A Switching Anti-windup Design Using Multiple Lyapunov Functions

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Abstract—This paper proposes a switching anti-windup design, which aims to enlarge the domain of attraction of the closed-loop system. Multiple anti-windup gains along with an index function that orchestrates the switching among these anti-windup gains are designed based on the min function of multiple quadratic Lyapunov functions. In comparison with the design of a non-switching anti-windup gain which maximizes a contractively invariant level set of a single quadratic Lyapunov function as a way to increase the size of the domain of attraction, the use of multiple Lyapunov functions and switching in the proposed design allows the union of the level sets of the multiple Lyapunov functions, each of which is not necessarily contractively invariant, to be contractively invariant within the domain of attraction. As a result, the resulting domain of attraction is expected to be significantly larger than the one resulting from a single anti-windup gain and a single Lyapunov function. Indeed, simulation results demonstrate such a significant improvement.

Keywords: Switching systems, anti-windup, composite Lyapunov functions, domain of attraction, actuator saturation.

I. INTRODUCTION

Anti-windup is a traditional approach to dealing with actuator saturation. The idea is to augment the closed-loop system that was designed without taking actuator saturation into consideration so that the negative effect of actuator saturation is weakened. Earlier works on anti-windup design try to minimize the effect of saturation in a direct way by reducing the difference between the input and output of the actuators (see, for example, [1], [8]).

Later works on anti-windup aim to reduce the effect of saturation indirectly by trying to improve the closed-loop stability and performances (see, for example, [3], [14], [16], [17], [18], [27] and the references therein). One of the earliest works that address the stability of control systems with anti-windup compensation involves the application of the scalar Popov and circle criteria ([9]). Stability analysis of multivariable systems with anti-windup compensation was carried out in [15], [22]. The $L_2$ formalism was introduced in [24] and adopted by several other authors (see, for example, [11], [25]).

More recently, an anti-windup design algorithm was developed with the explicit goal of enlarging the domain of attraction of the resulting system [3]. By expressing a saturated linear feedback law on the convex hull of some auxiliary linear feedbacks and using a single quadratic Lyapunov function, a set of conditions under which an ellipsoid, as a level set of the Lyapunov function, is contractively invariant and thus inside the domain of attraction, are established in terms of bilinear matrix inequalities in the auxiliary feedback gains, the positive definite matrix that defines the quadratic Lyapunov function and the anti-windup compensation gain. The design of the anti-windup gain to maximize the contractively invariant ellipsoid is then formulated as a constrained optimization problem with bilinear matrix inequalities. An iterative LMI algorithm is developed to solve this optimization problem. Numerical examples demonstrate that such a design procedure is indeed effective in achieving a large domain of attraction.

On the other hand, switching systems have been extensively studied in recent years. A large portion of the literature has focused on the stability analysis and controller design of switched systems (see, for example, [2], [5], [6], [13], [19], [23], [26] and the references therein). By employing either common or multiple Lyapunov functions, switching strategies are developed to make the resulting switched systems asymptotically stable. In a recent paper [13], three methods for composing a Lyapunov function from a group of quadratic functions for stabilization of switched systems composing of a number of linear systems are examined in detail. The resulting Lyapunov functions are referred to as the min function, the max function, and the convex hull function. In particular, the min function is defined at each state as the minimum value among all the quadratic functions in the group. Unlike the max and convex hull functions, both of which are convex, the min function is not a convex function and its level set is the union of the level sets of the individual quadratic functions. However, as explained in [13], the min function is more convenient to use in the synthesis of switched control systems.

There has also been effort on the design of switched systems in the presence of actuator saturation. For example, the idea of switching has been applied to a family of linear systems in the presence of actuator saturation with the objectives of enlarging the domain of attraction [20] and tolerating/rejecting disturbances [21].

In this paper, we explore the idea of switching among multiple anti-windup gains in order to further enlarge the domain of attraction of the resulting control systems. In particular, we will revisit the anti-windup design problem considered in [3], in which an algorithm is proposed to design a single anti-windup gain that enlarges the domain of attraction of the closed-loop system, which is estimated as a contractively invariant ellipsoid, a level set of a quadratic
Lyapunov function. Here in this paper we will use the min function approach developed in [13] to design multiple anti-windup gains and a switching strategy to further enlarge the domain of attraction beyond what can be achieved by the single anti-windup gain of [3]. The union of several ellipsoids, as the level set of the min function, is expected to help enlarge the domain of attraction. More significantly, thanks to switching, each of these individual ellipsoids is not required to be contractively invariant for their union to be contractively invariant. Thus, each of these individual ellipsoids is expected to be larger than the contractively invariant ellipsoid resulting from the single Lyapunov function design approach of [3].

The remainder of the paper is organized as follows. In Section II, we state the problem to be studied in this paper. Section III summarizes some tools that we will use to solve the problem. Section IV presents the algorithm for constructing anti-windup gains and a switching strategy that governs them. Numerical examples are presented in Section V. Section VI concludes the paper.

II. PROBLEM FORMULATION

Consider a linear system subject to actuator saturation

\[
\begin{align*}
\dot{x} &= Ax + B \text{sat}(u), \\
y &= Cx,
\end{align*}
\]

where \( x \in \mathbb{R}^n \) is the state, \( u \in \mathbb{R}^m \) is the control input, \( y \in \mathbb{R}^p \) the measured output, the function \( \text{sat}: \mathbb{R}^m \to \mathbb{R}^m \) is the vector valued standard saturation function defined as

\[
\text{sat}(u) = \left[ \text{sat}(u_1) \quad \text{sat}(u_2) \quad \cdots \quad \text{sat}(u_m) \right],
\]

\( \text{sat}(u_i) = \text{sign}(u_i) \min\{\{u_i\}, 1\} \). Here we have slightly abused the notation by using sat to denote both the scalar valued and the vector valued saturation functions. Also note that it is without loss of generality to assume unity saturation level. A nonunity saturation level can be absorbed into the \( B \) matrix and the feedback gain.

We assume that a linear dynamic controller of the form

\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c y, \\
x_c(0) &= 0, \\
\end{align*}
\]

has been designed that stabilizes the system (1) with desired performances in the absence of actuator saturation. When the actuator saturates, the control input delivered to the system \( \text{sat}(u) \) is different from the designed control input \( u \), causing the performance of the closed-loop systems, for example, the size of the stability region and the transient response, to degrade. To alleviate this degradation, an anti-windup compensator is designed that modifies the controller with a “correction” term \( E_c(\text{sat}(u) - u) \) as follows,

\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c y + E_c(\text{sat}(u) - u), \\
x_c(0) &= 0, \\
\end{align*}
\]

Under this compensated controller, the closed-loop system can be written as,

\[
\begin{align*}
\dot{x} &= \tilde{A} \tilde{x} + \tilde{B}(\text{sat}(u) - u), \\
u &= F \tilde{x},
\end{align*}
\]

where \( \tilde{x} = \begin{bmatrix} x & x_c \end{bmatrix}^T \) and

\[
\tilde{A} = \begin{bmatrix} A + BD_c C & BC \\ B_c C & A_c \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ E_c \end{bmatrix}, \quad F = [D_c \ C_c].
\]

In [3], an algorithm was developed based on the use of a single quadratic Lyapunov function to design the anti-windup gain matrix \( E_c \) that enlar\( \text{ge} \) the domain of attraction of the resulting closed-loop system (2). In this paper, we will develop a new algorithm based on multiple Lyapunov functions to design several anti-windup gains \( E_i, i \in [1, N] \), and a switching strategy \( i = \sigma(\tilde{x}) \) to govern these anti-windup gains so that the domain of attraction of the resulting closed-loop system is further enlarged. Here and throughout the paper, for two integers \( k_1 \) and \( k_2 \), \( I_{[k_1, k_2]} \) denotes the set of integers \( \{k_1, k_1 + 1, \ldots, k_2\} \). The function \( i = \sigma(\tilde{x}) \) determines which system is in operation according to the value of the state \( x \) and takes the form of \( \sigma(\tilde{x}) = i \) for \( \tilde{x} \in \Omega_i \) with \( \cup_{i=1}^N \Omega_i = \mathbb{R}^{n+c} \). Consequently, the resulting closed-loop system can be written as

\[
\begin{align*}
\dot{\tilde{x}} &= \tilde{A} \tilde{x} + \tilde{B}_i(\text{sat}(u) - u), \\
u &= F \tilde{x},
\end{align*}
\]

where \( \tilde{A} \) and \( F \) are as defined above and

\[
\tilde{B}_i = \begin{bmatrix} B \\ E_{ic} \end{bmatrix}.
\]

III. PRELIMINARIES

Given positive definite matrices \( P_j \in \mathbb{R}^{n \times n}, j \in [1, J] \), the min function can be defined as

\[
V_{\min}(x) = \min\{x'P_j x : j \in [1, J]\},
\]

Denote the 1-level set of \( V_{\min} \) as \( L_{V_{\min}} := \{x \in \mathbb{R}^n : V_{\min}(x) \leq 1\} \). Then we have \( L_{V_{\min}} = \bigcup_{j=1}^J E(P_j) \), where \( E(P_j) = \{x \in \mathbb{R}^n : x'P_j x \leq 1\} \).

Since \( V_{\min} \) is not differentiable everywhere, it is necessary to analyze the directional derivatives of \( V_{\min} \), which characterize the behavior of the system at the non-differentiable points. The one-side directional derivative of \( V_{\min} \) at \( x \) along \( \zeta \in \mathbb{R}^n \) is defined as

\[
V_{\min}(x; \zeta) := \lim_{\Delta t \downarrow 0} \frac{V_{\min}(x + \zeta \Delta t) - V_{\min}(x)}{\Delta t},
\]

where \( \Delta t \downarrow 0 \) denotes \( \Delta t \to 0 \), and \( \Delta t > 0 \). For a nonlinear system \( \dot{x} = h(x), \ V(x; h(x)) \) measures the time derivative of \( V \) at \( x \) along its trajectory. Let \( J_{\min}(x) := \{j \in [1, J] : V_j(x) = V_{\min}(x)\} \). Then, the directional derivative of \( V_{\min} \) is expressed as follows.

**Lemma 1 ([13]):** For a vector \( \zeta \in \mathbb{R}^n \), the directional derivative of \( V_{\min} \) at \( x_0 \in \mathbb{R}^n \) along \( \zeta \) is

\[
V_{\min}(x_0; \zeta) = \min\{V_j(x_0; \zeta) : j \in J_{\min}(x_0)\}.
\]

Consider a switched system constructed from \( N \) linear systems, \( \dot{x} = A_i x, i \in [1, N] \). Let a switching law be adopted as \( \sigma(x) = \arg \min_{i \in [1, N]} V_{\min}(x; A_i x) \), which results in a switched system

\[
\dot{x} = A_{\sigma(x)} x.
\]

Let \( \mu(x) := \min_{i \in [1, N]} V_{\min}(x; A_i x) \).

When sliding motions occur in a switched system, its trajectories move along the switching surface according to
Filippov’s convex combination [7], \( \dot{x} = \sum_{i \in I_m} \alpha_i A_i x \), with \( \alpha_i \geq 0 \) and \( \sum_{i \in I_m} \alpha_i = 1 \). Here \( I_m \) is the set of indices of all subsystems involved in the sliding motion. For the switched system (4), the derivatives of \( V_{\min} \) in the sliding mode is characterized as follows.

**Proposition 1 ([13]):** Consider a sliding mode involving subsystems \( \dot{x} = A_i x, i \in I_m \). Then for each \( x_0 \) in this sliding mode,

\[
\dot{V}_{\min}(x_0; A_i x) = \mu(x_0), \quad \forall i \in I_m.
\]

Moreover, along the sliding direction \( \sum_{i \in I_m} \alpha_i A_i x_0 = 0 \), where \( \alpha_i \geq 0 \) and \( \sum_{i \in I_m} \alpha_i = 1 \),

\[
\dot{V}_{\min}(x_0; \sum_{i \in I_m} \alpha_i A_i x_0) = \mu(x_0).
\]

In this paper, we consider the switched system (3), which results from a system under a saturated linear feedback and a switched anti-windup compensator. We recall a tool from [12] for expressing a saturated linear feedback \( u = \text{sat}(Fx) \) on a convex hull of a group of auxiliary feedbacks.

For an \( F \in \mathbb{R}^{m \times n} \), let \( \mathcal{L}(F) = \{x \in \mathbb{R}^{n} : \|Fx\| \leq 1, i \in I[1,m]\} \), where \( f_i \) represents the \( i \)th row of matrix \( F \). We note that \( \mathcal{L}(F) \) represents the region in \( \mathbb{R}^{n} \) where \( Fx \) does not saturate. Also, \( \mathcal{L}(F) \) is the set of \( m \times m \) diagonal matrices whose diagonal elements are either 1 or 0. There are \( 2^m \) elements in \( \mathcal{L}(F) \). Define \( E_s = I - E_s \). Clearly, \( E_s \in \mathcal{L}(F) \).

**Lemma 2 ([12]):** Let \( F, H \in \mathbb{R}^{m \times n} \). Then, \( \text{sat}(Fx) \in \text{co}\{E_s Fx + E_s H x, s \in I[1,2^m]\} \), \( \forall x \in \mathcal{L}(H) \), where \( \mathcal{L}(H) \) stands for the convex hull.

**IV. A SWITCHING ANTI-WINDUP COMPENSATOR**

Consider the switched system (3). We will use the min function composed from \( J \) quadratic functions \( V_j(\tilde{x}) = \tilde{x}^T P_j \tilde{x} \),

\[
V_{\min}(\tilde{x}) = \min \{\tilde{x}^T P_j \tilde{x} : j \in I[1,J]\},
\]

and the switching law that results from \( V_{\min} \),

\[
\sigma(\tilde{x}) = \arg \min_{i \in [1,N]} V_{\min}(\tilde{x}; (\tilde{A} - B_i F)\tilde{x} + B_i \text{sat}(F \tilde{x})).
\]

**Remark 1:** Proposition 2 indicates that, if \( P_j, j \in I[1,J] \), can be chosen such that \( \mu(\tilde{x}) < 0 \), for all \( \tilde{x} \in L_{V_{\min}} \), the system (3) is asymptotically stable at the origin with \( L_{V_{\min}} = \bigcup_{j=1}^J \mathcal{E}(P_j) \) contained in the domain of attraction, in disregard of the existence of the sliding motion. Theorem 1 below characterizes such matrices \( P_j \).

**Theorem 1:** Given \( P_j > 0, j \in I[1,J] \). If there exist matrices \( H_{ij} \in \mathbb{R}^{m \times n}, \alpha_{ij} \geq 0 \) and \( \beta_{jk} > 0, i \in I[1,N], j, k \in I[1,J] \), such that \( \Sigma_{i=1}^N \alpha_{ij} = 1 \) for each \( j \),

\[
\begin{align*}
(\Sigma_{i=1}^N \alpha_{ij} (\tilde{A} - B_i F + B_i (E_s F + E_s^{-1} H_{ij}))) & P_j \\
& + P_j (\Sigma_{i=1}^N \beta_{jk} (\tilde{A} - B_i F + B_i (E_s F + E_s^{-1} H_{ij}))) < 0,
\end{align*}
\]

and \( \mathcal{E}(P_j) \subset H_{ij}, i \in I[1,N], j \in I[1,J] \), then the system (3) with the switched anti-windup compensator as governed by the switching law (5) is asymptotically stable at the origin with \( \bigcup_{j=1}^J \mathcal{E}(P_j) \) contained in the domain of attraction.

**Proof:** In view of Remark 1, we need only to show that \( \mu(\tilde{x}) < 0 \) for all \( \tilde{x} \in (\bigcup_{j=1}^J \mathcal{E}(P_j)) \), \( i \in I[1,N] \), \( j \in I[1,J] \), by definition of \( J_{\min}(\tilde{x}) \),

\[
\tilde{x} \in \mathcal{E}(P_j) \subset H_{ij}, \quad \forall i \in I[1,N], j \in J_{\min}(\tilde{x}).
\]

Thus, it follows from Lemma 2 that

\[
\text{sat}(F \tilde{x}) \in \text{co}\{E_s F \tilde{x} + E_s^{-1} H_{ij} \tilde{x}, \; s \in I[1,2^m]\}, \quad \forall i \in I[1,N], j \in J_{\min}(\tilde{x}),
\]

and that

\[
(\tilde{A} - B_i F)\tilde{x} + B_i \text{sat}(F \tilde{x}) \in \text{co}\{(\tilde{A} - B_i F)\tilde{x} + B_i (E_s F + E_s^{-1} H_{ij})\tilde{x}, \; s \in I[1,2^m]\}, \quad \forall i \in I[1,N], j \in J_{\min}(\tilde{x}).
\]

Also by the definition of \( J_{\min}(\tilde{x}) \), we have

\[
\tilde{x}(P_j - P_k)\tilde{x} \leq 0, \quad \forall k \in I[1,J], j \in J_{\min}(\tilde{x}).
\]

It thus follows from (6) that

\[
2\tilde{x}^T P_j (\Sigma_{i=1}^N \alpha_{ij} (\tilde{A} - B_i F + B_i (E_s F + E_s^{-1} H_{ij}))) \tilde{x} < 0,
\]

\[
\begin{cases}
\text{since } \alpha_{ij} \geq 0 \text{ and } \Sigma_{i=1}^N \alpha_{ij} = 1 \text{ for each } j,
\text{it follows that}
\end{cases}
\]

\[
\min\{2\tilde{x}^T P_j (\tilde{A} - B_i F + B_i (E_s F + E_s^{-1} H_{ij})) \tilde{x} : i \in I[1,N], s \in I[1,2^m], j \in J_{\min}(\tilde{x}) \} < 0.
\]

By Lemma 1,

\[
\mu(\tilde{x}) : = \min\{V_{\min}(\tilde{x}; (\tilde{A} - B_i F)\tilde{x} + B_i \text{sat}(F \tilde{x})): i \in I[1,N]\}
\]

\[
\begin{cases}
\text{by definition of } J_{\min}(\tilde{x})
\end{cases}
\]

\[
\leq \max_{s \in I[1,2^m]} \min_{j \in J_{\min}(\tilde{x})} \left\{2\tilde{x}^T P_j (\tilde{A} - B_i F + B_i (E_s F + E_s^{-1} H_{ij})) \tilde{x} : i \in I[1,N] \right\} < 0.
\]
This completes the proof.

With Theorem 1, our design objective boils down to obtaining $E_{c_{i}, j} \in \mathcal{E}(P_{j})$, $j \in [1, N]$, and $p_{j} > 0$, $j \in [1, J]$, such that $\cup_{j=1}^{J} \mathcal{E}(P_{j})$ is maximized. This can be achieved by maximizing the individual $\mathcal{E}(P_{j})$. We will achieve the latter by maximizing the scalar $\alpha$ such that $\alpha r_{j} \in \mathcal{E}(P_{j})$, $j \in [1, J]$, where $r_{j} \in \mathbb{R}^{n}$ are some given vectors. Consequently, our design can be cast into the following optimization problem:

$$\sup_{p_{j}, E_{c_{i}, j}, \alpha_{i}, j} \alpha,$$

s.t. (a) $\alpha r_{j} \in \mathcal{E}(P_{j})$, $j \in [1, J]$,
(b) Inequalities (6),
(c) $\mathcal{E}(P_{j}) \subset \mathcal{L}(H_{i,j})$, $i \in [1, N], j \in [1, J]$,
(d) $p_{j} > 0, \beta_{jk} \geq 0, \alpha_{ij} \geq 0, \Sigma_{i=1}^{N} \alpha_{ij} = 1$,

where $h_{i,j,l}$ is the $l$th row of $H_{i,j}$.

Thus, letting $\nu = 1/\alpha^{2}$, we can rewrite the optimization problem (7) as the following BMI problem,

$$\inf_{p_{j}, E_{c_{i}, j}, \alpha_{i}, j, \beta_{jk}} \nu,$$

s.t. (a) $r_{j}^{T} P_{j} r_{j} \leq \nu$, $j \in [1, J]$,
(b) Inequalities (6),
(c) $\left[ \begin{array}{ll} 1 & h_{i,j,l}^{T} \\ h_{i,j,l} & P_{j} \end{array} \right] \geq 0,$

where $h_{i,j,l}$ is the $l$th row of $H_{i,j}$.

The essence of the iterative algorithm of [3] is the following. Denote

$$P_{j} = \left[ \begin{array}{ll} P_{j}(1, 1) & P_{j}(1, 2) \\ P_{j}(2, 1)^{T} & P_{j}(2, 2) \end{array} \right], \tilde{P}_{j} = \left[ \begin{array}{ll} P_{j}(1, 2) \\ P_{j}(2, 2) \end{array} \right], \tilde{B}_{0} = \left[ \begin{array}{l} B \\ 0 \end{array} \right],$$

where $P_{j}(1, 1) \in \mathbb{R}^{n \times n}$, $P_{j}(1, 2) \in \mathbb{R}^{n \times n}$ and $P_{j}(2, 2) \in \mathbb{R}^{n \times n}$. Then, $P_{j} \tilde{B}_{j} = \tilde{P}_{j} B_{0} + \tilde{P}_{j} E_{c_{j}}$. By setting $\beta_{jk} = 0$, the nonlinear matrix inequalities (6) can be written as

$$\tilde{A}^{T} P_{j} + P_{j} \tilde{A} + ((E_{F} F + E_{H} H_{j}) - F)^{T} (P_{j} \tilde{B}_{0} + \tilde{P}_{j} E_{c_{j}}) (E_{F} F + E_{H} H_{j}) - F) \leq 0,$$

where $E_{c_{j}}$ such that $P_{j}(1, 1)$ is as “small” as possible, making the region $\{x \in \mathbb{R}^{n} : x^{T} P_{j}(1, 1) x \leq 1\}$ as large as possible.

We will summarize our approach to solving the optimization problem (8) in the following algorithm.

**Algorithm for the Design of a Switched Anti-windup Compensator:**

Step 1) Set $j = 1$.

Step 2) Set a reference vector $r_{j}$ and $E_{c_{j}} = 0$, solve the optimization problem that is derived in [3]:

$$\inf_{q_{j} > 0, \nu_{j}, \nu_{j}} \nu_{j},$$

s.t. (a) $\left[ \begin{array}{ll} \nu_{j} & r_{j} \end{array} \right] \leq 0$, $j \in [1, J]$,
(b) $Q_{j}(\tilde{A} - \tilde{B}_{j} F)^{T} + (\tilde{A} - \tilde{B}_{j} F) Q_{j} + (E_{F} F Q_{j} + E_{H} H_{j}) \leq 0$,

where $Q_{j} = P_{j}^{-1}$, $G_{j} = H_{j} Q_{j}$ and $g_{j, k}$ is the $k$th row of $G_{j}$. Denote the solution as $\nu_{j, 0}$, $Q_{j, 0}$ and $G_{j, 0}$.

Step 3) Set $E_{c_{j}}$ with an initial value, $q = 1$ and $\nu_{q, j} = 1$.

Step 4) Solve the optimization problem (11) for $\nu_{j, q}, Q_{j, q}$, and $G_{j, q}$, respectively.

Step 5) Let $\nu_{q, j} = \nu_{j, q} * \nu_{q, j}, \nu_{j} = \nu_{j, q} * \nu_{j}, P_{j} = Q_{j}^{-1}$ and $H_{j} = G_{j} Q_{j}^{-1}$. where $\nu_{j} - 1 > \delta$, a pre-determined tolerance, GOTO Step 7), ELSE GOTO Step 8).

Step 7) Solve the following LMI problem:

$$\inf_{p_{j}(1, 1), \nu_{j}} P_{j}(1, 1),$$

s.t. (a) $\nu_{j}^{2} - r_{j}^{T} P_{j} r_{j} \geq 0$,
(b) $\tilde{A}^{T} P_{j} + P_{j} \tilde{A} + ((E_{F} F + E_{H} H_{j}) - F)^{T} (P_{j} \tilde{B}_{0} + \tilde{P}_{j} E_{c_{j}}) (E_{F} F + E_{H} H_{j}) - F) \leq 0$,

where $P_{j}(1, 1)$ and $\tilde{P}_{j}$ are defined in (9). Set the solution as $E_{c_{j}}$. Set $q = q + 1$. GOTO Step 4).

Step 8) IF $\nu_{q, j} < 1$, then, $\alpha_{q, j} = (\nu_{q, j})^{-1/2}$ and $E_{c_{j}}$ is a feasible solution and GOTO Step 9), ELSE set $E_{c_{j}}$ with another initial value and GOTO Step 3).

Step 9) IF $j < J$, set $j = j + 1$, and GOTO Step 2), ELSE GOTO Step 10).

Step 10) With the obtained $E_{c_{j}}$ and $P_{j}(1, 1), j \in [1, J]$, and fixing $\alpha_{i,j} = 1$, $i \in [1, N]$, we can solve the problem (8) to obtain the largest $\mathcal{E}(P_{j}), j \in [1, J]$, by sweeping over the parameters $\beta_{jk}$.

Remark 2: To prevent the anti-windup gain from being too high, we can constrain $E_{c_{j}} = \{e_{j}(p, q)\}_{n \times n}$ element-by-element as follows:

$$\psi_{j}(p, q) \leq e_{j}(p, q) \leq \psi_{j}(p, q), \quad p \in [1, n_{c}], q \in [1, m],$$

which are linear and can be readily added to the optimization process.
V. A NUMERICAL EXAMPLE

Example 1: Consider the benchmark example in [4], [3]:
\[
\dot{x}_1 = -0.1x_1 + 0.5\text{sat}(u_1) + 0.4\text{sat}(u_2),
\]
\[
\dot{x}_2 = -0.1x_2 + 0.4\text{sat}(u_1) + 0.3\text{sat}(u_2),
\]
where \(u_1\) and \(u_2\) are constrained to \([-3, 3]\) and \([-10, 10]\), respectively. At time \(t = 0\), the outputs \(x_1\) and \(x_2\) are subject to pulse set-point changes with the duration of 20 seconds and the magnitudes of 2 and 1, respectively. The PI controller considered in [4], [3] is:
\[
\begin{align*}
\dot{x}_1 &= y_{na} - x_1 + e_{11}(\text{sat}(u_1) - u_1) + e_{12}(\text{sat}(u_2) - u_2), \\
\dot{x}_2 &= y_{na} - x_2 + e_{21}(\text{sat}(u_1) - u_1) + e_{22}(\text{sat}(u_2) - u_2), \\
u_1 &= 10(y_{na} - x_1) + x_1, \\
u_2 &= -10(y_{na} - x_2) - x_2,
\end{align*}
\]
where \(y_{na}\) and \(y_{na}\) are the set point points for outputs. In the absence of actuator saturation, this PI controller places the closed-loop system poles at \([-1, -1, -0.1, -0.1]\).

To apply our algorithm, we set
\[
A = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},
C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
B = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.3 \end{bmatrix},
\begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix},
A_c = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix},
B_c = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix},
D_c = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix},
\]
\[
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\end{bmatrix},
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
We let \(r_1 = [0.6, 0.4, 0, 0]^T\), and set \(E_{c1} = 0\). We obtain \(\alpha_{1,0} = 1/\sqrt{P_{1,0}} = 0.7844\), which implies that the estimate of the domain of attraction does not include the given initial point. We then set \(E_{c1} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\), and proceed with our iterative algorithm, with the tolerance set to \(\delta = 0.0001\) and each element of \(E_{c1}\) limited by \(\pm 100\). We obtain \(\alpha_{1} = 2482.1278\), with
\[
E_{c1} = \begin{bmatrix} 13.1876 & -87.5656 \\ 12.6626 & -93.0210 \end{bmatrix},
\]
\[
P_{1}(1, 1) = 10^{-5} \times \begin{bmatrix} 0.5029 & -0.5750 \\ -0.5750 & 0.7574 \end{bmatrix}.
\]
Next, we let \(r_2 = [-0.4, 0.6, 0, 0]^T\), and set \(E_{c2} = 0\). We obtain \(\alpha_{2,0} = 1/\sqrt{P_{2,0}} = 0.1933\), which again indicates that the estimate of the domain of attraction does not include the given initial point. We then set \(E_{c2} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}\), and proceed with our iterative algorithm, with the tolerance set to \(\delta = 0.0001\) and each element of \(E_{c2}\) limited by \(\pm 100\). We obtain \(\alpha_{2} = 2682.9072\), with
\[
E_{c2} = \begin{bmatrix} 35.9142 & -57.6678 \\ 43.6532 & -97.8258 \end{bmatrix},
\]
\[
P_{2}(1, 1) = 10^{-5} \times \begin{bmatrix} 0.4255 & -0.3500 \\ -0.3500 & 0.3675 \end{bmatrix}.
\]
Implementing Step 10 of the algorithm, we obtain
\[
P_1 = 10^{-5} \times \begin{bmatrix} 0.2508 & -0.2807 & -0.0269 & 0.0266 \\ -0.2807 & 0.3620 & 0.0306 & -0.0341 \\ -0.0269 & 0.0306 & 0.0029 & -0.0029 \\ 0.0266 & -0.0341 & -0.0029 & 0.0032 \end{bmatrix},
\]
\[
P_2 = 10^{-5} \times \begin{bmatrix} 0.3293 & -0.2899 & -0.0342 & 0.0280 \\ -0.2899 & 0.3058 & 0.0308 & -0.0291 \\ -0.0342 & 0.0308 & 0.0037 & -0.0030 \\ 0.0280 & -0.0291 & -0.0030 & 0.0029 \end{bmatrix}.
\]

Plotted in Fig. 1 are the ellipsoids \(E(P_1(1, 1), 1)\) and \(E(P_2(1, 1), 1)\), and their union. Also plotted in the figure in a dash-dotted line is the ellipsoid obtained in [3]. Clearly, the union of the two larger ellipsoids is significantly larger than the ellipsoid enclosed by the dash-dotted line. In fact, numerical computation indicates that the area of the union of the two larger ellipsoids is \(1.0552 \times 10^6\) and that of the smaller ellipsoid is \(4.4731 \times 10^5\). We note that an alternative anti-windup design approach [10] was proposed after [3]. While no systematic numerical comparison is available on these two approaches, the approach in [10] is capable of resulting in global asymptotic stability.

Plotted in Fig. 2 are the evolution of \(V_{\text{min}}\) as a function of time and the switching history corresponding to the trajectory shown in Fig. 1. Shown in Fig. 3 are state responses under the pulse set-point changes mentioned above, for the two different anti-windup designs, the proposed switched anti-windup gain design and the single anti-windup gain design of [3], and when the actuator saturation is absent.

We next consider the situation when the open loop system is unstable and global stabilization is not possible. Let us replace the first equation of the system with
\[
\dot{x}_1 = 0.1x_1 - 0.1x_2 + 0.5\text{sat}(u_1) + 0.4\text{sat}(u_2).
\]
Under the same PI controller, and with the elements of the anti-windup gain matrix limited by \(\pm 100\), the single anti-windup gain design algorithm of [3] results in a contractively invariant ellipsoid of area \(1.6326 \times 10^3\). The algorithm of
VI. CONCLUSIONS

This paper revisited the problem of anti-windup compensator design and proposed an algorithm for designing a switched anti-windup compensator. A switched anti-windup compensator consists of a group of anti-windup gains and a switching law that governs the switching among these anti-windup gains. Our design is based on the min function defined from a group of quadratic Lyapunov functions. Such a min function was recently recognized to facilitate the design and analysis of switched systems composed of linear systems. Our simulation results indicate that the proposed switched anti-windup compensator has the ability to enlarge the domain of attraction of the resulting closed-loop system significantly beyond what a single anti-windup gain is able to achieve.

REFERENCES