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MITTAG-LEFFLER-HYERS-ULAM STABILITY OF A LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER USING LAPLACE TRANSFORMS

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Abstract. In this paper, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of a homogeneous and non-homogeneous linear differential equation of first order by using the Laplace Transforms.

1. Introduction

A classical question in the theory of functional equations is the following : "when is it true that a function which approximately satisfies a functional equation g must be close to an exact solution of g?" If the problem accepts a solution, we say that the equation g is stable.

A simulating and famous talk presented by Ulam [1] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. Among those, the following question concerns the stability of homomorphisms.

Theorem 1.1. *(Ulam [1]) Let G_1 be a group and let G_2 be a group endowed with a metric ρ. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)\ f(y)) < \delta$, for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?*

If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [2] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered

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the question of Ulam, the problem for the case of approximately additive mappings, when G_1 and G_2 are assumed to be Banach spaces. The result of Hyers is stated in the following celebrated Theorem.

Theorem 1.2. *(Hyers [2]) Assume that G_1 and G_2 are Banach spaces. If a function $f : G_1 \to G_2$ satisfies the inequality $\|f(x+y) - f(x) - f(y)\| \le \epsilon$ for some $\epsilon > 0$ and for all $x, y \in G_1$, then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

exists for each $x \in G_1$ and $A : G_1 \to G_2$ is the unique additive function such that

$$\|f(x) - A(x)\| \le \epsilon \tag{1.1}$$

for all $x \in G_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in G_1$, then A is linear.

Taking the above fact into account, the additive functional equation

$$f(x+y) = f(x) + f(y)$$

is said to have the *Hyers-Ulam stability* on (G_1, G_2). In the above Theorem, an additive function A satisfying inequality (1.1) is constructed directly from the given function f and it is the most powerful tool to study the stability of several functional equations. In course of time, the theorem formulated by Hyers was generalized by Aoki [3] for additive mappings.

There is no reason for the Cauchy difference $f(x+y) - f(x) - f(y)$ to be bounded as in the expression of (1.1). Towards this point, in the year of 1978, Rassias [4] tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the Hyers-Ulam-stability for the Additive Cauchy Equation. This terminology is justified because the theorem of Rassias has strongly influenced Mathematicians studying stability problems of functional equation. In fact, Rassias proved the following Theorem.

Theorem 1.3. *(Rassias [4]) Let X and Y be Banach spaces. Let $\theta \in (0, \infty)$ and let $p \in [0, 1)$. If a function $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \theta \left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2 - 2^p} \|x\|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

The findings of Rassias have exercised a delectable influence on the development of what is addressed as the generalized Hyers-Ulam stability of functional equations. For decades, many researchers have extended the theory of the Hyers-Ulam stability to other functional equations, and generalized the Hyers result in different directions (See, for example, [5–11]).

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', ...x^{(n)}\right) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function x such that

$$\left|\phi\left(f, x, x', x'', ...x^{(n)}\right)\right| \leq \epsilon,$$

there exists a solution x_a of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon\to 0} K(\epsilon) = 0$. If the preceding statement is also true when we replace ϵ and $K(\epsilon)$ by $\phi(t)$ and $\varphi(t)$, where ϕ, φ are appropriate functions not depending on x and x_a explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability.

Obloza seems to be the first author who investigated the Hyers-Ulam stability of linear differential equations [12, 13]. In 1998, Alsina and Ger [14] first investigated the Hyers-Ulam stability of differential equations. They proved in [14] the following Theorem.

Theorem 1.4. *Assume that a differentiable function $f : I \to R$ is a solution of the differential inequality $\|x'(t) - x(t)\| \leq \epsilon$, where I is an open sub interval of $\mathbb{R}$. Then there exists a solution $g : I \to R$ of the differential equation $x'(t) = x(t)$ such that for any $t \in I$, we have $\|f(t) - g(t)\| \leq 3\epsilon$.*

This result of C. Alsina and R. Ger [14] has been generalized by Takahasi [15]. They proved in [15] that the Hyers-Ulam stability holds true for the Banach Space valued differential equation $y'(t) = \lambda y(t)$. Indeed, the Hyers-Ulam stability has been proved for the first order linear differential equations in more general settings [16–20].

In 2006, Jung [21] investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients by using matrix method. In 2007, Wang, Zhou and Sun [22] studied the Hyers-Ulam stability of a class of first-order linear differential equations. Rus [23] discussed four types of Ulam stability: Ulam-Hyers stability, Generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and Generalized Ulam-Hyers-Rassias stability of the Ordinary Differential Equation

$$u'(t) = A(u(t)) + f(t, u(t)), t \in [a, b].$$

In 2014, Alqifiary and Jung [24] proved the Generalized Hyers-Ulam stability of linear differential equation of the form

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k \, x^{(k)}(t) \;=\; f(t)$$

by using the Laplace Transform method, where α_k are scalars and x and f are n times continuously differentiable function and of the exponential order, respectively. Recently, the Ulam stability of first order, second order and third order differential equations were investigated in a series of papers [25–35] and the investigation is going on.

In 2019, Murali and Ponmana Selvan [36] investigated the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of a homogeneous and nonhomogeneous general linear differential equations of first order and second order by using Fourier transform method. Very recently, Rassias, Murali and Ponmana Selvan [37] established the Mittag-Leffler-Hyers-Ulam stability of homogeneous and nonhomogeneous linear differential equations of first order and second order by using Fourier transform method.

Motivated by the above results, by using Laplace Transforms Method, we study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of a general homogeneous and non-homogeneous linear differential equations of first order

$$u'(t) + l\ u(t) = 0 \tag{1.2}$$

and

$$u'(t) + l\ u(t) = r(t) \tag{1.3}$$

for all $t \in I$, $u(t) \in C(I)$ and $r(t) \in C(I)$ where $I = [a, b]$, $-\infty < a < b < \infty$.

2. Preliminaries

In this section, we introduce some notations, definitions and preliminaries which are used throughout this paper.

Throughout this paper, $\mathbb{F}$ denotes the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : (0, \infty) \to \mathbb{F}$ of exponential order if there exists a constants $M(> 0) \in \mathbb{R}$ such that $|f(t)| \leq Me^{at}$ for all $t > 0$. For each function $f : (0, \infty) \to \mathbb{F}$ of exponential order, we define the Laplace Transform of f by

$$F(s) = \int_0^\infty f(t)\ e^{-st}\ dt.$$

The Laplace transform of f is sometimes denoted by $\mathcal{L}(f)$. It is also well known that $\mathcal{L}$ is linear and one-to-one. Then, at points of continuity of f, we have

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\alpha-iT}^{\alpha+iT} F(s)e^{st}\ ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\alpha+iy} F(\alpha+iy)\ dy, \end{aligned}$$

which is called the inverse Laplace transforms.

Definition 2.1. (Convolution). Given two functions f and g, both are Lebesgue integrable on $(-\infty, +\infty)$. Let S denote the set of x for which the Lebesgue integral

$$h(x) = \int_{-\infty}^{\infty} f(t)\ g(x-t)\ dt$$

exists. This integral defines a function h on S called the convolution of f and g. We also write $h = f * g$ to denote this function.

Theorem 2.2. *The Laplace transform of the convolution of $f(x)$ and $g(x)$ is the product of the Laplace transform of $f(x)$ and $g(x)$. That is,*

$$\mathcal{L}\{f(x) * g(x)\} = \mathcal{L}\{f(x)\}\ \mathcal{L}\{g(x)\} = F(s)\ G(s)$$

or

$$\mathcal{L}\left\{\int_0^\infty f(t)\ g(x-t)\ dt\right\} = \mathcal{L}(f)\mathcal{L}(g) = F(s)\ G(s),$$

where $F(s)$ and $G(s)$ are Laplace transform of $f(x)$ and $g(x)$ respectively.

Definition 2.3. [38] The Mittag-Leffler function of one parameter is denoted by $E_\alpha(z)$ and defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + 1)} z^k$$

where $z, \alpha \in \mathbb{C}$ and $Re(\alpha) > 0$. If we put $\alpha = 1$, then the above equation becomes

$$E_1(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)} z^k = \sum_{k=0}^{\infty} \frac{z^k}{k} = e^z.$$

Definition 2.4. [38] The generalization of $E_\alpha(z)$ is defined as a function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\alpha k + \beta)} z^k$$

where $z, \alpha, \beta \in \mathbb{C}$, $Re(\alpha) > 0$ and $Re(\beta) > 0$.

Let $I, J \subseteq \mathbb{R}$. Throughout this paper, we denote the space of k continuously differentiable functions from I to J by $C^k(I, J)$ and denote $C^k(I, I)$ by $C^k(I)$. Furthermore, $C(I, J) = C^0(I, J)$ denotes the space of continuous functions from I to J. In addition, $\mathbb{R}_+ := [0, \infty)$. From now on, we assume that $I = [a, b]$, where $-\infty < a < b < \infty$.

We firstly give some definitions of various forms of Mittag-Leffler-Hyers-Ulam stability of the first order differential equations (1.2) and (1.3).

Definition 2.5. We say that the differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam stability, if there exists a positive constant K satisfies the following conditions: For every $\epsilon > 0$ and there exists $u(t) \in C(I)$ satisfying the inequality

$$|u'(t) + lu(t)| \leq \epsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C(I)$ satisfying $v'(t) + l\ v(t) = 0$ such that

$$|u(t) - v(t)| \leq K\epsilon E_\alpha(t),$$

for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for (1.2).

Definition 2.6. We say that the differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam stability, if there exists a positive constant K satisfies the following conditions: For every $\epsilon > 0$ and there exists $u(t) \in C(I)$ satisfying the inequality

$$|u'(t) + lu(t) - r(t)| \leq \epsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C(I)$ satisfying the linear differential equation $v'(t) + l\ v(t) = r(t)$ such that

$$|u(t) - v(t)| \leq K\epsilon E_\alpha(t),$$

for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam stability constant for (1.3).

Definition 2.7. We say that the differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability with respect to $\phi : (0, \infty) \to (0, \infty)$, if there exists a positive constant K satisfies the following conditions: For every $\epsilon > 0$ and there exists $u(t) \in C(I)$ satisfying the inequality

$$|u'(t) + lu(t)| \leq \phi(t)\epsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C(I)$ satisfies $v'(t) + l\ v(t) = 0$ such that

$$|u(t) - v(t)| \leq K\phi(t)\epsilon E_\alpha(t),$$

for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for (1.2).

Definition 2.8. We say that the differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability with respect to $\phi : (0, \infty) \to (0, \infty)$, if there exists a positive constant K satisfies the following conditions: For every $\epsilon > 0$ and there exists $u(t) \in C(I)$ satisfying the inequality

$$|u'(t) + lu(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t),$$

for all $t \in I$. Then there exists a solution $v \in C(I)$ satisfies the linear differential equation $v'(t) + l\ v(t) = r(t)$ such that $|u(t) - v(t)| \leq K\phi(t)\epsilon E_\alpha(t)$, for all $t \in I$. We call such K as the Mittag-Leffler-Hyers-Ulam-Rassias stability constant for (1.3).

3. Mittag-Leffler-Hyers-Ulam Stabilities for (1.2)

In this section, we prove the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the first order differential equation (1.2) by using the Laplace transforms.

Theorem 3.1. *The differential equation* (1.2) *is Mittag-Leffler-Hyers-Ulam stable.*

Proof. Given $\epsilon > 0$, we suppose that $u(t) \in C(I)$ satisfies

$$|u'(t) + lu(t)| \leq \epsilon E_\alpha(t), \tag{3.1}$$

for all $t \in I$. We aim to prove that there exists real number $K > 0$, which is independent of ϵ and u, such that $|u(t) - v(t)| \leq K\epsilon E_\alpha(t)$, for some $v \in C(I)$ satisfying $v'(t) + lv(t) = 0$ for all $t \in I$. Define a function $p : (0, \infty) \to \mathbb{R}$ such that $p(t) =: u'(t) + lu(t)$ for all $t > 0$. In view of (3.1), we have $|p(t)| \leq \epsilon E_\alpha(t)$. Taking Laplace transform to $p(t)$, we have

$$\mathcal{L}\{p\} = (s+l)\mathcal{L}\{u\} - \ u(0). \tag{3.2}$$

Thus

$$\mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + u(0)}{s+l}. \tag{3.3}$$

In view of the (3.2), a function $u_0 : (0, \infty) \longrightarrow \mathbb{R}$ is a solution of (1.2) if and only if

$$(s+l)\mathcal{L}\{u_0\} - u_0(0) = 0.$$

Setting $v(t) = u(0)\ e^{-lt}$, we have $v(0) = u(0)$. Taking Laplace transform to $v(t)$, we obtain

$$\mathcal{L}\{v\} = \frac{u(0)}{(s+l)}. \tag{3.4}$$

On the other hand,

$$\mathcal{L}\{v'(t) + l\ v(t)\} = (s+l)\mathcal{L}\{v\} - v(0).$$

Using (3.4), we get $\mathcal{L}\{v'(t) + l\ v(t)\} = 0$. Since $\mathcal{L}$ is one-to-one operator and linear, then $v'(t) + l\ v(t) = 0$. This means that $v(t)$ is a solution of (1.2). It follows from (3.3) and (3.4) that

$$\begin{aligned}\mathcal{L}\{u\} - \mathcal{L}\{v\} &= \frac{\mathcal{L}\{p\} + u(0)}{(s+l)} - \frac{u(0)}{(s+l)} = \frac{\mathcal{L}\{p\}}{(s+l)} \\ \mathcal{L}\{u(t) - v(t)\} &= \mathcal{L}\left\{p(t) * e^{-lt}\right\}.\end{aligned}$$

The above equalities show that

$$u(t) - v(t) = p(t) * e^{-lt}.$$

Taking modulus on both sides and using $|p(t)| \leq \epsilon E_\alpha(t)$, we get

$$\begin{aligned}|u(t) - v(t)| = \left|p(t) * e^{-lt}\right| &\leq \left|\int_0^t p(t)\ e^{-l(t-x)}\ dx\right| \\ &\leq |p(t)| \left|\int_0^t e^{-l(t-x)}\ dx\right| \\ &\leq \epsilon E_\alpha(t) \left|\int_0^t e^{-l(t-x)}\ dx\right|\end{aligned}$$

for all $t > 0$, where $K = \left|\int\limits_0^t e^{-l(t-x)}\ dx\right|$ exists. Hence, $|u(t) - v(t)| \leq K\epsilon E_\alpha(t)$. By the virtue of Definition 2.5, the linear differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam stability. This completes the proof. □

By using the same technique in Theorem 3.1, we can also prove that the following theorem, which shows the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (1.2). The method of proof is similar, but we still state it for the sake of completeness.

Theorem 3.2. *The linear differential equation* (1.2) *has the Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Given $\epsilon > 0$, we suppose that $u(t) \in C(I)$ and $\phi(t) : (0,\infty) \to (0,\infty)$ satisfies

$$|u'(t) + lu(t)| \leq \phi(t)\epsilon E_\alpha(t), \tag{3.5}$$

for all $t \in I$. We aim to prove that there exists real number $K > 0$, which is independent of ϵ and u such that $|u(t) - v(t)| \leq K\phi(t)\epsilon E_\alpha(t)$, for some $v \in C(I)$ satisfies $v'(t) + lv(t) = 0$ for all $t \in I$. Define a function $p : (0,\infty) \to \mathbb{R}$ such that $p(t) =: u'(t) + lu(t)$ for all $t > 0$. In view of (3.5), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t)$. Taking Laplace transform to $p(t)$, we have

$$\mathcal{L}\{p\} = (s + l)\mathcal{L}\{u\} - \ u(0), \tag{3.6}$$

and thus

$$\mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + u(0)}{s + l}. \tag{3.7}$$

In view of the (3.6), a function $u_0 : (0,\infty) \longrightarrow \mathbb{R}$ is a solution of (1.2) if and only if

$$(s + l)\mathcal{L}\{u_0\} - u_0(0) = 0.$$

Setting $v(t) = u(0)\ e^{-lt}$, we have $v(0) = u(0)$. Taking Laplace transform to $v(t)$, we obtain

$$\mathcal{L}\{v\} = \frac{u(0)}{(s + l)}. \tag{3.8}$$

On the other hand,

$$\mathcal{L}\{v'(t) + l\ v(t)\} = (s + l)\mathcal{L}\{v\} - v(0).$$

Using (3.8), we get $\mathcal{L}\{v'(t) + l\ v(t)\} = 0$. Since $\mathcal{L}$ is one-to-one operator and linear, then t $v'(t) + l\ v(t) = 0$. This means that $v(t)$ is a solution of (1.2). It follows from (3.7) and (3.8) that

$$\mathcal{L}\{u\} - \mathcal{L}\{v\} = \frac{\mathcal{L}\{p\} + u(0)}{(s + l)} - \frac{u(0)}{(s + l)} = \frac{\mathcal{L}\{p\}}{(s + l)}$$
$$\mathcal{L}\{u(t) - v(t)\} = \mathcal{L}\left\{p(t) * e^{-lt}\right\}.$$

The above equalities show that

$$u(t) - v(t) = p(t) * e^{-lt}.$$

Taking modulus on both sides and using $|p(t)| \leq \phi(t)\epsilon E_\alpha(t)$, we get

$$\begin{aligned}
|u(t) - v(t)| = \left|p(t) * e^{-lt}\right| &\leq \left|\int_0^t p(t)\ e^{-l(t-x)}\ dx\right| \\
&\leq |p(t)|\ \left|\int_0^t e^{-l(t-x)}\ dx\right| \\
&\leq \phi(t)\epsilon E_\alpha(t)\ \left|\int_0^t e^{-l(t-x)}\ dx\right|
\end{aligned}$$

for all $t > 0$, where $K = \left|\int_0^t e^{-l(t-x)}\ dx\right|$ exists. Hence, $|u(t) - v(t)| \leq K\phi(t)\epsilon E_\alpha(t)$. By the virtue of Definition 2.7, the linear differential equation (1.2) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. This completes the proof. □

4. Mittag-Leffler-Hyers-Ulam Stabilities for (1.3)

In this section, we investigate the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (1.3). First, we prove the Mittag-Leffler-Hyers-Ulam stability of the non-homogeneous linear differential equation (1.3).

Theorem 4.1. *The differential equation* (1.3) *has Mittag-Leffler-Hyers-Ulam stability.*

Proof. Given $\epsilon > 0$, we suppose that $u(t) \in C(I)$ satisfies

$$|u'(t) + l\ u(t) - r(t)| \leq \epsilon E_\alpha(t), \tag{4.1}$$

for all $t \in I$. We aim to prove that there exists real number $K > 0$, which is independent of ϵ and $u(t)$ such that $|u(t) - v(t)| \leq K\epsilon E_\alpha(t)$, for some $v \in C(I)$ satisfies $v'(t) + lv(t) = r(t)$ for all $t \in I$. Define a function $p : (0, \infty) \to \mathbb{R}$ such that $p(t) =: u'(t) + l\ u(t) - r(t)$ for all $t > 0$. In view of (4.1), we have $|p(t)| \leq \epsilon E_\alpha(t)$. Taking Laplace transform to $p(t)$, we have

$$\mathcal{L}\{p\} = (s + l)\mathcal{L}\{u\} - \ u(0) - \mathcal{L}(r), \tag{4.2}$$

and thus

$$\mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + u(0) + \mathcal{L}(r)}{s + l}. \tag{4.3}$$

In view of the (4.2), a function $u_0 : (0, \infty) \longrightarrow \mathbb{R}$ is a solution of (1.3) if and only if

$$(s + l)\mathcal{L}\{u_0\} - u_0(0) = \mathcal{L}(r).$$

Set $v(t) = u(0)\ e^{-lt} + (q * r)(t)$, where $q(t) = e^{-lt}$. Then we have $v(0) = u(0)$. Taking Laplace transform to $v(t)$, we obtain

$$\mathcal{L}\{v\} = \frac{u(0) + \mathcal{L}(r)}{(s + l)} = \frac{v(0) + \mathcal{L}(r)}{(s + l)}. \tag{4.4}$$

On the other hand,

$$\mathcal{L}\{v'(t) + l\ v(t) - r(t)\} = (s + l)\mathcal{L}\{v\} - v(0) - \mathcal{L}(r).$$

Using (4.4), we get $\mathcal{L}\{v'(t) + l\ v(t) - r(t)\} = 0$. Since $\mathcal{L}$ is one-to-one operator and linear, then

$$v'(t) + l\ v(t) = r(t).$$

This means that $v(t)$ is a solution of (1.3). It follows from (4.3) and (4.4) that

$$\mathcal{L}\{u\} - \mathcal{L}\{v\} = \frac{\mathcal{L}\{p\} + u(0) + \mathcal{L}(r)}{(s + l)} - \frac{u(0) + \mathcal{L}(r)}{(s + l)} = \frac{\mathcal{L}\{p\}}{(s + l)}.$$

The above equalities show that

$$\mathcal{L}\{u(t) - v(t)\} = \mathcal{L}\left\{p(t) * q(t)\right\},$$

then it gives that $u(t)-v(t)=(p*q)(t)$. Taking modulus on both sides and using $|p(t)| \leq \epsilon E_\alpha(t)$, we get

$$\begin{aligned} |u(t)-v(t)| = |(p*q)(t)| &\leq \left| \int_0^t p(t)\ e^{-l(t-x)}\ dx \right| \\ &\leq |p(t)| \ \left| \int_0^t e^{-l(t-x)}\ dx \right| \\ &\leq \epsilon E_\alpha(t) \ \left| \int_0^t e^{-l(t-x)}\ dx \right| \end{aligned}$$

for all $t>0$, where $K = \left| \int\limits_0^t e^{-l(t-x)}\ dx \right|$ exists for $t>0$. Then we have

$$|u(t)-v(t)| \leq K\epsilon E_\alpha(t).$$

By the virtue of Definition 2.6, the linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam stability. This completes the proof. □

In analogous to Theorem 4.1, we have the following result which shows the Mittag-Leffler-Hyers-Ulam-Rassias stability of the differential equation (1.3).

Theorem 4.2. *The non-homogeneous linear differential equation* (1.3) *has Mittag-Leffler-Hyers-Ulam-Rassias stability.*

Proof. Given $\epsilon > 0$, we suppose that $u(t) \in C(I)$ and $\phi(t) : (0,\infty) \to (0,\infty)$ satisfies

$$|u'(t) + l\ u(t) - r(t)| \leq \phi(t)\epsilon E_\alpha(t), \tag{4.5}$$

for all $t \in I$. We aim to prove that there exists real number $K > 0$ which is independent of ϵ and u such that $|u(t)-v(t)| \leq K\phi(t)\epsilon E_\alpha(t)$, for some $v \in C(I)$ satisfies $v'(t) + lv(t) = r(t)$ for all $t \in I$. Define a function $p : (0,\infty) \to \mathbb{R}$ such that $p(t) =: u'(t) + l\ u(t) - r(t)$ for all $t > 0$. In view of (4.5), we have $|p(t)| \leq \phi(t)\epsilon E_\alpha(t)$. Taking Laplace transform from $p(t)$, we have

$$\mathcal{L}\{p\} = (s+l)\mathcal{L}\{u\} -\ u(0) - \mathcal{L}(r), \tag{4.6}$$

and thus

$$\mathcal{L}\{u\} = \frac{\mathcal{L}\{p\} + u(0) + \mathcal{L}(r)}{s+l}. \tag{4.7}$$

In view of the (4.6), a function $u_0 : (0,\infty) \longrightarrow \mathbb{R}$ is a solution of (1.3) if and only if

$$(s+l)\mathcal{L}\{u_0\} - u_0(0) = \mathcal{L}(r).$$

Set $v(t) = u(0)\ e^{-lt} + (q*r)(t)$, where $q(t) = e^{-lt}$. Then we have $v(0) = u(0)$. Taking Laplace transform to $v(t)$, we obtain

$$\mathcal{L}\{v\} = \frac{u(0) + \mathcal{L}(r)}{(s+l)} = \frac{v(0) + \mathcal{L}(r)}{(s+l)}. \tag{4.8}$$

On the other hand,

$$\mathcal{L}\{v'(t) + l\ v(t) - r(t)\} = (s+l)\mathcal{L}\{v\} - v(0) - \mathcal{L}(r).$$

Using (4.4), we get

$$\mathcal{L}\{v'(t) + l\ v(t) - r(t)\} = 0.$$

Since $\mathcal{L}$ is one-to-one operator and linear, then we get $v'(t) + l\ v(t) = r(t)$. This means that $v(t)$ is a solution of (1.3). It follows from (4.7) and (4.8) that

$$\mathcal{L}\{u\} - \mathcal{L}\{v\} = \frac{\mathcal{L}\{p\} + u(0) + \mathcal{L}(r)}{(s+l)} - \frac{u(0) + \mathcal{L}(r)}{(s+l)} = \frac{\mathcal{L}\{p\}}{(s+l)}.$$

The above equalities show that

$$u(t) - v(t) = (p * q)(t).$$

Taking modulus on both sides and using $|p(t)| \le \phi(t)\epsilon E_\alpha(t)$, we get

$$\begin{aligned} |u(t) - v(t)| &\le \left| \int_0^t p(t)\ e^{-l(t-x)}\ dx \right| \\ &\le |p(t)| \left| \int_0^t e^{-l(t-x)}\ dx \right| \\ &\le \phi(t)\epsilon E_\alpha(t) \left| \int_0^t e^{-l(t-x)}\ dx \right| \end{aligned}$$

for all $t > 0$, where $K = \left| \int\limits_0^t e^{-l(t-x)}\ dx \right|$ exists. Hence,

$$|u(t) - v(t)| \le K\phi(t)\epsilon E_\alpha(t).$$

By the virtue of Definition 2.8, the linear differential equation (1.3) has the Mittag-Leffler-Hyers-Ulam-Rassias stability. This completes the proof. □

Conclusion

In this paper, we proved the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equations of first order with constant co-efficient using the Laplace Transforms method. That is, we established the sufficient criteria for Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation of first order with constant co-efficient using Laplace Transforms method. Additionally, this paper also provides another method to study the Mittag-Leffler-Hyers-Ulam stability of differential equations. Also, this paper shows that the Laplace Transform method is more convenient to study the Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam-Rassias stability of the linear differential equation with constant co-efficients.

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