A Jacobi–Jacobi dual-Petrov–Galerkin method for third- and fifth-order differential equations

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This paper analyzes a method for solving the third- and fifth-order differential equations with constant coefficients using a Jacobi dual-Petrov–Galerkin method, which is more reasonable than the standard Galerkin one. The spatial approximation is based on Jacobi polynomials $P_n^{(α,β)}$ with $α, β \in (-1, ∞)$ and $n$ is the polynomial degree. By choosing appropriate base functions, the resulting system is sparse and the method can be implemented efficiently. A Jacobi–Jacobi dual-Petrov–Galerkin method for the differential equations with variable coefficients is developed. This method is based on the Petrov–Galerkin variational form of one Jacobi polynomial class, but the variable coefficients and the right-hand terms are treated by using the Gauss–Lobatto quadrature form of another Jacobi class. Numerical results illustrate the theory and constitute a convincing argument for the feasibility of the proposed numerical methods.

A R T I C L E   I N F O

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1. Introduction

The classical Jacobi Polynomials $P_n^{(α,β)}$ are a family of orthogonal polynomials [1] that have been used in a variety of applications for their ability to approximate general classes of functions, some of which are the resolution of the Gibbs’ phenomenon [2], electrocardiogram data compression [3] and the solution to differential equations [4–8]. In addition, Jacobi polynomials have a very close connection to the associated Legendre functions that are widely used in the Spherical Harmonic expansions. Due to the range of applications for Jacobi polynomials, it is desirable to perform spectral approximations as accurately as possible.

The spectral methods are preferable in numerical solutions of ordinary and partial differential equations due to their high-order accuracy whenever they work [9–12]. Standard spectral and collocation methods have been extensively investigated for solving second- and fourth-order differential equations. In a sequence of papers [13,4–7,14] the authors have constructed efficient spectral-Galerkin algorithms for second-, fourth-, and 2nth-order differential equations subject to various boundary conditions.

The use of general Jacobi polynomials has the advantage of obtaining the solutions of differential equations in terms of the Jacobi parameters $α$ and $β$. Of these polynomials, the most commonly used are the ultraspherical polynomials $C_n^{(α)}(x)$; the Chebyshev polynomials of the first kind $T_n(x)$; the Legendre polynomials $P_n(x)$; the Chebyshev polynomials of the second kind $U_n(x)$; the Chebyshev polynomials of the third kind $V_n(x)$; and the Chebyshev polynomials of the fourth kind $W_n(x)$ (see, [15]). Each of these polynomials has been used separately, by many authors, for solving approximately differential
equations for each particular pairs of $\alpha$ and $\beta$; among them, Heinrichs [16], Shen [17,18], Ma and Sun [19,20], Doha and Bhrawy [5], Livermore [21], Livermore and Lerley [8] Fernandino et al. [22] and Bialecki et al. [12].

Hence to generalize and instead of developing approximation results for each particular pair of indices, it would be very useful to carry out a systematic study on Jacobi polynomials with general indices $\alpha, \beta > -1$ which can then be directly applied to other applications. It is with this motivation that we introduce in this paper a family of Jacobi polynomials with general indices $\alpha, \beta > -1$.

The main aim of this paper is to propose a suitable way to approximate the third- and fifth-order differential equations by convenient spectral method-based on Jacobi polynomials such that it can be implemented efficiently and at the same time has a good convergence property like that in the case of spectral methods for second- and fourth-order differential equations [4–7]. It is worth noting here that odd-order problems lack the symmetry of even-order ones, so we propose a Jacobi dual-Petrov–Galerkin (JDPG) method, which is more reasonable and suitable than the standard Galerkin one. By choosing appropriate base functions, the system in (JDPG) method is sparse and can be implemented efficiently. We apply the method to the third- and fifth-order differential equations by constructing efficient spectral-Galerkin algorithms using compact combinations of Jacobi polynomials, which satisfy essentially all the underlying homogeneous boundary conditions. Since the main differential operators in these odd-order equations are not symmetric, it is quite natural to employ a Jacobi dual-Petrov–Galerkin (JDPG) method. To be more precise, we choose the trial functions to satisfy the underlying boundary conditions of the differential equations, and we choose the test functions to satisfy the dual boundary conditions.

The various matrix systems resulting from these discretizations are carefully investigated, especially their condition numbers. An algebraic preconditioning yields a condition number of $O(N)$, which is an improvement with respect to the well-known condition number $O(N^5)$ of collocation methods for third-order differential equations, and often exhibit unstable modes if the collocation points are not properly chosen (see, for instance, [23,24]).

Dealing with third- and fifth-order differential equations with variable coefficients, we introduce the pseudospectral Jacobi dual-Petrov–Galerkin (P-JDPG), and Jacobi–Jacobi dual-Petrov–Galerkin (J–JDPG) methods. The methods are basically formulated in the Jacobi spectral form with general indices $\alpha, \beta > -1$ but the variable coefficient terms being treated by the Jacobi collocation method with other two general indices $\theta, \vartheta > -1$ so that the schemes can be implemented at Jacobi–Gauss–Lobatto points efficiently. Therefore, we can generalize the Chebyshev–Legendre–Galerkin method to the Jacobi–Jacobi–Galerkin method. Some other cases, can be obtained directly as special cases from our proposed Jacobi–Jacobi dual-Petrov–Galerkin approximations. We, therefore, motivated our interest in Jacobi–Jacobi dual-Petrov–Galerkin approximations. Finally, numerical results are presented in which the usual exponential convergence behaviour of spectral approximations is exhibited.

The remainder of this paper is organized as follows. In Section 2 we give an overview of Jacobi polynomials and their relevant properties needed hereafter. Sections 3 and 4 are devoted to the theoretical derivation of the (JDPG) and (P-JDPG), (J–JDPG) methods for third-order differential equations with constant and variable coefficients subject to homogeneous boundary conditions. Section 5 gives the corresponding results for those obtained in Sections 3 and 4, but for the fifth-order differential equations. In Section 6, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithms. Some concluding remarks are given in the final section.

2. Some properties of Jacobi polynomials

The Jacobi polynomials associated with the real parameters ($\alpha > -1, \beta > -1$) (see, [25,1]), are a sequence of polynomials $P_n^{(\alpha,\beta)}(x)$ $(n = 0, 1, 2, \ldots )$, each respectively of degree $n$, satisfying the orthogonality relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) \, dx = \begin{cases} 0, & m \neq n, \\ h_n, & m = n, \end{cases}$$

where

$$h_n = \frac{2^{\alpha + \beta + 1} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1)! \Gamma(n + \alpha + \beta + 1)}.$$  (2.1)

For our present purposes it is convenient to standardize the Jacobi polynomials so that

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha + 1)_n}{n!}, \quad P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n(\beta + 1)_n}{n!},$$  (2.2)

where $(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}$. In this form the polynomials may be generated using the standard recurrence relation of Jacobi polynomials starting from $P_0^{(\alpha,\beta)}(x) = 1$ and $P_1^{(\alpha,\beta)}(x) = \frac{1}{2}[\alpha - \beta + (\lambda + 1)x]$, or obtained from Rodrigue's formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\alpha}(1+x)^{-\beta} D^n \left( (1-x)^{\alpha+n}(1+x)^{\beta+n} \right),$$

where $\lambda = \alpha + \beta + 1$, and $D \equiv \frac{d}{dx}$. For $\alpha = \beta$ one recovers the ultraspherical polynomials (symmetric Jacobi polynomials) and for $\alpha = \beta = \pm \frac{1}{2}, \alpha = \beta = 0$, the Chebyshev of the first and second kinds and Legendre polynomials respectively; and
for the nonsymmetric Jacobi polynomials, the two important special cases \( \alpha = -\beta = \pm \frac{1}{2} \) (Chebyshev polynomials of the third and fourth kinds) are also recovered.

The special values

\[
D^n p_n^{(\alpha, \beta)}(1) = \prod_{i=0}^{n-1} \frac{\Gamma(n + \alpha + 1)(n + \lambda + i)}{2^i(n - q)! \Gamma(q + \alpha + 1)}, \quad D^n p_n^{(\alpha, \beta)}(-1) = (-1)^{n+q} D^n p_n^{(\beta, \alpha)}(1),
\]

will be of important use later.

The following lemma will be of fundamental importance in what follows.

**Lemma 2.1.** The \( q \)th derivative of \( p_n^{(\alpha, \beta)}(x) \) can be written as

\[
D^q p_n^{(\alpha, \beta)}(x) = \sum_{i=0}^{k-q} C_q(k, i, \alpha, \beta) p_i^{(\alpha, \beta)}(x),
\]

where

\[
C_q(k, i, \alpha, \beta) = \frac{(k + \lambda)_q(k + \lambda + q)_i(i + \alpha + q + 1)_{k-i-q}}{2^i(k-i-q)! \Gamma(2i + \lambda)} \times \binom{2i}{i, i + \alpha + q + 1, 2i + \lambda + 1; 1}.
\]

(For the proof, see [26], and for the general definition of a generalized hypergeometric series and special \( 3F_2 \), see [25], p. 41 and pp. 103–104, respectively.)

### 3. Third-order differential equations with constant coefficients

We are interested in using the JDPG to solve the third-order differential equation

\[
u^{(3)} - \gamma_1 u^{(2)} - \gamma_2 u^{(1)} + \gamma_3 u = f(x), \quad \text{in } I = (-1, 1),
\]

subject to

\[u(\pm 1) = u^{(1)}(1) = 0,
\]

where \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are constants, and \( f(x) \) is a given source function.

The dual-Petrov–Galerkin method generates a sequence of approximate solutions that satisfy a weak form of the original differential equations as tested against polynomials in a dual space. To describe this method, we introduce some basic notation that will be used in the upcoming sections. We set

\[
S_N = \text{span}\{p_0^{(\alpha, \beta)}(x), p_1^{(\alpha, \beta)}(x), \ldots, p_N^{(\alpha, \beta)}(x)\},
\]

\[
W_N = \{u \in S_N : u(\pm 1) = u^{(1)}(1) = 0\},
\]

and

\[
W_N^* = \{u \in S_N : u(\pm 1) = u^{(1)}(-1) = 0\}.
\]

Then the Jacobi dual-Petrov–Galerkin approximation to (3.1) is, to find \( u_N \in W_N^* \) such that

\[
(u_N^{(3)}, v_N)_{w^{\alpha, \beta}} - \gamma_1(u_N^{(2)}, v_N)_{w^{\alpha, \beta}} - \gamma_2(u_N^{(1)}, u_N, v_N)_{w^{\alpha, \beta}} + \gamma_3(u_N, v_N)_{w^{\alpha, \beta}} = (f, v_N)_{w^{\alpha, \beta}}, \quad \forall v_N \in W_N^*,
\]

where \( w^{\alpha, \beta}(x) = (1 - x)^\alpha (1 + x)^\beta \) and \( (u, v)_{w^{\alpha, \beta}} = \int_I u v w^{\alpha, \beta} \, dx \) is the inner product in the weighted space \( L^2_{w^{\alpha, \beta}}(I) \). The norm in \( L^2_{w^{\alpha, \beta}}(I) \) will be denoted by \( \| \cdot \|_{w^{\alpha, \beta}} \).

#### 3.1. Dual-Petrov–Galerkin method

We choose compact combinations of Jacobi polynomials as basis functions aiming to minimize the bandwidth and the condition number of the coefficient matrix corresponding to (3.1). We choose the test basis and trial functions of expansion \( \phi_k(x) \) and \( \psi_k(x) \) to be of the form:

\[
\phi_k(x) = \eta_k[p_k^{(\alpha, \beta)}(x) + \epsilon_k p_{k+1}^{(\alpha, \beta)}(x) + \epsilon_k p_{k+2}^{(\alpha, \beta)}(x) + \xi_k p_{k-1}^{(\alpha, \beta)}(x)],
\]

\[
\psi_k(x) = \eta_k[p_k^{(\alpha, \beta)}(x) + \rho_k p_{k+1}^{(\alpha, \beta)}(x) + \rho_k p_{k+2}^{(\alpha, \beta)}(x) + \sigma_k p_{k+3}^{(\alpha, \beta)}(x)],
\]
where $\eta_k$ is a normalization constant—to be determined later—to make the main diagonal elements of the matrix corresponding to the third-order derivative operator equal to unity (see Theorem 3.1), and $\epsilon_k, \delta_k, \zeta_k, \rho_k, \theta_k$ and $\sigma_k$ are the unique constants such that $\phi_k(x) \in W_N$ and $\psi_k(x) \in W_N, \forall k = 0, 1, \ldots, N - 3$. From the boundary conditions; $\phi_k(\pm 1) = \phi_k^{(1)}(1) = 0$ and the two relations (2.2) and (2.3), we have the following system

$$
\begin{align*}
\epsilon_k & \frac{(k + \alpha + 1)}{(k + 1)} + \epsilon_k \frac{(k + \alpha + 1)}{(k + 1)^2} + \zeta_k \frac{(k + \alpha + 1)}{(k + 1)^3} = -1, \\
-\epsilon_k & \frac{(k + \beta + 1)}{(k + 1)} + \epsilon_k \frac{(k + \beta + 1)}{(k + 1)^2} - \zeta_k \frac{(k + \beta + 1)}{(k + 1)^3} = -1, \\
\epsilon_k & \frac{(k + \alpha + 1)(k + \lambda + 1)}{k(k + \lambda)} + \epsilon_k \frac{(k + \alpha + 1)(k + \lambda + 2)}{(k + 1)(k + \lambda)} + \zeta_k \frac{(k + \alpha + 1)(k + \lambda + 3)}{(k + 1)(k + \lambda)} = -1.
\end{align*}
$$

(3.7)

(3.8)

(3.9)

Hence $\epsilon_k, \delta_k$ and $\zeta_k$ can be uniquely determined to give

$$
\epsilon_k = \frac{-(k + 1)(2k + \lambda + 2)(k - \alpha + 2\beta + 1)}{(k + \alpha + 1)(k + \beta + 1)(2k + \lambda + 4)},
$$

$$
\delta_k = \frac{-(k + 1)2(2k + \lambda + 1)(k - \beta + 2\alpha + 3)}{(k + \alpha + 1)(k + \beta + 1)(2k + \lambda + 5)}
$$

and

$$
\zeta_k = \frac{(k + 1)3(2k + \lambda + 1)}{(k + \alpha + 1)(k + \beta + 1)(2k + \lambda + 4)}.
$$

Similarly, one can easily verify that

$$
\rho_k = \frac{(k + 1)(2k + \lambda + 2)(k - \beta + 2\alpha + 1)}{(k + \alpha + 1)(k + \beta + 1)(2k + \lambda + 4)},
$$

$$
\theta_k = \frac{-(k + 1)2(2k + \lambda + 1)(k - \alpha + 2\beta + 3)}{(k + \beta + 1)(k + \alpha + 1)(2k + \lambda + 5)},
$$

and

$$
\sigma_k = \frac{-(k + 1)3(2k + \lambda + 1)}{(k + \beta + 1)(k + \alpha + 1)(2k + \lambda + 4)}.
$$

It is clear that the two sets of basis functions $\phi_k(x) \in W_{k+3}$ and $\psi_k(x) \in W_{k+3}, \ k = 0, 1, 2, \ldots, N - 3$, are linearly independent. Therefore by the dimension argument and for $N \geq 3$, we have

$$
W_N = \text{span}\{\phi_k(x) : k = 0, 1, 2, \ldots, N - 3\},
$$

$$
W_N^* = \text{span}\{\psi_k(x) : k = 0, 1, 2, \ldots, N - 3\}.
$$

### 3.2. Derivation of the matrix systems

#### 3.2.1. Linear system for third-order differential equations with constant coefficients

Now it is clear that the variational formulation of (3.4) is equivalent to

$$
(u_N^{(3)}, \psi_k(x))_{w^\alpha,\beta} - \gamma_1(u_N^{(2)}, \psi_k(x))_{w^\alpha,\beta} - \gamma_2(u_N^{(1)}, \psi_k(x))_{w^\alpha,\beta} + \gamma_3(u_N, \psi_k(x))_{w^\alpha,\beta} = (f, \psi_k(x))_{w^\alpha,\beta}, \quad k = 0, 1, \ldots, N - 3.
$$

(3.10)

Let us denote

$$
\begin{align*}
f_k &= (f, \psi_k(x))_{w^\alpha,\beta}, \quad f = (f_0, f_1, \ldots, f_{N-3})^T, \\
u_N(x) &= \sum_{n=0}^{N-3} a_n \phi_n(x), \quad a = (a_0, a_1, \ldots, a_{N-3})^T, \\
A &= (a_{ij}), \quad B = (b_{ij}), \quad C = (c_{ij}), \quad D = (d_{ij}), \quad 0 \leq k, j \leq N - 3.
\end{align*}
$$

Then Eq. (3.10) is equivalent to the following matrix equation

$$
(A + \gamma_1 B + \gamma_2 C + \gamma_3 D) a = f.
$$

(3.11)

where the nonzero elements of the matrices $A, B, C$ and $D$ are given explicitly in the following theorem.
Theorem 3.1. If we take \( \phi_k(x) \) and \( \psi_k(x) \) as defined in (3.5) and (3.6) respectively, and if we denote \( a_{ij} = (\phi_i^{(3)}(x), \psi_j(x))_{w_{\alpha, \beta}} \), \( b_{ij} = (-\phi_i^{(2)}(x), \psi_j(x))_{w_{\alpha, \beta}} \), \( c_{ij} = -\phi_i^{(1)}(x), \psi_j(x))_{w_{\alpha, \beta}} \), and \( d_{ij} = (\phi_i(x), \psi_j(x))_{w_{\alpha, \beta}} \). Then the nonzero elements \( a_{kk}, b_{kj}, c_{kj}, \) and \( d_{kj} \) for \( 0 \leq k, j \leq N - 3 \) are given as follows:

\[
\begin{align*}
a_{kk} & = 1, \\
a_{kj} & = \eta_k \eta_j [O_2(j, k, \alpha, \beta) h_k + O_2(j, k + 1, \alpha, \beta) \rho_k h_{k+1} + O_3(j, k + 2, \alpha, \beta) \rho_k h_{k+2} + O_3(j, k + 3, \alpha, \beta) \sigma_k h_{k+3}], \\
b_{k+1,k} & = -\eta_k \eta_{k+1} \xi_k C_2(k + 3, k + 1, \alpha, \beta) h_{k+1}, \\
b_{kk} & = -\eta_k \eta_{k+1} \zeta_k C_2(k + 3, k + 1, \alpha, \beta) h_{k+1}, \\
b_{kj} & = -\eta_k \eta_j [O_2(j, k, \alpha, \beta) h_k + O_2(j, k + 1, \alpha, \beta) \rho_k h_{k+1} + O_2(j, k + 2, \alpha, \beta) \rho_k h_{k+2} + O_2(j, k + 3, \alpha, \beta) \sigma_k h_{k+3}], \\
c_{k+2,k} & = -\eta_k \eta_{k+2} \zeta_k C_1(k + 3, k + 2, \alpha, \beta) h_{k+2}, \\
c_{kk} & = -\eta_k \eta_{k+1} \xi_k C_1(k + 3, k + 1, \alpha, \beta) h_{k+1} + \xi_k \rho_{k+1} C_1(k + 3, k + 2, \alpha, \beta) h_{k+2}, \\
c_{kj} & = -\eta_k \eta_j [O_1(j, k, \alpha, \beta) h_k + O_1(j, k + 1, \alpha, \beta) \rho_k h_{k+1} + O_1(j, k + 2, \alpha, \beta) \rho_k h_{k+2} + O_1(j, k + 3, \alpha, \beta) \sigma_k h_{k+3}], \\
d_{k+3,k} & = \eta_k \eta_{k+3} \xi_k h_{k+3}, \\
d_{k,k+1} & = \eta_k \eta_{k+1} \xi_k C_2(k + 3, k, \alpha, \beta) h_k + \xi_k \rho_k C_2(k + 3, k, \alpha, \beta) h_{k+1}, \\
d_{k+1,k} & = \eta_k \eta_{k+1} \xi_k C_1(k + 3, k, \alpha, \beta) h_k + \xi_k \rho_k C_1(k + 3, k, \alpha, \beta) h_{k+1}, \\
d_{k+2,k} & = \eta_k \eta_{k+2} \zeta_k C_1(k + 3, k, \alpha, \beta) h_k + \xi_k \rho_k C_1(k + 3, k, \alpha, \beta) h_{k+1}. \\
\end{align*}
\]

where

\[
O_i(j, k, \alpha, \beta) = C_i(j, k, \alpha, \beta) + \epsilon_j C_i(j + 1, k, \alpha, \beta) + \epsilon_j C_i(j + 2, k, \alpha, \beta) + \epsilon_j C(j + 3, k, \alpha, \beta).
\]

Proof. The basis functions \( \phi_k(x) \) are chosen such that \( \phi_k(x) \in W_N \) for \( k = 0, 1, \ldots, N - 3 \). On the other hand, it is clear that \( \{\phi_k(x)\} \) and \( \{\psi_k(x)\} \) are linearly independent and the dimension of both \( W_N \) and \( W_N^* \) is equal to \( (N - 2) \). The nonzero elements \( a_{ij} \) for \( 0 \leq k, j \leq N - 3 \) can be obtained by direct computations using the properties of Jacobi polynomials.

It can be easily proved that the diagonal elements of the matrix A take the form

\[
a_{kk} = \eta_k^2 \xi_k C_3(k + 3, k, \alpha, \beta) h_k,
\]

and imposing the condition \( a_{kk} = 1 \), enables one to determine the normalization constant \( \eta_k \) to be

\[
\eta_k^2 = \frac{1}{\xi_k C_3(k + 3, k, \alpha, \beta) h_k},
\]

which in turn yields the following explicit expression for \( \eta_k \) as

\[
\eta_k = \left( \frac{(k + \alpha + 1)_2(k + \beta + 1)\Gamma(k + 1)\Gamma(k + \lambda + 3)}{2^{\lambda - 3}(1 + \lambda)_2(2k + \lambda + 1)_2 \Gamma(2k + \lambda + 3)\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)} \right)^{\frac{1}{2}}.
\]

It can be easily shown, that all other formulae can be obtained by direct computations using the properties of Jacobi polynomials.

In the case of \( \gamma_1, \gamma_2, \gamma_3 \neq 0, \alpha, \beta \in (-1, \infty) \), the linear system (3.11), can be solved by forming explicitly the LU factorization, i.e., \( A + \gamma_1 B + \gamma_2 C + \gamma_3 D = LU \). The special structure of L and U enables us to obtain the solution in \( O(N^2) \) operations.

3.2.2. Special cases

The special cases for ultraspherical basis \( \alpha = \beta \) and each is replaced by \( (\alpha - \frac{1}{2}) \) and for the Chebyshev basis of the first, second, third and fourth kinds may be obtained directly by taking \( \alpha = \beta = \pm \frac{1}{2}, \alpha = -\beta = \pm \frac{1}{2}, \) respectively, and for the Legendre basis by taking \( \alpha = \beta = 0 \). Results for the three cases \( \alpha = \beta = 0, \alpha = \beta = \pm \frac{1}{2} \) and \( \alpha = \beta = -\frac{1}{2} \) are given in detail as corollaries to the previous theorem as follows:

Corollary 3.2. If \( \alpha = \beta = 0 \), then the nonzero elements \( a_{ij}, b_{ij}, c_{ij}, d_{ij} \) for \( 0 \leq k, j \leq N - 3 \) are given as follows:

\[
\begin{align*}
a_{kk} & = 1, \\
b_{kk} & = \frac{1}{2} \left( \frac{1}{k + \frac{3}{2}} \right)_2, \\
b_{k,k+1} & = -b_{k+1,k} = \frac{1}{2} \left( \frac{1}{k + 5} \right)_2, \\
c_{kk} & = \frac{1}{2} \left( \frac{1}{k + \frac{3}{2}} \right)_2.
\end{align*}
\]
Corollary 3.3. If \( \alpha = \beta = \frac{1}{2} \) then the nonzero elements \((a_{ij}), (b_{ij}), (c_{ij}), (d_{ij})\) for \(0 \leq k, j \leq N - 3\) are given as follows:

\[
\begin{align*}
a_{kk} &= 1, & a_{ij} &= \frac{(j(j - 2k) + k(3k + 10) + 11)\sqrt{k + 4}}{2(j + 1)(j + 2)(j + 3)\sqrt{j + 4}}, & \text{for } j = k + 2n + 1, \ n \geq 0, \\
a_{ij} &= \frac{-(j^2 + (2k + 10) + k(3k + 20) + 36)\sqrt{k + 4}}{2(j + 1)(j + 2)(j + 3)\sqrt{j + 4}}, & \text{for } j = k + 2n, \ n \geq 1, \\
b_{kk} &= \frac{1}{(k + 2)(k + 3)}, & b_{k+1,k} &= \frac{-\sqrt{k + 5}}{2(k + 3)\sqrt{k + 4}}, \\
b_{ij} &= \frac{3\sqrt{k + 4}}{(j + 1)(j + 2)(j + 3)\sqrt{j + 4}}, & j = k + n, \ n \geq 2, & b_{k+1,k} &= \frac{-\sqrt{k + 5}}{2(k + 3)\sqrt{k + 4}}, \\
c_{kk} &= \frac{(k + 4)(2k + 5)}{4(k + 2)^2(k + 3)^2}, & c_{k+1,k} &= \frac{-(3k + 8)\sqrt{k + 5}}{4(k + 2)^2(k + 3)^2\sqrt{k + 4}}, & c_{k+2,k} &= \frac{-\sqrt{k + 6}}{4(k + 3)^2\sqrt{k + 4}}, \\
c_{k+1,k} &= \frac{(3k + 10)\sqrt{k + 5}}{4(k + 2)^2(k + 3)\sqrt{k + 4}}, & c_{k+2,k} &= \frac{-\sqrt{k + 6}}{4(k + 3)^2\sqrt{k + 4}}, \\
d_{kk} &= \frac{(2k + 5)(k^2 + 5k + 7)}{2(k + 1)(k + 3)^2}, & d_{k+1,k} &= \frac{-2k + 7}{2(k + 1)(k + 3)^2\sqrt{k + 4}}, & d_{k+3,k} &= \frac{-2k + 7}{2(k + 1)(k + 3)^2\sqrt{k + 4}}.
\end{align*}
\]

Corollary 3.4. If \( \alpha = \beta = -\frac{1}{2} \) then the nonzero elements \((a_{ij}), (b_{ij}), (c_{ij}), (d_{ij})\) for \(0 \leq k, j \leq N - 3\) are given as follows:

\[
\begin{align*}
a_{kk} &= 1, & a_{ij} &= \frac{(3j^2 + 6(j - 2k) + k^2 + 3)\sqrt{j + 3}(k + 3)}{2(j + 1)(j + 2)(j + 3)(k + 3)}, & \text{for } j = k + 2n + 1, \ n \geq 0, \\
a_{ij} &= \frac{(3j^2 + (j + k)(k + 6) + 12)}{2(j + 1)(j + 2)(j + 3)(k + 3)}, & \text{for } j = k + 2n, \ n \geq 1, & b_{kk} &= \frac{1}{2(k + 1)^3}, \\
b_{k+1,k} &= \frac{k^2 + 4k + 5}{2(k + 2)^2\sqrt{k + 3}}, & b_{ij} &= \frac{1}{(j + 1)(j + 2)(j + 3)(k + 3)}, & j = k + n, \ n \geq 2, \\
b_{k+1,k} &= \frac{-\sqrt{k + 3}}{2(k + 2)^2\sqrt{k + 4}}, & c_{kk} &= \frac{2k + 3}{4(k + 1)^3}, & c_{k+1,k} &= \frac{1}{4(k + 2)^2\sqrt{k + 3}}, \\
c_{k+2,k} &= \frac{-1}{4(k + 3)^2\sqrt{k + 3}}, & c_{k+1,k} &= \frac{1}{4(k + 2)^2\sqrt{k + 3}}, \\
d_{k+1,k} &= \frac{-2k + 7}{2(k + 1)^3}, & d_{kk} &= \frac{1}{2(k + 1)^3}, & d_{k+2,k} &= \frac{-2k + 7}{2(k + 1)^3}, \\
d_{k+3,k} &= \frac{-2k + 7}{2(k + 1)^3}. & d_{k+3,k} &= \frac{-2k + 7}{2(k + 1)^3}.
\end{align*}
\]

If \( \gamma_1 = \gamma_2 = \gamma_3 = 0 \), then (3.11) reduces to \( Av = f \). From Theorem 3.1, we see that \( A \) is an upper triangular matrix. Hence the linear system can be solved by special backward substitution.

3.2.3. Condition number

We can define the condition number of a real \( n \times n \) nonsingular matrix \( A \) as \( \text{Cond}(A) = \|A\|_{\text{w},\alpha,\beta} \|A^{-1}\|_{\text{w},\alpha,\beta} \), from which one can easily deduce that \( \text{Cond}(A) \geq 1 \).
23 turns the nonhomogeneous boundary conditions (3.14) and there reference therein). For example, the Gauss–Lobatto points are not suitable for third-order equations (cf. [23] and the references therein). However, the Jacobi dual-Petrov–Galerkin approximations presented in this paper lead to systems with small condition numbers and are numerically stable. The system resulting from the equation \( u^{(3)} = f(x) \) is \( Av = f \), where \( A \) is the identity matrix for \( \alpha = \beta = 0 \) and is an upper triangular matrix for other values of \( \alpha, \beta \), \( \gamma > 1 \), so we note that the matrix \( A \) is well-conditioned for all values of \( \alpha, \beta \). For the case \( \gamma_1 = \gamma_2 = \gamma_3 = 1 \) in system (3.11), one can easily show that the condition number of the matrix \( E = A + \gamma_1 B + \gamma_2 C + \gamma_3 D \) behaves like \( O(k) \) for large values of \( k \) and \( \alpha, \beta > -1 \). Hence the propagation of roundoff errors should not be very significant. The numerical examples presented in Section 6 confirm that our algorithms are numerically stable.

\[ \text{Table 3.1} \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>Cond(( A ))</th>
<th>( \gamma_1 )</th>
<th>( \gamma_2 )</th>
<th>( \gamma_3 )</th>
<th>Cond(( E ))</th>
<th>Cond(( E ))/( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
<td></td>
<td>8.767</td>
<td></td>
<td></td>
<td>1.095</td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td></td>
<td>24.252</td>
<td>1.515</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0</td>
<td>0</td>
<td>57.164</td>
<td>1.786</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>1</td>
<td></td>
<td>125.184</td>
<td>1.955</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>1</td>
<td></td>
<td>263.773</td>
<td>2.060</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td></td>
<td>21.051</td>
<td>2.631</td>
<td></td>
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</tr>
<tr>
<td>16</td>
<td>1</td>
<td></td>
<td>67.033</td>
<td>4.189</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>1/2</td>
<td>1/2</td>
<td>168.491</td>
<td>5.265</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>1/2</td>
<td>1/2</td>
<td>379.741</td>
<td>5.933</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>1/2</td>
<td>1/2</td>
<td>810.963</td>
<td>6.335</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td></td>
<td>3.438</td>
<td>4.298 (10^{-1})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>1</td>
<td></td>
<td>8.138</td>
<td>5.086 (10^{-1})</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>1/2</td>
<td>1/2</td>
<td>18.152</td>
<td>6.072 (10^{-1})</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
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<td>1/2</td>
<td>1/2</td>
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<td>6.078 (10^{-1})</td>
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<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>1/2</td>
<td>1/2</td>
<td>81.211</td>
<td>6.344 (10^{-1})</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

However the condition numbers of the two matrices \( A \) and \( E \) in (3.11) for some different values of \( \alpha, \beta \) and \( N \).

3.3. Treatment of the nonhomogeneous boundary conditions

In the following we can always modify the right-hand side to take care of the nonhomogeneous boundary conditions. Let us consider for instance the one-dimensional third-order differential equation (3.1) subject to the nonhomogeneous boundary conditions:

\[
\begin{align*}
    u(\pm 1) &= a_{\pm}, & u^{(1)}(1) &= \tilde{a}_+. \\
\end{align*}
\]

(3.12)

We proceed as follows:

Set

\[
    V(x) = u(x) + b_0 + b_1 x + b_2 x^2,
\]

(3.13)

where

\[
    b_0 = \frac{-3a_+ - a_- + 2\tilde{a}_+}{4}, \quad b_1 = \frac{-a_+ + a_-}{2}, \quad b_2 = \frac{a_+ - a_- - 2\tilde{a}_+}{4}.
\]

The transformation (3.13) turns the nonhomogeneous boundary conditions (3.12) into the homogeneous boundary conditions

\[
    V(\pm 1) = V^{(1)}(1) = 0.
\]

(3.14)

Hence it suffices to solve the following modified one-dimensional third-order differential equation:

\[
    V^{(3)} - \gamma_1 V^{(2)} - \gamma_2 V^{(1)} + \gamma_3 V = f^*(x) \quad \text{in} \quad I = (-1, 1),
\]

(3.15)

subject to the homogeneous boundary conditions (3.14), where \( V(x) \) is given by (3.13), and

\[
    f^*(x) = f(x) + (\gamma_2 b_0 - \gamma_2 b_1 - 2\gamma_1 b_2) + (\gamma_3 b_1 - 2\gamma_2 b_2)x + \gamma_3 b_2 x^2.
\]

4. Third-order differential equation with variable coefficients

In this section we use the pseudospectral dual-Petrov–Galerkin method in modal basis and J–JDPG method. In the case of using the pseudospectral with the nodal basis, the choice of quadrature rules/collocation points plays an important role and should be made in accordance with the underlying differential equations and boundary conditions (see, for instance, [23] and the references therein). For example, the Gauss–Lobatto points are not suitable for third-order equations (cf. [24]). With the modal basis, since the use of the quadrature rule is merely to approximate the integrals in the variational formulation,
the choice of quadrature rules/collocation points is not important. Therefore, for the third-order equation with variable coefficients, we can still use the usual Jacobi–Gauss–Lobatto quadrature (cf. [18]).

Let \( x_{N,j}^{(α,β)} \), \( 0 \leq j \leq N \), be the zeros of \((1 - x^2)\phi_N^{(α,β)}(x)\). Denote by \( σ_N^{(α,β)} \), \( 0 \leq j \leq N \), the weights of the corresponding Gauss–Lobatto quadrature formula, which are arranged in decreasing order. We define the discrete inner product and norm as follows:

\[
(u, v)_{w^α, β, N} = \sum_{k=0}^{N} u(x_{N,k}^{(α,β)}) v(x_{N,k}^{(α,β)}) σ_N^{(α,β)}, \quad \|u\|_{w^α, β, N} = \sqrt{(u, u)_{w^α, β, N}}.
\]

(4.1)

Obviously,

\[
(u, v)_{w^α, β, N} = (u, v)_{w^α, β} \quad \forall u, v ∈ S_{2N−1}.
\]

(4.2)

Thus, for any \( u ∈ S_ν \), the norms \( \|u\|_{w^α, β, N} \) and \( \|u\|_{w^α, β} \) coincide.

Associating with this quadrature rule, we denote by \( I_N^{β(α,β)} \) the Jacobi–Gauss–Lobatto interpolation (cf. [27–29]),

\[
I_N^{β(α,β)} u(x_{N,k}^{(α,β)}) = u(x_{N,k}^{(α,β)}), \quad 0 \leq j \leq N.
\]

We denote by \( I_N^β = I_N^{β(−1,1)} \) and \( I_N^0 = I_N^{0(0,0)} \) the Chebyshev–Gauss–Lobatto and Legendre–Gauss–Lobatto interpolation operators, respectively.

4.1. Pseudospectral method in modal basis

Let us consider, the following third-order differential equation with variable coefficients

\[
\begin{align*}
&u^{(3)} - γ_1(x) u^{(2)} - γ_2(x) u^{(1)} + γ_3(x) u = f(x), \quad x ∈ I, \\
&u(±1) = u^{(1)}(1) = 0.
\end{align*}
\]

(4.3)

The pseudospectral dual-Petrov–Galerkin method for (4.3) is to find \( u_N ∈ W_N \) such that

\[
\begin{align*}
&\left(u_N^{(3)} , v_N\right)_{w^α, β, N} - (γ_1(x)u_N^{(2)} , v_N)_{w^α, β, N} - (γ_2(x)u_N^{(1)} , v_N)_{w^α, β, N} + (γ_3(x)u_N , v_N)_{w^α, β, N} \\
&= (f , v_N)_{w^α, β, N} \quad \forall v_N ∈ W_N^∗.
\end{align*}
\]

(4.4)

where \( (u, v)_{w^α, β, N} \) is the discrete inner product of \( u \) and \( v \) associated with the Jacobi–Gauss–Lobatto quadrature.

Hence, by setting

\[
\begin{align*}
&u_N = \sum_{k=0}^{N−3} \tilde{a}_k ϕ_k, \\
&\tilde{f}_k = (f, ϕ_k)_{w^α, β, N}, \\
&\tilde{ų}_k = (γ_1(x)ϕ_k^{(2)})_{w^α, β, N}, \\
&\tilde{ų}_ij = -(γ_2(x)ϕ_j^{(1)})_{w^α, β, N}, \\
&\tilde{ų}_k = -(γ_3(x)ϕ_k)_{w^α, β, N},
\end{align*}
\]

and

\[
\begin{align*}
&\tilde{A} = (\tilde{ų}_k), \quad \tilde{B} = (\tilde{ų}_{ij}), \quad \tilde{C} = (\tilde{ų}_{ij}), \quad \tilde{D} = (\tilde{ų}_{ij}), \quad 0 ≤ k, j ≤ N−3.
\end{align*}
\]

Then the linear system (4.4) becomes

\[
(\tilde{A} + \tilde{B} + \tilde{C} + \tilde{D})\tilde{a} = \tilde{f}.
\]

(4.5)

It is clear that the matrices \( \tilde{B}, \tilde{C} \) and \( \tilde{D} \) are full and their formation involves \( N^3 \) operations as well as the inversion of (4.5). Hence a direct approach is advisable only if one uses a small or moderate number of modes.

4.2. J-DPG method

We can generalize the so-called Chebyshev–Legendre approach [30,17,31], i.e., using the Legendre formulation and Chebyshev–Gauss–Lobatto points to the Jacobi–Jacobi approach. The Chebyshev–Legendre method has a quasi-optimal computational complexity to permit the use of very large numbers of modes without suffering from large round-off errors, which are necessary for simulations of very complex dynamics of challenging scientific and engineering problems.

The number of operations can be reduced, if we use the following Chebyshev–Jacobi dual-Petrov–Galerkin method: Find \( u_N ∈ W_N \) such that

\[
\begin{align*}
&(u_N^{(3)} , v_N)_{w^α, β, N} - (I_N^{β}(γ_1(x)u_N^{(2)}) , v_N)_{w^α, β, N} - (I_N^{β}(γ_2(x)u_N^{(1)}) , v_N)_{w^α, β, N} + (I_N^{β}(γ_3(x)u_N) , v_N)_{w^α, β, N} \\
&= (f , v_N)_{w^α, β, N} \quad \forall v_N ∈ W_N^∗.
\end{align*}
\]

(4.6)
where \( I_k^\alpha \) is the interpolation operator based on the Chebyshev–Gauss–Lobatto points, while \((\ldots)_{\alpha,\beta,N}\) is still the discrete inner product of \( u \) and \( v \) associated with the Jacobi–Gauss–Lobatto quadrature. Hence the only difference between (4.4) and (4.6) is that \( \gamma_1(x)u_N^{(2)} \), \( \gamma_2(x)u_N^{(1)} \) and \( \gamma_3(x)u_N \) in (4.6) are replaced by \( I_k^\alpha(\gamma_1(x)u_N^{(2)}), I_k^\alpha(\gamma_2(x)u_N^{(1)}) \) and \( I_k^\alpha(\gamma_3(x)u_N) \) respectively.

In the special case of \( \alpha = \beta = 0 \), the discrete inner product of \( u \) and \( v \) is associated with the Legendre–Gauss–Lobatto quadrature. Thanks to the fast Fourier transform (FFT) and the fast Chebyshev–Legendre transform \[ (4.6) \]

\[ 1828 \]

For \( \Phi \)

\[ \text{is that} \]

\[ I_k^\alpha(\gamma_1(x)u_N^{(2)}), I_k^\alpha(\gamma_2(x)u_N^{(1)}) \] and \( I_k^\alpha(\gamma_3(x)u_N) \) can be computed in \( O(N \log N) \) operations (see [17] for details).

We can generalize the Chebyshev–Jacobi dual-Petrov–Galerkin method to the Jacobi–Jacobi dual-Petrov–Galerkin method: Find \( u_N \in W_N \) such that

\[
(u_N^{(3)}, v_N)_{\alpha,\beta,N} - (I_k^{(\alpha,\beta)}(\gamma_1(x)u_N^{(2)}), v_N)_{\alpha,\beta,N} - (I_k^{(\alpha,\beta)}(\gamma_2(x)u_N^{(1)}), v_N)_{\alpha,\beta,N} + (I_k^{(\alpha,\beta)}(\gamma_3(x)u_N), v_N)_{\alpha,\beta,N} = (f, v_N)_{\alpha,\beta,N} \quad \forall v_N \in W_N^*.
\]

(4.7)

where \( I_k^{(\alpha,\beta)} \) is the interpolation operator based on the Jacobi–Gauss–Lobatto points with the Jacobi parameters \( \theta \) and \( \vartheta \), while the discrete inner product of \( u \) and \( v \) is associated with the Legendre–Gauss–Lobatto quadrature. This statement is confirmed by our numerical results (see Section 6).

5. Fifth-order differential equations

In this section, we consider the fifth-order differential equation of the form

\[
-u^{(5)} + \gamma_1 u^{(3)} + \gamma_2 u^{(1)} + \gamma_3 u = f(x), \quad \forall x \in I.
\]

(5.1)

\[ u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(1) = 0. \]

We define

\[ V_N = \{ u \in S_N : u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(1) = 0 \}, \]

\[ V_N^* = \{ u \in S_N : u(\pm 1) = u^{(1)}(\pm 1) = u^{(2)}(-1) = 0 \}. \]

The results for fifth-order differential equations will be given without proofs.

5.1. JDPG method for constant coefficients

For \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are constants, we consider the following Jacobi dual-Petrov–Galerkin procedure for (5.1): Find \( u_N \in V_N \) such that

\[
-(u_N^{(5)}, v_N)_{\alpha,\beta,N} + \gamma_1(u_N^{(3)}, v_N)_{\alpha,\beta,N} + \gamma_2(u_N^{(1)}, v_N)_{\alpha,\beta,N} + \gamma_3(u_N, v_N)_{\alpha,\beta,N} = (f, v_N)_{\alpha,\beta,N} \quad \forall v_N \in V_N^*. \]

(5.2)

Now, we choose the basis and the dual basis functions \( \Phi_k(x) \) and \( \Psi_k(x) \) to be of the form

\[
\Phi_k(x) = \sum_{j=0}^{k} \phi_{k+j}(\alpha,\beta)(x) + \sum_{j=0}^{k} \phi_{k+j+1}(\alpha,\beta)(x) + \sum_{j=0}^{k} \phi_{k+j+2}(\alpha,\beta)(x) + \sum_{j=0}^{k} \phi_{k+j+3}(\alpha,\beta)(x) + \sum_{j=0}^{k} \phi_{k+j+4}(\alpha,\beta)(x) + \sum_{j=0}^{k} \phi_{k+j+5}(\alpha,\beta)(x),
\]

(5.3)

\[
\Psi_k(x) = \sum_{j=0}^{k} \psi_{k+j}(\alpha,\beta)(x) + \sum_{j=0}^{k} \psi_{k+j+1}(\alpha,\beta)(x) + \sum_{j=0}^{k} \psi_{k+j+2}(\alpha,\beta)(x) + \sum_{j=0}^{k} \psi_{k+j+3}(\alpha,\beta)(x) + \sum_{j=0}^{k} \psi_{k+j+4}(\alpha,\beta)(x) + \sum_{j=0}^{k} \psi_{k+j+5}(\alpha,\beta)(x),
\]

(5.4)

It is not difficult to show that the basis functions \( \Phi_k(x) \in V_{k+5} \) and the dual basis functions \( \Psi_k(x) \in V_{k+5}^* \) are given by

\[
\Phi_k(x) = \sum_{j=0}^{k} \phi_{k+j}(\alpha,\beta)(x) = \frac{(k+1)(k+2)(k+3)(k+4)(k+5)}{(k+1)(k+2)(k+3)(k+4)(k+5)} P_{k+1}^{(\alpha,\beta)}(x)
\]

(5.5)
and
\[ \Psi_k(x) = \hat{\eta}_k \left[ p_k^{(\alpha, \beta)}(x) + \frac{(k + 1)(2k + \lambda + 2)(k + 3\alpha - 2\beta + 1)}{(k + \alpha + 1)(k + \beta + 1)(2k + \lambda + 6)} p_{k+1}^{(\alpha, \beta)}(x) \right. \]
\[ \left. \quad - \frac{(k + 1)_2(2k + \lambda + 1)(2k + \lambda + 4)}{(k + \alpha + 1)_2(k + \beta + 1)_2(2k + \lambda + 6)_2} \times [2(k^2 + 6k + 8) + (9 + 4k + 6\alpha - \beta)\beta - 3(\alpha - 1)\alpha] p_{k+2}^{(\alpha, \beta)}(x) \right. \]
\[ \left. \quad - \frac{(k + 1)_3(2k + \lambda + 1)_2}{(k + \alpha + 1)_2(k + \beta + 1)_3(2k + \lambda + 7)_2} \times [2(k^2 + 6k + 8) + (15 + 4k + 6\beta - \alpha)\alpha - 3\beta(\beta + 1)] p_{k+3}^{(\alpha, \beta)}(x) \right. \]
\[ \left. \quad + \frac{(k + 1)_4(k + 3\beta - 2\alpha + 5)(2k + \lambda + 1)_3}{(k + \alpha + 1)_2(k + \beta + 1)_1(2k + \lambda + 6)_2(2k + \lambda + 9)_2} p_{k+4}^{(\alpha, \beta)}(x) \right. \]
\[ \left. \quad + \frac{(k + 1)_5(2k + \lambda + 1)_4}{(k + \alpha + 1)_2(k + \beta + 1)_3(2k + \lambda + 6)_4} p_{k+5}^{(\alpha, \beta)}(x) \right] \]
(5.6)

where
\[ \hat{\eta}_k = \left( \frac{(k + \alpha + 1)_3(k + \beta + 1)_2k!\Gamma(k + \lambda + 5)}{2^{1-k}(k + \alpha + 1)_5(2k + \lambda + 1)_5(2k + \lambda + 1)_4\Gamma(k + \alpha + 1)\Gamma(k + \beta + 1)} \right)^{\frac{1}{2}} \]
is the normalization constant to make the main diagonal elements of the matrix corresponding to the fifth-order derivative operator equal to unity.

Therefore, for \( N \geq 5 \), we have
\[ V_N = \text{span}\{\phi_0, \phi_1, \ldots, \phi_{N-5}\}, \]
\[ V'_N = \text{span}\{\psi_0, \psi_1, \ldots, \psi_{N-5}\}. \]

5.1.1. The linear system and its coefficient matrices

It is clear that the variational formulation of (5.2) is equivalent to
\[ - (u_N^{(5)}, \Psi_k(x))_{w, \alpha, \beta} + \gamma_1 (u_N^{(3)}, \Psi_k(x))_{w, \alpha, \beta} + \gamma_2 (u_N^{(1)}, \Psi_k(x))_{w, \alpha, \beta} + \gamma_3 (u_N, \Psi_k(x))_{w, \alpha, \beta} = (f, \Psi_k(x))_{w, \alpha, \beta}, \quad k = 0, 1, \ldots, N - 5. \]
(5.7)

Let us denote
\[ f_k = (f, \Psi_k(x))_w, \quad f = (f_0, f_1, \ldots, f_{N-5})^T, \]
\[ u_N(x) = \sum_{n=0}^{N-5} v_n \Phi_n(x), \quad v = (v_0, v_1, \ldots, v_{N-5})^T. \]
\[ p_{ij} = - (\Phi_j^{(5)}, \Psi_i)_{w, \alpha, \beta}, \quad q_{ij} = (\Phi_j^{(3)}, \Psi_i)_{w, \alpha, \beta}, \quad r_{ij} = (\Phi_j^{(1)}, \Psi_i)_{w, \alpha, \beta}, \quad s_{ij} = (\Phi_j, \Psi_i)_{w, \alpha, \beta}. \]

then equation (5.7) is equivalent to the following matrix equation
\[ (P + \gamma_1 Q + \gamma_2 R + \gamma_3 S)v = f, \]
(5.8)

where the nonzero elements of the matrices \( P, Q, R \) and \( S \) are given explicitly in the following theorem.

Theorem 5.1. If we take \( \Phi_k(x) \) and \( \Psi_k(x) \) as defined in (5.3) and (5.4) respectively, and if we denote \( p_{ij} = - (\Phi_j^{(5)}, \Psi_i)_{w, \alpha, \beta}, \quad q_{ij} = (\Phi_j^{(3)}, \Psi_i)_{w, \alpha, \beta}, \quad r_{ij} = (\Phi_j^{(1)}, \Psi_i)_{w, \alpha, \beta}, \quad s_{ij} = (\Phi_j, \Psi_i)_{w, \alpha, \beta}. \) Then the nonzero elements \( (p_{ij}), (q_{ij}), (r_{ij}) \) and \( (s_{ij}) \) for \( 0 \leq k, j \leq N - 5 \) are given as follows:
\[ p_{kj} = 1, \]
\[ p_{kj} = - \hat{\eta}_k \hat{\eta}_j [O_5(j, k, \alpha, \beta)h_k + O_5(j, k + 1, \alpha, \beta)\hat{\eta}_{k+1} + O_5(j, k + 2, \alpha, \beta)\hat{\eta}_{k+2} + O_5(j, k + 3, \alpha, \beta)\hat{\eta}_{k+3} + O_5(j, k + 4, \alpha, \beta)\hat{\eta}_{k+4} + O_5(j, k + 5, \alpha, \beta)\hat{\eta}_{k+5}], \quad j = k + n, n \geq 1, \]
\[ q_{kj} = \hat{\eta}_k \hat{\eta}_j [O_3(j, k, \alpha, \beta)h_k + O_3(j, k + 1, \alpha, \beta)\hat{\eta}_{k+1} + O_3(j, k + 2, \alpha, \beta)\hat{\eta}_{k+2} + O_3(j, k + 3, \alpha, \beta)\hat{\eta}_{k+3} + O_3(j, k + 4, \alpha, \beta)\hat{\eta}_{k+4} + O_3(j, k + 5, \alpha, \beta)\hat{\eta}_{k+5}], \quad j = k + n - 2, n \geq 1, \]
\[ r_{kj} = \hat{\eta}_k \hat{\eta}_j [O_1(j, k, \alpha, \beta)h_k + O_1(j, k + 1, \alpha, \beta)\hat{\eta}_{k+1} + O_1(j, k + 2, \alpha, \beta)\hat{\eta}_{k+2} + O_1(j, k + 3, \alpha, \beta)\hat{\eta}_{k+3} + O_1(j, k + 4, \alpha, \beta)\hat{\eta}_{k+4} + O_1(j, k + 5, \alpha, \beta)\hat{\eta}_{k+5}], \quad j = k + n - 4, n \geq 0, \]
\[ s_{kj} = \hat{\eta}_k \hat{\eta}_j [O_1(j, k, 0, 0)h_k + O_1(j, k + 1, 0, 0)\hat{\eta}_{k+1} + O_1(j, k + 2, 0, 0)\hat{\eta}_{k+2} + O_1(j, k + 3, 0, 0)\hat{\eta}_{k+3} + O_1(j, k + 4, 0, 0)\hat{\eta}_{k+4} + O_1(j, k + 5, 0, 0)\hat{\eta}_{k+5}], \quad j = k + n - 5, n \geq 0. \]
Consider the third-order differential equation with variable coefficients.

\[ \frac{d^3y}{dt^3} + \alpha \frac{dy}{dt} + \beta y = f(t), \]

where \( \alpha, \beta \) are variables.

**Proof.** The proof of this theorem is not difficult and it can be accomplished by following the same procedure used in proving Theorem 3.1.

\[ O_i(j, k, \alpha, \beta) = C_i(j, k, \alpha, \beta) + \tilde{\psi}_j C_i(j + 1, k, \alpha, \beta) + \tilde{\psi}_j C_i(j + 2, k, \alpha, \beta) + \tilde{\psi}_j C_i(j + 3, k, \alpha, \beta) + \tilde{\psi}_j C_i(j + 4, k, \alpha, \beta) + \tilde{\psi}_j C_i(j + 5, k, \alpha, \beta). \]

5.2. Fifth-order equations with variable coefficients

Let us consider the fifth-order differential equation (5.1) with \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are variables.

5.2.1. P-JDPG method

The pseudospectral dual-Petrov–Galerkin method for (5.1) is to find \( u_N \in V_N \) such that

\[
- (w^{(5)}_N, u_N \omega, \beta_N) + (\gamma_1(x) u^{(3)}_N, v_N \omega, \beta_N) + (\gamma_2(x) u^{(1)}_N, v_N \omega, \beta_N) + (\gamma_3(x) u_N, v_N \omega, \beta_N) \\
= (f, v_N \omega, \beta_N) \quad \forall v_N \in V^*_N, \tag{5.9}
\]

where \( (u, v) \omega, \beta_N \) is the discrete inner product of \( u \) and \( v \) associated with the Jacobi–Gauss–Lobatto quadrature (for detail, see Section 4).

5.2.2. J-JDPG method

The Jacobi–Jacobi dual-Petrov–Galerkin method for (5.1), is to find \( u_N \in V_N \) such that

\[
- (w^{(5)}_N, v_N \omega, \beta_N) + (I_{N,\ell}^{(5,0)}(\gamma_1(x) u^{(3)}_N), v_N \omega, \beta_N) + (I_{N,\ell}^{(5,0)}(\gamma_2(x) u^{(1)}_N), v_N \omega, \beta_N) + (I_{N,\ell}^{(5,0)}(\gamma_3(x) u_N), v_N \omega, \beta_N) \\
= (f, v_N \omega, \beta_N) \quad \forall v_N \in V^*_N, \tag{5.10}
\]

where \( I_{N,\ell}^{(5,0)} \) is the interpolation operator based on the Jacobi–Gauss–Lobatto points and the discrete inner product of \( u \) and \( v \) is associated with the Jacobi–Gauss–Lobatto quadrature.

6. Numerical results

We report on two numerical examples by using the algorithms presented in the previous sections.

**Example 1.** Consider the third-order differential equation with variable coefficients

\[
u^{(3)} - \cos(4x)u^{(2)} - e^{3x}u^{(1)} + (\sin(x) + x^3)u = f(x), \quad \text{with the exact solution } u(x) = (1 - x^2)(1 - x)e^{2x}.
\]

It should be noted that for a general right-hand side function \( f \) one is unable to compute exactly its representation by Jacobi polynomials. In fact, the so-called pseudospectral method is used to treat the right-hand side; i.e., we replace \( f \) in (6.1) by its polynomial interpolation over the set of Jacobi–Gauss–Lobatto points. This should slightly improve the round-off error.

Tables 6.1 and 6.2 list the maximum pointwise error, using the P-JDPG and J-JDPG methods with various choices of \( \alpha, \beta, N \) and \( \alpha, \beta, \theta, \vartheta, N \) respectively. Numerical results of third-order differential equations with variable coefficients show that the P-JDPG and J-JDPG methods converge exponentially.
Consider the one-dimensional fifth-order differential problem

\[-u^{(5)} + \tau_0 u^{(3)} + \tau_1 u^{(1)} + \tau_2 u = f(x), \]

\[u(\pm 1) = \cos(m), \quad u^{(1)}(\pm 1) = \mp m \sin(m), \quad u^{(2)}(1) = -m^2 \cos(m), \quad x \in I,\]

with the exact solution \(u(x) = \cos(mx)\).

Table 6.3 lists the maximum pointwise error, using the JDPG method with various choices of \(\alpha, \beta, m, \tau_0, \tau_1, \tau_2\) and \(N\). Numerical results of fifth-order differential equation with constant coefficients and nonhomogeneous boundary conditions show that the JDPG method converges exponentially.

In Table 6.4, we list the maximum pointwise error of \(u - u_b\) by the P-JDPG with two choices of the coefficients \(\tau_0, \tau_1, \tau_2\) and various choices of \(\alpha, \beta, N\), to examine the algorithm proposed in Section 5.2. The results indicate that the spectral accuracy is achieved and that the effect of roundoff errors is very limited.

7. Concluding remarks

A Jacobi dual-Petrov–Galerkin method is presented to solve third- and fifth-order differential equations subject to homogeneous and nonhomogeneous boundary conditions. The homogeneous boundary conditions are satisfied exactly by expanding the unknown variable into a polynomial basis of functions which are built upon the Jacobi polynomials. In this paper the matrix elements of the discrete operators are provided explicitly, and this in turn greatly simplifies the steps of obtaining solutions.
Table 6.4
Maximum pointwise error of $u - u_0$ for $N = 8, 16, 24, 32.$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$m$</th>
<th>$\omega$</th>
<th>$\beta$</th>
<th>$\tau_0$</th>
<th>$\tau_1$</th>
<th>$\tau_2$</th>
<th>P-JDPG</th>
<th>$r_0$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>P-JDPG</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$3\pi$</td>
<td>$7.113 \times 10^{-1}$</td>
<td>$2.127 \times 10^{-5}$</td>
<td>$\sin(\pi x)$</td>
<td>$x^3$</td>
<td>$e^{5x}$</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$3\pi$</td>
<td>$8.132 \times 10^{-12}$</td>
<td>$1.744 \times 10^{-6}$</td>
<td>$\sin(\pi x)$</td>
<td>$x^3$</td>
<td>$e^{5x}$</td>
</tr>
<tr>
<td>24</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$3\pi$</td>
<td>$2.273 \times 10^{-1}$</td>
<td>$1.927 \times 10^{-13}$</td>
<td>$\sin(\pi x)$</td>
<td>$x^3$</td>
<td>$e^{5x}$</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$3\pi$</td>
<td>$3.081 \times 10^{-1}$</td>
<td>$4.495 \times 10^{-6}$</td>
<td>$\sin(\pi x)$</td>
<td>$x^3$</td>
<td>$e^{5x}$</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$3\pi$</td>
<td>$9.102 \times 10^{-13}$</td>
<td>$4.612 \times 10^{-6}$</td>
<td>$\sin(\pi x)$</td>
<td>$x^3$</td>
<td>$e^{5x}$</td>
</tr>
<tr>
<td>24</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\pi$</td>
<td>$2\pi$</td>
<td>$3\pi$</td>
<td>$9.102 \times 10^{-13}$</td>
<td>$4.612 \times 10^{-6}$</td>
<td>$\sin(\pi x)$</td>
<td>$x^3$</td>
<td>$e^{5x}$</td>
</tr>
</tbody>
</table>

We have also presented some efficient direct solvers for the same equations with variable coefficients using P-JDPG and J-JDPG methods. Numerical results exhibit the high accuracy of the proposed numerical methods of solutions.

References