Asymptotic Performance of Dual-hop Non-regenerative Cooperative Systems With or Without Direct Path

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Abstract—We derive simple asymptotic expressions for the outage probability (OP) and the average error probability (AEP) of dual-hop fixed-gain amplify-and-forward relaying system with or without direct path in a Rayleigh fading environment and high signal-to-noise ratio (SNR). We show that, when direct path is not utilized, both OP and EP decay as \( \log(SNR)/SNR \), whereas, when the direct path is utilized, OP and EP decay as \( \log(SNR)/SNR^2 \) at high SNRs.

Keywords—Outage probability, error probability, Rayleigh fading, diversity order

I. INTRODUCTION

The concept of cooperative communications is considered to be one of the building blocks for the next generation wireless communication systems, such as LTE Advanced [1]-[2]. The cooperation is realized by relaying the information between the source and destination via a partner node (relay), which is willing to share its limited resources to facilitate the source-destination communication. It is proven that cooperation significantly improves the system capacity (sum rate) and/or system reliability because it introduces cooperative diversity into the system [1]-[2]. The simplest relaying technique is the amplify-and-forward (AF) method (also known as the non-regenerative method), where the relay amplifies the broadcasted signal from the source and retransmits it to the destination, without decoding or detecting the relayed information. The AF relaying utilizes either variable-gain relays [4]-[6] or fixed-gain relays [3], [7]-[11]. The variable-gain relaying requires the relay’s knowledge of the channel state information on at least the source-relay link, while the fixed-gain relaying does not require such information. In this paper, we focus on the AF systems with fixed-gain relays because of their simplicity and performance close to the variable-gain relay systems given the fixed gain relaying factor is properly chosen [3].

In recent years, many variants of fixed-gain AF wireless systems have been studied, ranging from dual-hop [3] to multi-hop [6]-[7] systems, systems with single-relay [3], [9]-[11], or multiple relay [8] at any given hop, for various fading channels, such as, Rayleigh [3], [7] or Nakagami-m channels [9]-[11].

Part of these works present exact performance studies of these systems by using relatively complex analytical expressions for the outage probabilities (OP) and average error probabilities (AEP) [3], [6], [8]-[9]. Other important group of works study these systems for high signal-to-noise ratios (SNR), which is particularly important because such asymptotic analysis typically results in relatively simple yet accurate approximate expressions for moderate and high average SNRs [9]-[11]. These high SNR approximations are important because they explain how the system depends on the crucial system parameters (such as mean per-hop SNR, number of relays and/or the channel's fading parameters), which is very important for the system design [12].

In this paper, we present simple high SNR approximations for the OP and the AEP of dual-hop AF systems with fixed-gain relaying that operate over Rayleigh fading channels. We analyze two scenarios: (1) the source and the destination communicate only through the relay path (single diversity scenario), and (2) the source and the destination communicate over the direct path and the relay path (dual diversity scenario). The derived expressions prove that the presence of the direct path improves the OP and AEP asymptotic performances but the diversity order is less than 2 [11].

II. SYSTEM MODEL

We consider a dual-hop wireless communication system where the communication between two the source (A) and the destination (C) can be realized over two independent paths: the direct path between A and C, and the dual hop relay path traversing over the node B (Fig. 1). The node B is a half-duplex relay, which utilizes amplify and forward strategy with fixed gain G. The fixed gain G can be selected according to different criteria, two of which are maintaining (1) fixed transmit power of the relay or (2) same average fixed gain (i.e. semi-blind relays). The communication protocol is divided in two successive phases: During the first phase, the node A transmits (broadcasts) its information block towards the destination C and/or the relay B; During the second phase, the relay B forwards the received signal from A towards C.

In section III, we consider the single diversity scenario, where only the relay path is utilized (in absence of the direct path). Section IV considers the dual diversity scenario, where both the direct path and the relay path are utilized as C combines the incoming signals from A and B by using maximal ratio combination (MRC).
We assume that the source node A transmits the information block $x(t)$ with an average power $E_t$ during the first phase. The fading statistics are assumed to be slowly varying, so the random fading amplitudes remain static for the duration of each information block $x(t)$. The signals received at nodes B and C are respectively given by

$$y_b(t) = \alpha_b x(t) + n_b(t)$$

$$y_c(t) = \alpha_c x(t) + n_c(t)$$

where $\alpha_b$ and $\alpha_c$ are the Rayleigh fading amplitude of the A-C and A-B links, respectively, with mean squared values $E[\alpha^2_b] = \Omega_b$ and $E[\alpha^2_c] = \Omega_c$. In (1), $n_b(t)$ and $n_c(t)$ are the AWGN at the input of the nodes C and B, respectively, with mean power $N_0$ and $N_0$, respectively. In the second phase, the signal is then amplified by the fixed gain $G$ and forwarded to C. The received signal at C is given by

$$y_{c1}(t) = \alpha_c G y_b(t) + n_2(t)$$

$$= \alpha_c G (\alpha_b x(t) + n_b(t)) + n_2(t)$$

where $\alpha_c$ is the Rayleigh fading amplitude of the B-C link with mean squared value $E[\alpha^2_c] = \Omega_c$, and $n_2(t)$ is the AWGN at the input of C with mean power $N_0$.

III. ASYMPOTIC PERFORMANCE OF THE SINGLE DIVERSITY SCENARIO

The instantaneous SNR of the signal received at C traversing the relay path is given by [3, Eq. (3)],

$$\gamma_1 = \frac{\alpha_b^2 G^2 E_t}{\alpha_c^2 G^2 N_{01} + N_{02}}$$

which can be rewritten as

$$\gamma_1 = \Gamma \frac{XY}{Y + 1}$$

where $\Gamma = E_t / N_0$ is denoted as the transmit SNR. In (4), $X$ and $Y$ are random variables defined as

$$X = \alpha_b^2$$

$$Y = G^2 \alpha_c^2 N_{01} / N_{02}$$

both of which follow the exponential distribution with mean values $E[X] = \Omega_b$ and $E[Y] = G^2 \Omega_c$, respectively. Without loss in generality, we assume $N_{01} = N_{02} = N_0$.

A. Outage Probability

The OP is defined as the probability that the instantaneous SNR before the receiver C is below some predefined threshold $\gamma_{th}$. The particular value of $\gamma_{th}$ depends on the particular implementation of the receiver in C. In terms of $\gamma_1$, the OP is actually the cumulative distribution function (CDF) of $\gamma_1$ evaluated at threshold $\gamma_{th}$,

$$P_{out} = F_{\gamma_1}(\gamma_{th}) = \Pr\{\gamma \leq \gamma_{th}\}$$

The analytical expression for the OP of the single diversity scenario is well known and is given in closed form as [3, Eq. (9)]

$$P_{out,1} = 1 - 2 \sqrt{\frac{\gamma_{th}}{G^2 \Gamma \Omega_1 \Omega_2}} \exp\left(-\frac{\gamma_{th}}{\Gamma \Omega_1}\right) K_1\left(2 \sqrt{\frac{\gamma_{th}}{G^2 \Gamma \Omega_1 \Omega_2}}\right)$$

where $K_1(\cdot)$ is the first-order modified Bessel function of the second kind [13, Eq. (8.432)].

Based on (7), it is not easy to develop an intuition on the behavior of the OP in function of each of the involved parameters; for example, one cannot decide whether the OP is increasing or decreasing with some input parameter, or how sensitive is OP from the changes of the threshold and/or the channel gains. For this, we need to draw curves for various input parameters. So it is important to have simple expressions from which is easy to define the performances of the system. However, we expect that increasing the transmit SNR and decreasing the outage threshold indicates decreasing outage probability. Next we will show a simple relationship between the above mentioned parameters and the OP when the SNR is high.

For the purpose of deriving our high SNR approximations, we use the power series expansion of the function $K_1(\cdot)$ [13, Eq. (8.446)]

$$K_1(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2^{2n+1} n! (n+1)!} \left[\log\left(\frac{z}{2}\right) + \mu\right] - \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{2^{2n+1} n! (n+1)!} \left[\sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n+1} \frac{1}{k} \right] + \frac{1}{z},$$

where $\mu = 0.5772\ldots$ is the Euler’s constant and $\log(x)$ is natural logarithm.

Since as $\Gamma \rightarrow \infty$, $z \rightarrow 0$, we consider only the first terms of the sum that appears in (8), i.e., we keep only the term for $n = 0$. Thus,

$$K_1(z) \approx \frac{1}{z} + \frac{z}{2} \left[\log\left(\frac{z}{2}\right) + \mu\right]$$

Other terms can be neglected because they are almost negligible at high SNR. We also use McLauren series...
expansion for exponential function. Inserting these approximations, after some algebraic manipulations, (7) at high SNR is tightly approximated as

\[ P_{\text{out},1} \approx a \gamma_{th}^{\gamma_{th}} + b \gamma_{th} \log \left( \frac{1}{b} \gamma_{th} \right) . \]  

(10)

where

\[ a = \frac{1}{\Omega_1} - (2\mu - 1)b \]

\[ b = \frac{1}{G^2 \Omega_1 \Omega_2} . \]

(11)

It is important to notice that the system behavior can be determined from this equation. As we expected, if we decrease \( \gamma_{th} \), the OP also decreases and if we increase the average SNR, the outage probability decreases. These results are validated by Monte Carlo simulations.

**B. Average Error Probability**

The error probability of a communication system depends on the modulation/demodulation schemes. The error probability of a binary linear modulation and coherent demodulation in an uncoded system is expressed in the form \( Q(c \gamma_{1}) \), where

\[ Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} \exp \left( -\frac{t^2}{2} \right) dt , \]

(12)

and \( c \) depends on the modulation/demodulation scheme (e.g., \( c = 2 \) for BPSK and coherent BFSK).

In presence of slow channel fading, the received SNR \( \gamma_{t} \) is a slowly fading random variable, in which case the system performance is expressed as the average error probability (AEP). The AEP is determined as

\[ P_{\text{e},1} = E_{\gamma_{t}}[Q(c \gamma_{1})] . \]

(13)

where the \( Q \) function is averaged with respect to the instantaneous SNR given by (3). Eq. (13) can be alternatively expressed as [14, Eq. (8)]

\[ P_{\text{e},1} = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} F_{\gamma_{1}} \left( \frac{x^2}{c} \right) e^{-\frac{x^2}{2}} dx = \frac{1}{2} \Pr \left( X^2 > c \gamma_{1} \right) = \frac{1}{2} \Pr \left( X > \sqrt{c \gamma_{1}} \right) = \frac{1}{2} \Pr \left( X > \Gamma Z \right) . \]

(14)

As a reference for the tightness of our AEP approximation, firstly we derive the exact AEP by introducing (7) into (14) and applying [13, Eqs. (3.461.2) and (6.614.5)], which yields a closed form solution, as

\[ \bar{P}_{e,1} = \frac{1}{2} \Pr \left( X > \Gamma Z \right) = \frac{1}{2} \Pr \left( X > \gamma_{th} / \Gamma \right) = \frac{1}{2} \left( 1 - Q \left( \frac{\gamma_{th}}{\Gamma} \right) \right) . \]

(15)

Combining (10) and (14) and solving some elementary integrals according to the procedure presented in the Appendix, we obtain the approximation for the AEP for the system of Fig. 1 (without a direct path) under high SNR as

\[ P_{\text{e},1} \approx \frac{1}{2} \Pr \left( X > \gamma_{th} / \Gamma \right) \approx \frac{1}{2} \left( 1 - Q \left( \frac{\gamma_{th}}{\Gamma} \right) \right) . \]

(16)

**IV. ASYMPTOTIC PERFORMANCE OF THE DUAL DIVERSITY SCENARIO**

The instantaneous SNR at the input of C, which originates from the A-C link is

\[ \gamma_{0} = \frac{\alpha^2 Z}{N_{0}} = \frac{\Gamma Z}{N_{0}} . \]

(17)

where \( Z = \alpha^2 Z \) is an exponential random variable with mean \( E[Z] = \Omega_{0} \).

In the dual diversity scenario, the signal transmitted from A and received at C over the direct path (during the first phase) and the signal transmitted from B and received at C (during the second phase) are combined using MRC. Thus, the total received SNR is a sum of instantaneous SNRs of signals over each of the diversity paths, i.e.,

\[ \gamma = \gamma_{1} + \gamma_{0} = \Gamma \left( \frac{XY}{Y + 1} + Z \right) . \]

(18)

**A. Outage Probability**

The OP is given by

\[ P_{\text{out},2} = P \left( \gamma < \gamma_{th} \right) = P \left\{ \frac{XY}{Y + 1} + Z < \delta \right\} . \]

(19)

where \( \delta = \gamma_{th} / \Gamma \). Note, \( \delta \rightarrow 0 \) when \( \Gamma \rightarrow \infty \). Since \( Z \) is an exponentially distributed random variable with mean \( \Omega_{0} \), we have

\[ P_{\text{out},2} = \frac{1}{\Omega_{0}} \int_{0}^{\infty} F_{\gamma_{1}} \left( \frac{x}{\Omega_{0}} \right) e^{-\frac{x}{\Omega_{0}}} dx = \frac{1}{\Omega_{0}} \int_{0}^{\delta} \frac{1}{\Omega_{0}} \exp \left( -\frac{z}{\Omega_{0}} \right) dz . \]

(20)

Note, when \( \delta \rightarrow 0 \), the exponential function \( \exp(-z / \Omega_{0}) \) is nearly 1 in the tight integration interval \((0, \delta)\). By using [15 Eq. (37)] and after some simple algebraic manipulations, (20) can be tightly approximated as
\[ P_{\text{out},2} \approx \frac{\gamma_{th}^2}{2\Omega_0 \Gamma^2} \left( a + \frac{b}{2} \right) - \frac{\gamma_{th}^2 b}{2\Omega_0 \Gamma^2} \log \left( \frac{b \gamma_{th}}{\Gamma} \right), \]  

where \( a \) and \( b \) are defined according to (11).

B. Average Error Probability

To derive the AEP for the dual diversity scenario, we use the same procedure and the same definite integrals from [13]. Therefore, combining (14) and (21) we have

\[ \overline{\gamma} \approx \frac{1}{c^2 \Gamma^2 \Omega_0} \left( \frac{3a}{4} + \left( \frac{3\mu}{4} - \frac{13}{8} \right) b \right) + \frac{3b}{4c^2 \Gamma^2 \Omega_0} \log \left( \frac{2c\Gamma}{b} \right). \]  

V. NUMERICAL EXAMPLES

Figs. 2 and 3 illustrate the tightness of our approximations. The presented numerical example curves show an excellent match between the exact and the asymptotic OP and asymptotic AEP for three different values of the threshold parameter \( \gamma_{th} \) when both scenarios are considered.

![Fig. 2 Outage probability for three different values of \( \gamma_{th} \), when \( G=10, \Omega_1=\Omega_2=1 \)](image)

![Fig. 3 Average Error Probability for three different values of \( G \), when \( \Omega_1=\Omega_2=1 \)](image)

Figs. 2 and 3 show that the presence of two paths from the source to the destination improves the reliability of communication, i.e., for given transmit SNR, the OP is significantly decreased in the case of dual diversity scenario. More particularly, in the case of single diversity scenario, OP and AEP decay as \( \log(\Gamma)/\Gamma \), whereas, in the case of dual diversity scenario, OP and AEP decay as \( \log(\Gamma)/\Gamma^2 \) at high SNR. In contrast to what was expected, the diversity gain of the two considered scenarios is below 1 and 2, respectively. In the two scenarios, contrary to the expected, the diversity gain of the communication between the source and the destination is below 1 and 2, respectively.

VI. CONCLUSIONS

In this paper, we study the asymptotic behavior of the OP and the AEP of a dual hop relay system with fixed gain amplify-and-forward half-duplex relay with or without the presence of the direct link between the source and the destination. We derived simple approximations for these two performance parameters, which are very tight to the exact values for moderate and high average SNRs.

To the best of authors’ knowledge, the OP and AEP approximations are new, which, unlike other related results in literature, are valid for arbitrary fixed gain of the relay. It was shown that both the OP and AEP of the dual-hop fixed-gain AF relaying system in Rayleigh fading decay as \( \log(\Gamma)/\Gamma^d \), where \( d \) is the number of independent signal paths (i.e., \( d = 1 \) for the single diversity scenario and \( d = 2 \) for the dual diversity scenario).

APPENDIX

Combining (10) and (14), we have

\[ P_{\text{er}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} \, dx + \frac{1}{\sqrt{2\pi} c\Gamma} \int_{0}^{\infty} b x^2 \log \left( \frac{c\Gamma}{bx} \right) e^{-x^2/2} \, dx = \]

\[ = \frac{1}{\sqrt{2\pi} c\Gamma} \int_{0}^{\infty} x^2 e^{-x^2/2} \, dx - \frac{2b}{\sqrt{2\pi} c\Gamma} \int_{0}^{\infty} x^2 \log \left( \frac{b}{\sqrt{c\Gamma}} \right) e^{-x^2/2} \, dx \]  

(A.1)

The integral that appears in the first summand of (A.1) can be solved by using [13, Eq. (2.461.2)],

\[ \int_{0}^{\infty} x^2 e^{-x^2/2} \, dx = \frac{1}{2} \sqrt{2\pi} \]  

(A.2)

The integral in the second summand of (A.1) is solved by integration by parts, the change of variables \( x\sqrt{b/c\Gamma} = y \) and then by the application of [13, Eq. (4.333)], as

\[ \int_{0}^{\infty} x^2 \log \left( x\sqrt{b/c\Gamma} \right) e^{-x^2/2} \, dx = \]
\[
\frac{1}{2} \sqrt{2 \pi} \frac{1}{4} \sqrt{2 \pi \left( \mu + \log (2 c G^2 \Gamma \Omega, \Omega_2) \right)} \]

(A.3)

Combining (A.1), (A.2) and (A.3) we derive (16).

REFERENCES