial y_i \), \( i = 1 \ldots n \), \( j = 1 \ldots m \), where \( a_i \) solves \( E = 0 \). Further if the coefficients \( A_i \) are constants, then in addition the translation symmetry \( \partial_i \) is inherited by the derived system of ODEs.

It will also be of interest for further study to investigate inherited symmetries of derived systems of linear ODEs which are of mixed type not considered above.

Before we terminate this section, we state an analogous result as for scalar \( nth \)-order ODEs for systems regarding integrable equations.

Proposition 7. If an \( nth \)-order, \( n \geq 2 \), system of ODEs \( E_u(x,y, \ldots y^{(r)}) = 0 \), \( u = 1 \ldots, m \), \( y = (y_1, \ldots, y_m) \), has exact solutions \( \phi_u(x,y) = 0 \) or \( \phi_u(x,y, C_1, \ldots, C_t) = 0 \), where \( r \) ranges from 1 to \( r \leq mn \), then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves.

The proof is similar to that of Proposition 2 except that we have \( \phi_u \) for each \( u \). In this context there are \( r \) first integrals yielding \( r \) essential constants in the solution or no first integral just giving a particular solution devoid of any essential constants.

As an example, we take the system of two ODEs considered in the work [3] which drew attention to the misunderstanding that a system integrable by quadratures must have a non-trivial symmetry group. It was demonstrated in [3] that the system

\[
x'' = c(x), \quad x' = \frac{d}{dt} x
\]

is integrable by quadratures but in general has a trivial symmetry group.

The system [94] for arbitrary \( c \) and \( g \) has conditional symmetry:

\[
X = \partial_t + s(x) \partial_x + \left( yf \left( \int s^{-1}(x) \ dx - c_2, x, s(x) \right) + \int g \left( \int s^{-1}(x) \ dx - c_2, x, s(x) \right) \ s^{-1}(x) \ dx + c_3 \right) \partial_y,
\]

where \( s(x) = \pm \sqrt{2 \int c(x) dx + 2c_3} \) with \( c_1, c_2 \) and \( c_3 \) being arbitrary constants, subject to the invariant curve conditions:

\[
x' = s(x),
\]

\[
y' = f(s^{-1}(x) \ dx - c_2, x, s(x)) + g \left( \int s^{-1}(x) \ dx - c_2, x, s(x) \right) s^{-1}(x) \ dx + c_3.
\]

The invariant curve conditions have three arbitrary constants and their integrability gives one further constant.

6. Conditional symmetries subject to linearizable ODEs

Conditional linearizability of scalar third-order ODEs subject to Lie linearizable second-order ODEs was completely invariantly characterized in terms of the coefficients of the equation in the papers [26,27] wherein the derived third-order ODEs before and after substitution of the second derivative were investigated. Here we briefly focus on the conditional symmetries of two conditionally linearizable equations subject to second-order Lie linearizable root ODEs.

(1) The non-linear third-order ODE which is of class given in [26]

\[
y^2 y'' - 3 y^3 = 0
\]

subject to the linearizable second-order ODE (43) has conditional symmetries (44).

Here clearly \( X_1 \) is a inherited symmetry of (97). Now consider the symmetry \( X_2 \) and its third prolongation and operating it on ODE (97), we find

\[
X_2^{(3)}(y^2 y'' - 3 y^3) = y^3 y'' y'' + 2 y^4 y'' + 6 y^5 y''.
\]

Now putting \( y'' = y' \) in (98), we obtain

\[
X_2^{(3)}(y^2 y'' - 3 y'^3) = y'^3 y'' - 3 y'^3 = 0
\]

on solutions of the equation. Thus \( X_2 \) is a proper conditional symmetry when we substitute the value of \( y'' \).

Consider the generator \( X_2 \) with its third prolongation and operating it on ODE (97), we get

\[
X_2^{(3)}(y^2 y'' - 3 y'^3) = -3 y^3 y'' y'' - 3 y^3 = 0.
\]

Hence \( X_2 \) is an inherited symmetry of ODE (97).

The symmetry \( X_4 \) is also an inherited symmetry for ODE (97). Since \( X_4 = y^2 \partial_y \) is a conditional symmetry by Proposition 1 with \( \lambda = -6 y' \), thus it would also remain as a conditional symmetry for a third-order ODE (97) after substitution of \( y'' \).

Consider now the symmetry \( X_5 = (x/y) \partial_y \) with its third prolongation and operating on ODE (97) we find

\[
X_5^{(3)}(y^2 y'' - 3 y'^3) = x y'^3 y'' - 3 y'^3 = y^3 y'' y'' + 2 y^4 y'' + 6 y^5 y'' y''/y''.
\]

Now putting the value of \( y'' \) in (101), we deduce

\[
X_5^{(3)}(y^2 y'' - 3 y'^3) = x y'^3 y'' - 3 y'^3 = x y^2 (y^2 y'' - 3 y'^3) = 0.
\]
Hence $X_0$ is a proper conditional symmetry of a third-order ODE (97) after substitution.

Now for $X_1 = x^2 \delta_x + \frac{1}{2} y \phi_y \phi_x$, we have found that $\lambda = -6x$. Thus $X_1$ is a conditional symmetry for the third-order equation before and hence it remains the same for the third-order equation (97) after substitution of $y'$. Finally we have $X_3 = x y^2 \delta_x + \frac{1}{2} y^3 \phi_y \phi_x$ with $\lambda = -6xy'$. Here $X_3$ is also a conditional symmetry of the third-order equation before and it remains the same for the third-order equation (97) by substituting the value of $y'$. Here we see that only $X_1, X_3$ and $X_4$ are inherited symmetries.

(2) Another example given by

$$y'' - \frac{6}{x^2} y' - 7y^3 - 3x^2 y^5 = 0 \tag{103}$$

is of the class given in [26] and has conditional symmetries (33) subject to the linearizable root equation (32). We have the same symmetries $X_2$ and $X_6$ of (33) as inherited symmetries.

We have seen that the inherited symmetries are different for linear and non-linear ODEs. That is, a derived linear second-order equation has more inherited symmetries than a conditionally linearizable equation as in (103). Moreover, we see from the above two examples that conditional symmetries of the derived ODE before and after substitution need not be the same. In Example 2 above the inherited symmetries are the same as that of Example 3 given in Section 2. The same does not apply for Example 1 above which has $X_1, X_3$ and $X_4$ as inherited symmetries and in Example 4 of Section 2, we have that $X_1$ to $X_4$ and $X_6$ are inherited symmetries.

7. Physical applications

It is well-known that conditional symmetries of partial differential equations are of paramount importance as they unearth physical relevant solutions. Recently this has been applied to non-Newtonian fluid flow problems [33,34].

Here we demonstrate the utility of conditional symmetries for ODEs that arise in the non-Newtonian fluid flow of a third grade fluid as well as in classical mechanics.

(1) The third-order non-linear ODE for the steady-state solution $u = f(y)$ of the unsteady flow of a third grade fluid through a porous wall without a modified pressure gradient is (see [33])

$$\mu f'' - \alpha_W f' + 3\beta f' - r_b f'' + W f'/K_+ = 0, \quad \text{or} \quad \frac{df}{dx}, \tag{104}$$

where the relevant constants in (104) are the parameters of the fluid problem. The boundary conditions are

$$f(0) = b_2, \quad b_2 = \text{const}, \quad f(\infty) = 0, \quad f'(\infty) = 0. \tag{105}$$

We show here that the physical solution of (104) subject to (105) arises from a conditional symmetry. Indeed, the conditional symmetry of the ODE (104) subject to the first-order linear ODE

$$f' + \frac{\sqrt{c_x}}{\sqrt{2} y} f = 0 \tag{106}$$

is

$$X = \frac{\partial X_2}{\sqrt{2} y} \quad \frac{\partial X_3}{\sqrt{2} y} \tag{107}$$

The solution of the corresponding characteristics system of (107) results in the solution of (104) as

$$u = f(y) = b_2 \exp \left(-\frac{\sqrt{c_x}}{\sqrt{2} y} \right) \tag{108}$$

which is precisely of the form given in [33]. The substitution of this solution (108) into the ODE (104) gives the parameters constraint

$$\frac{r_b}{\sqrt{2} y} + \frac{\sqrt{c_x}}{\sqrt{2} y} X_2 - W \frac{\sqrt{c_x}}{\sqrt{2} y} \frac{1}{K_+} = 0 \tag{109}$$

as is the case [33]. In this example the one constant in the first-order ODE (106) which is a compatible side condition satisfies the boundary conditions with a further constraint on the parameters as given in (109).

(2) We consider the modified Emden equation:

$$y'' + \left( \frac{n-1}{n+3} \right)^{-1} y' + y^n = 0, \quad n \neq 0, 1, -1, -3 \tag{110}$$

where $k$ is an arbitrary constant. This ODE (110) is contained in the general class

$$y'' + a(x)y'y^n = 0 \tag{111}$$

where $a(x)$ is an arbitrary function, which has been studied by various authors, notably Leach [35], Berkovich [36] recently Mahomed and Momoniat [37]. Also extensions have been investigated [38]. Leach [35] investigated the first integrals of (111) via the Noether theorem. Berkovich [36] showed that one can reduce Eq. (111) to an autonomous canonical form by means of the Kummer–Liouville transformation:

$$y = v(x), \quad dt = u(x) dx. \tag{112}$$

In [36], the point symmetry properties and exact solutions of the particular form

$$y = \rho(x), \quad \text{for constant } \rho \text{ which satisfies a polynomial equation were also determined. The recent work [37] focused on a special case of (111), viz.}$$

$$y'' + \frac{3x}{k+x} y' + y^3 = 0, \quad k = \text{const.} \tag{114}$$

The authors of [37] then found the new general solution of (114) utilizing a symmetry of a first integral of (114).

Note that our ODE (110) has particular solutions of the form (113) as is easily constructible from the results of Berkovich [36]. These are

$$y = \left( \frac{2}{n+3} \right)^{2/(n-1)} \left( \frac{n-1}{n+3} \right)^{-2/(n-1)}. \tag{115}$$

We now obtain the new general solution of (110) by using a conditional symmetry of (110) subject to the first-order ODE:

$$\frac{1}{2} \left( \frac{n-1}{n+3} \right)^{2(n+1)/(n-1)} y'' + \frac{2}{n+3} \left( \frac{n-1}{n+3} \right)^{(n+3)/(n-1)} \frac{yy'}{n+1} \left( \frac{n-1}{n+3} \right)^{(n+1)/(n-1)} y^{n+1} = C, \quad C = \text{const.} \tag{116}$$

This is easily deduced and we have

$$X = \frac{n+3}{2} \left( \frac{n-1}{n+3} \right) \delta_y - y \delta_y. \tag{117}$$

We thus have an invariant of this operator (117):

$$u = y \left( \frac{n-1}{n+3} \right)^{2/(n-1)} \tag{118}$$

This invariant (118) after substitution in the ODE (116) results in the separable ODE

$$\left( \frac{n-1}{n+3} \right) u' = \pm \sqrt{2C + \frac{4u^2}{(n+3)^2} - \frac{2}{n+1} u^{n+1}} \tag{119}$$

which is integrable by quadratures. Thus the new general solution of (110) is given by (119). Note here that the first-order ODE (116) is a first integral which has the arbitrary constant $C$ in it. The solution of this then gives the complete integrability of our modified Emden equation (110).

(3) The fifth-order non-linear ODE for the steady-state solution $u = h(y)$ of the unsteady flow of a fourth grade fluid over an infinite
moved plate with suction/injection without a modified pressure gradient is (see [34])
\[
W_0 h^r + h^r - \alpha_1 W_0 h^r + \alpha_2 W_0^2 h^r + \beta_1 h^{r'} h^r - r W_0 h^r - 2 r W_0 \frac{d}{d \sigma} \left( h^{r'} h^r \right) = 0,
\]
(120)
where the relevant constants in (120) are the parameters of the fluid problem.

We show here that the physical solution of fifth-order ODE (120) arises from a conditional symmetry. Indeed, the conditional symmetry of the ODE (120) subject to the first-order linear ODE
\[
\frac{dh}{dy} - \frac{\beta}{3 W_0} h = 0,
\]
(121)
is
\[
X = \frac{\partial}{\partial y} + \frac{\beta}{3 W_0} \frac{\partial}{\partial h}.
\]
(122)
By solving the corresponding characteristics system of (122), the solution of the ODE (120) is
\[
u = h(y) = \exp \left( \frac{\beta y}{3 W_0} \right)
\]
(123)
which is precisely the form mentioned in [34]. The substitution of this solution (123) into the ODE (120) gives the solution on the physical parameters:
\[
-\frac{\beta}{3 T} + \frac{\beta^2}{(3 T)^2 (W_0)^2} + \alpha_1 \beta^3 (3 T)^2 (W_0)^2 - \frac{\beta_1 r_0^2}{(3 T)^2 (W_0)^2} - \frac{r_0^5}{(3 T)^2 (W_0)^2}
\]
(124)
as given in [34]. Similar to Example 1 above, we have a first-order side condition which yields one arbitrary constant and the solution of this satisfies the boundary conditions with the above constant on the parameters.

8. Concluding remarks

Conditional symmetries of ODEs are of paramount importance when the ODEs under investigation have insufficient number of Lie point symmetries. These are useful for integration as we have seen.

We have shown by means of an algorithm as to how one calculates the conditional symmetries of ODEs subject to lower order ODEs. Many simple examples as well as the properties of scalar linear and linear systems of ODEs were considered for their conditional symmetries. We explained how the symmetries of the root ODE are inherited by the derived ODE by means of a four propositions. Moreover we have proved that if a system of ODEs has exact solutions, then it admits a conditional symmetry subject to the first-order ODEs related to the invariant curve conditions which arises from the known solution curves.

We have also investigated third-order non-linear ODEs which admit eight conditional symmetries subject to a Lie linearizable second-order ODEs.

Finally we presented applications of conditional symmetries to ODEs that arise in mechanics and have obtained new solutions.

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