Robust Nonlinear Control of a Voltage-Controlled Magnetic Levitation System with Disturbance Observer

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Abstract—This paper considers the control problem of a popular magnetic levitation system, which is open-loop unstable and strongly nonlinear associated with the electromechanical dynamics. The system dynamics is governed by a third-order nonlinear differential equation. The overall controller is designed through a backstepping manner by combining both the robust control and disturbance observer techniques. With the help of nonlinear damping terms, the input-to-state stability (ISS) property of the overall nonlinear control system is proved. Rigorous analysis of the ISS property is given, and experimental results are included to show the excellent position tracking performance of the designed control system.

I. INTRODUCTION

Due to strong open-loop instability and inherent nonlinearities associated with the electromechanical dynamics, the control problem of a magnetic levitation system is usually quite challenging to the control engineers. In our previous works, we designed adaptive robust controller for a magnetic levitation system and verified excellent position tracking performance through experimental studies [6], [7]. However, a major drawback is that the controller is very complicated.

In this paper, we propose a robust nonlinear controller of a voltage-controlled magnetic levitation system using disturbance observer (DOB). The DOB based motion controllers have been widely accepted in the industrial side, due to their simplicity and transparency of design, and excellent disturbance compensation ability. However, the DOB based motion controllers are usually designed according to the linear control theory [1], [4], even if the actual controlled plant may be strongly nonlinear. Unfortunately, the rigorous stability of these controllers for nonlinear systems has not been well studied in the literature.

The overall controller is designed through a backstepping manner by combining both the robust control and DOB techniques. With the help of nonlinear damping terms, the input-to-state stability (ISS) property of the overall nonlinear control system is proved. Rigorous analysis of the ISS property is given, and experimental results are included to show the excellent position tracking performance of the designed control system.

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II. STATEMENT OF THE PROBLEM

Consider the magnetic levitation system shown in Fig. 1, whose dynamics are described in the following equations [3],

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \alpha(x) + \beta(x) + \gamma(x) u + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\dot{x}_3 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\alpha(x) &= -\frac{Q x_3^2}{2M (X_0 + x_1)^2} \\
\beta(x) &= \frac{x_3}{M} (R (X_0 + x_1)^2 - x_2) \\
\gamma(x) &= \frac{x_3}{Q} (X_0 + x_1) + \frac{L_\infty (X_0 + x_1)^2}{Q + L_\infty (X_0 + x_1)}
\end{align*}
\]

where \( x = [x_1, x_2, x_3]^T = [x, \dot{x}, i]^T \) is state variable vector. And, \( x \): air gap (vertical position) of the steel ball; \( i \): coil current; \( g \): gravity acceleration; \( M \): mass of the steel ball; \( R \): electrical resistance; \( u \): voltage control input; \( L_\infty \), \( Q \) and \( X_\infty \): positive constants determined by the characteristics of the coil, magnetic core and steel ball.

Denote the nominal physical parameters as \( g_0 \), \( M_0 \), \( R_0 \), \( L_\infty \), \( Q_0 \) and \( X_\infty \). Then we have the nominal nonlinear functions and the modelling errors respectively as the fol-
following.
\[
\alpha_0(x) = - \frac{Q_0 x_2^2}{2 M_0 (x_{\infty} + x_1)^2} \\
\beta_0(x) = \frac{x_3 (Q_0 x_2 - R_0 (x_{\infty} + x_1)^2)}{Q_0 (x_{\infty} + x_1) + L_0 (x_{\infty} + x_1)^2} \\
\gamma_0(x) = \frac{Q_0 + L_\infty (x_{\infty} + x_1)}{x_{\infty} + x_1}
\]
(2)

\[
\Delta_\alpha(x) = \alpha(x) - \alpha_0(x) \\
\Delta_\beta(x) = \beta(x) - \beta_0(x) \\
\Delta_\gamma(x) = \gamma(x) - \gamma_0(x)
\]

III. COORDINATE TRANSFORMATION
To convert the original nonlinear system into a system that is “simpler” in the sense that controller synthesis is more straightforward, we adopt the following nonlinear coordinate transformation.
\[
\xi = [\xi_1, \xi_2, \xi_3]^T = [x_2, \alpha_0(x)]^T
\]
(3)

Notice that \(\xi = T(x)\) is only locally defined in a compact feasible region \(\Omega_x = \{x | 0 \leq x_1 \leq x_{1,M}, x_3 > 0 \} \subset \mathbb{R}^3\), no matter what the control strategy is. The restriction \(x_3 > 0\) is in order to avoid the singular point of the control input \(u\), see equation (26).

Hence the system model (1) is transformed into
\[
\dot{\xi}_1 = \xi_2 \\
\dot{\xi}_2 = g_0 + \Delta_g + \xi_3 \left(1 + \frac{\Delta_\alpha(x)}{\alpha_0(x)}\right) \\
\dot{\xi}_3 = F_1(x) + F_0(x) + \Delta F(x) + u \left(G_0(x) + \Delta G(x)\right)
\]
(4)

where
\[
F_1(x) = \frac{Q_0 x_2^2}{M_0 (x_{\infty} + x_1)^3} x_2
\]
(5)
\[
F_0(x) = -\frac{Q_0 x_2^2 \left(Q_0 x_2 - R_0 (x_{\infty} + x_1)^2\right)}{(x_{\infty} + x_1)^3 \left(Q_0 + L_\infty (x_{\infty} + x_1)\right)}
\]
(6)
\[
G_0(x) = -\frac{Q_0 x_3}{M_0 (x_{\infty} + x_1) \left(Q_0 + L_\infty (x_{\infty} + x_1)\right)}
\]
(7)
\[
\Delta F(x) = -\frac{Q_0 x_3}{M_0 (x_{\infty} + x_1)^2} \Delta_\beta(x)
\]
(8)
\[
\Delta G(x) = -\frac{Q_0 x_3}{M_0 (x_{\infty} + x_1)^2} \Delta_\gamma(x)
\]
(9)

IV. CONTROLLER DESIGN
In this section, we show the design procedure of the robust nonlinear controller. It is assumed here that the reference position \(y_r\) of the steel ball and its first, second and third derivatives, i.e., \(\dot{y}_r, \ddot{y}_r, \dddot{y}_r\) are uniformly bounded, and available.

The concrete design procedure is given as follows.

**Step 1:** Define the position error signal and velocity error signal respectively as
\[
z_1 = \xi_1 - y_r, \quad z_2 = \xi_2 - \alpha_1
\]
(10)

where \(\alpha_1\) is the virtual input to stabilize \(z_1\).

Then we have subsystem \(S_1\) as the following.
\[
\dot{z}_1 = \alpha_1 + \xi_2 - \dot{y}_r
\]
(11)

The virtual input \(\alpha_1\) is designed based on the common PI control technique.
\[
\alpha_1 = -c_{1p} z_1 - c_{1i} \int_0^t z_1 dt + \dot{y}_r
\]
(12)

where \(c_{1p} > 0, c_{1i} > 0\).

Notice that
\[
\dot{\alpha}_1 = -c_{1p} (x_2 - \dot{y}_r) - c_{1i} \xi_1 + \dot{y}_r
\]
(13)

**Step 2:** Define the error signal of the nominal acceleration exerted by the electromagnet as
\[
z_3 = \xi_3 - \alpha_2
\]
(14)

where \(\alpha_2\) is a virtual input to stabilize \(z_2\). Then we have the subsystem \(S_2\) as
\[
\dot{z}_2 = -\dot{\alpha}_1 + \alpha_2 + g_0 + \alpha_2 \frac{\Delta_\alpha(x)}{\alpha_0(x)} + \Delta_g + z_3 \frac{\alpha_0(x)}{\alpha_0(x)}
\]
(15)

Lump all the uncertain terms in this (mechanical) subsystem into \(w_2\):
\[
w_2 = z_2 - \left(g_0 + \alpha_2 - \dot{\alpha}_1\right)
\]
(16a)
\[
= \left(\frac{\Delta_\alpha(x)}{\alpha_0(x)} + \Delta_g\right) + \left(z_3 \frac{\alpha(x)}{\alpha_0(x)}\right)
\]
(16b)
\[
= w_{2\Delta} + w_{2z}
\]
(16c)

where, \(w_{2\Delta}\) is due to the modeling error, \(w_{2z}\), is due to the control error \(z_3\) of the next (electrical) subsystem:
\[
w_{2\Delta} = \alpha_2 \frac{\Delta_\alpha(x)}{\alpha_0(x)} + \Delta_g, \quad w_{2z} = z_3 \frac{\alpha(x)}{\alpha_0(x)}
\]
(17)

Since \(\dot{z}_2\) in (16a) is usually noisy, we pass \(w_2\) through a low-pass filter to obtain its estimate \(\hat{w}\):
\[
\hat{w}_2 = Q(s) w_2 = \left[Q(s) w_{2\Delta}\right] + \left[Q(s) w_{2z}\right]
\]
(18)

This is the so called DOB studied extensively in the literature [1], [4]. In the low-frequency domain, we can expect \(w_2 \approx \hat{w}_2\). In is paper, we adopt a simple second-order filter:
\[
Q(s) = \frac{1}{(1 + \tau_2 s)^2}
\]
(19)

Since at the next step, we have to calculate \(\dot{w}_2 = s \hat{w}_2\), it is necessary to let the relative degree of \(Q(s)\) higher than 1.

By compensating the virtual input \(\alpha_2\) by \(\hat{w}_2\), a simple controller can be designed for the approximated nominal model. Replacing \(\alpha_2\) by \(\alpha_2 = v_2 + \dot{\alpha}_1 - g_0\), and assuming \(\dot{w}_2 \approx w_2\), we have
\[
\dot{\alpha}_1 = -c_{1p} (x_2 - \dot{y}_r) - c_{1i} \xi_1 + \dot{y}_r
\]
(20)
where \( v_2 \) is a nominal linear input. This paves the way to design a simple controller. The simplest design is to let \( v_2 = -c_2 z_2 \). However, it should be pointed out that we can only expect \( \tilde{z}_2 \approx w_2 \) at low-frequencies. If the disturbance and model mismatch are fast changing, the estimation error \( w_2 - \tilde{w}_2 \) can not be neglected and even can destroy the stability of the closed-loop in the case of large model mismatch [8].

To stabilize subsystem \( S_2 \), we design the robust virtual input \( \alpha_2 \) as follows.

\[
\begin{align*}
\alpha_2 &= \alpha_{20} - \alpha_{2w} - \alpha_{21} - \alpha_{22} - \alpha_{23} \\
\alpha_{2w} &= -c_2 z_2 + \alpha_1 - g_0 \\
\alpha_{21} &= \kappa_2 g_0 z_2 \\
\alpha_{22} &= \kappa_2 \sqrt{\frac{\alpha_2^2 + \nu z_2^2}{\tilde{w}_2^2 + \nu z_2^2}} \\
\alpha_{23} &= \kappa_3 \sqrt{\frac{\alpha_2^2 + \nu z_2^2}{\tilde{w}_2^2 + \nu z_2^2}}
\end{align*}
\] (21)

where, \( c_2, \kappa_{21}, \kappa_{22}, \kappa_{23} > 0; \nu \) is a small positive number, and is given as \( \nu = 0.01; \alpha_{20} \) is a nominal feedback controller with nominal model compensation; \( \alpha_{2w} \) is a compensating term by the DOB’s output; \( \alpha_{21} \) is a linear damping term to counteract \( \Delta \gamma \); \( \alpha_{22} \) is a nonlinear damping term to counteract \( \Delta \gamma(x)/\alpha_0(x) \); \( \alpha_{23} \) is a nonlinear damping term to ensure boundedness of \( z_2 \) when \( \tilde{w}_2 \) is used. Notice that the nonlinear damping terms employ time-varying control gains so that they grow at least as the same order as the corresponding uncertain terms grow.

**Step 3:** The derivative of \( \alpha_2 \) is calculated as follows.

\[
\dot{\alpha}_2 = F_2 - F_3 (\alpha_0(x) + g_0) - F_3 (\Delta \gamma(x) + \Delta \gamma) \] (22)

where

\[
F_2 = \left[ 1 - \kappa_2 (\alpha_2^2 + \nu)^{-0.5} \alpha_{20} z_2 \right] \times
\begin{align*}
&\left[ c_1 p \dot{z}_1 - c_1 z_1 + \dot{y}^{(3)} \right] + c_2 \dot{\alpha}_1 \\
&\left[ 1 + \kappa_3 (\tilde{w}_2^2 + \nu)^{-0.5} \alpha_{23} \tilde{w}_2 \\
&+ (\kappa_2 g_0 + \kappa_3 (\alpha_2^2 + \nu)^{0.5} + \kappa_3 (\tilde{w}_2^2 + \nu)^{0.5}) \alpha_1 \right]
\end{align*}
\]

\[
F_3 = c_2 + c_1 p + \kappa_2 g_0 + \kappa_3 (\alpha_2^2 + \nu)^{0.5} + \kappa_3 (\tilde{w}_2^2 + \nu)^{0.5} - (c_2 + c_1 p) \kappa_2 (\alpha_2^2 + \nu)^{-0.5} \alpha_{20} z_2
\] (23)

Therefore, we have the electrical subsystem \( S_3 \) as

\[
\dot{z}_3 = \dot{\Psi}_0 + \Delta \psi + G_0(x) u + \Delta G(x) u
\] (24)

where

\[
\dot{\Psi}_0 = F_0(x) + F_1(x) - F_2 - F_3 (\alpha_0(x) + g_0) \\
\Delta \psi = \Delta F(x) + F_3 (\Delta \gamma(x) + \Delta \gamma)
\] (25)

Then we design the control voltage as follows.

\[
u = \frac{\alpha_{30} - \alpha_{31} - \alpha_{32} - \alpha_{33}}{G_0(x)}
\] (26)

\[
\begin{align*}
\alpha_{30} &= -c_3 z_3 - \Psi_0 \\
\alpha_{31} &= \kappa_{31} \left( 1 - 0.5 e^{-\lambda_1 |z_3|} \right) F_{0d} z_3 \\
\alpha_{32} &= \kappa_{32} \left( 1 - 0.5 e^{-\lambda_2 |z_3|} \right) |F_3| (\|\alpha_0(x)\| + g_0) z_3 \\
\alpha_{33} &= \kappa_{33} \left( 1 - 0.5 e^{-\lambda_3 |z_3|} \right) |\alpha_{30}| z_3
\end{align*}
\] (27)

\[
F_{0d} = \frac{|F_e|}{Q_0} \left( \frac{x_3^2 (Q_0 |z_2| + R_0 (X_{\infty 0} + x_1)^2)}{x_3^2 (Q_0 + L_{\infty 0} (X_{\infty 0} + x_1))} \right)
\] (28)

where, \( c_3, \kappa_{31}, \kappa_{32}, \kappa_{33} > 0; \alpha_{30} \) is a nominal feedback controller with nominal model compensation; \( \alpha_{31}, \alpha_{32} \) and \( \alpha_{33} \) are nonlinear damping terms employed to counteract the modelling errors. Also, notice that \( (1 - 0.5 e^{-\lambda_i |z_3|}), i = 1, 2, 3 \) are introduced to reduce control efforts due to the nonlinear damping terms, when \( |z_3| \) is relatively small. In this study, we choose \( \lambda_1 = \lambda_2 = \lambda_3 = 0.1 \).

**Remark 1:** Since the signals of the electrical subsystem change much faster than those of the mechanical subsystem, the DOB which uses a low-pass filter is not so effective. Therefore, we do not employ a DOB for the electrical subsystem. However, as can be seen in (16b), since the error signal \( z_3 \) of the electrical subsystem is included in the lumped disturbance term \( w_2 \), the low-frequency components of \( z_3 \) can be compensated by the DOB employed at step 2, so that the influences by \( z_3 \) to the mechanical subsystem can be reduced at low-frequencies.

**V. Stability Analysis**

**Step 1:** Applying \( \alpha_1 \) to subsystem \( S_1 \), we have

\[
\dot{z}_1 = z_2 - c_{1p} z_1 - c_{1i} \int_0^t z_1 dt
\] (29)

Rewrite it into the state-space form:

\[
\dot{z}_{1a} = A \ z_{1a} + B \ z_2
\] (30)

where \( z_{1a} = [\int_0^t z_1 dt, z_1]^T \).

\[
A = \begin{bmatrix} 0 & 1 \\ -c_{1i} & -c_{1p} \end{bmatrix}, \ B = [0 \ 1]^T
\] (31)

The ISS property of subsystem \( S_1 \) can be described in the following lemma [5]:

**Lemma 1:** If the virtual input \( \alpha_1 \) is applied to subsystem \( S_1 \), and if \( z_2 \) is made uniformly bounded at the next step, then \( S_1 \) is ISS, i.e., for \( \exists \lambda_0 > 0, \exists \rho_0 > 0 \),

\[
|z_{1a}(t)| \leq \lambda_0 e^{-\rho_0 t} |z_{1a}(0)| + \lambda_0 \sup_{0 \leq \tau \leq t} |z_2(\tau)|
\]

**Step 2:** Applying \( \alpha_2 \) to subsystem \( S_2 \), we have

\[
\dot{z}_2 = -c_2 z_2 - (\alpha_{21} + \alpha_{22} + \alpha_{23}) - \tilde{w}_2 + w_2
\] (32a)

\[
= -c_2 z_2 - (\alpha_{21} + \alpha_{22} + \alpha_{23}) \frac{\alpha(x)}{\alpha_0(x)} \\
+ \alpha_{20} \frac{\Delta \gamma(x)}{\alpha_0(x)} + \Delta \gamma + \frac{\alpha(x)}{\alpha_0(x)} z_3 - \frac{\alpha(x)}{\alpha_0(x)} \tilde{w}_2
\] (32b)
And then,
\[ \frac{d}{dt} \left( \frac{z_2^2}{2} \right) \leq -\frac{c_2}{2} z_2^2 - \left[ \frac{c_2}{2} + D_2 \right] \|z_2\| \|z_2\| - \mu_2(t) \]
\[ \mu_2(t) = \frac{\alpha(x)}{\alpha_0(x)} \left( \kappa_{21} g_0 + \kappa_{22} \sqrt{\alpha_0^2 + \nu + \kappa_{23} \sqrt{w_2^2 + \nu}} \right) \]
\[ D_2 = \frac{\alpha(x)}{\alpha_0(x)} \left( \kappa_{21} g_0 + \kappa_{22} \sqrt{\alpha_0^2 + \nu + \kappa_{23} \sqrt{w_2^2 + \nu}} \right) \]
(33)

It can be verified that the nonlinear damping terms in the denominator grow at least as the same order as the uncertain terms in the numerator grow. Furthermore, if \( z_3 \) is made uniformly bounded at step 3, we can conclude that \( \mu_2(t) \) is uniformly bounded. More precisely, we have
\[ |z_2(t)| \geq \mu_2(t) \Rightarrow \frac{d}{dt} \left( \frac{z_2^2}{2} \right) \leq -c_2 z_2^2 \]
(34)
\[ |z_2(t)| \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_2(\tau) \]
(35)

In the above analysis, the main attention is to show the boundedness of the internal signals. No analysis yet be done for the attenuation effects of \( w_2 - \bar{w}_2 \). Without such an analysis, we cannot clearly see how the DOB’s output \( \bar{w}_2 \) can bring improvement. We now attempt to make such an effort. Rewrite (16)~(18):
\[ w_2 - \bar{w}_2 = \alpha_2 \frac{\alpha_0(x)}{\alpha_0(x)} + z_3 \frac{\alpha_0(x)}{\alpha_0(x)} + \Delta_g - \bar{w}_2 \]
(36a)
\[ = \frac{\Delta_\alpha(x)}{\alpha_0(x)} \left( -D_{2w} z_2 + \alpha_{20} - \bar{w}_2 \right) + \Delta_g \]
\[ + z_3 \frac{\alpha_0(x)}{\alpha_0(x)} - \bar{w}_2 \]
\[ = w_2\Delta - w_2\Delta + w_2z - \bar{w}_2z \]
(36b)
\[ = \frac{\alpha_0(x)}{\alpha_0(x)} \left( \kappa_{21} g_0 + \kappa_{22} \sqrt{\alpha_0^2 + \nu + \kappa_{23} \sqrt{w_2^2 + \nu}} \right) \]
\[ D_{2w} = \left( \kappa_{21} g_0 + \kappa_{22} \sqrt{\alpha_0^2 + \nu + \kappa_{23} \sqrt{w_2^2 + \nu}} \right) \]
\[ \text{And from (32a) and (36), we have} \]
\[ \frac{d}{dt} \left( \frac{z_2^2}{2} \right) \leq -\frac{c_2}{2} z_2^2 - \left[ \frac{c_2}{2} + D_{2w} \right] \|z_2\| \|z_2\| - \mu_{2wa}(t) \]
(38)
\[ \mu_{2wa}(t) = \frac{|\bar{w}_2 - w_2|}{\frac{c_2}{2} + D_{2w}} \]
(39a)
\[ \leq \frac{|w_2\Delta - w_2\Delta| + |w_2z - w_2z|}{\frac{c_2}{2} + D_{2w}} + \frac{|w_2z - w_2z|}{\frac{c_2}{2} + D_{2w}} \]
\[ = \mu_{2wb}(t) + \mu_{2wb}(t) \]
\[ = \mu_{2wb}(t) \]
(39b)

and
\[ \mu_{2wb}(t) = \left( \frac{1 - Q_2(s)}{2} \right) \left[ \frac{(\alpha(x)/\alpha_0(x))z_3}{\frac{c_2}{2} + D_{2w}} \right] \]
(40)

Therefore, we have
\[ |z_2(t)| \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_{2wa}(\tau) \]
(41a)
\[ \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_{2wb}(\tau) \]
(41b)

**Remark 2:** So far, we have obtained (35), (41a) and (41b) for \( z_2 \), which seem confusing. These results are explained here. The inequality (35) is to show the boundedness of the internal signals of the first two subsystems. It can be seen in (36b), (37) and (39a) that in the numerator of \( \mu_{2wa}(t) \), there is a term \( D_{2w} z_2 \). However, in the denominator, the corresponding term is \( D_{2w} \). Therefore, we should at first ensure the boundedness of \( z_2 \) as shown in (35), then we can discuss the boundedness of \( \mu_{2wa}(t) \) in (41a). Notice that \( \mu_{2wa}(t) \) has very transparent physical meaning. At low-frequencies, we can expect \( \mu_{2w} \approx 0 \). And any nonzero \( w_2 - \bar{w}_2 \) at high-frequencies is counteracted by \( c_2/2 + D_{2w} \) so that \( z_2 \) is quite robust against \( w_2 - \bar{w}_2 \). However, for the sake of proving the ISS property of the overall error system, we have to express the overall error system as a cascade of the three subsystems [2]. Therefore, as shown in (39c), we separate the effects of the next control error \( z_3 \) from the modeling uncertainty in \( \mu_{2wb}(t) \).

Then we have
\[ \text{Lemma 2: If the virtual input } \alpha_2 \text{ is applied to subsystem S2, and if } z_3 \text{ is made uniformly bounded at the next step, then S2 is ISS:} \]
\[ |z_2(t)| \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_{2wb}(\tau) \]
\[ \text{Step 3: Applying the control voltage } u \text{ to the subsystem S3, we have} \]
\[ z_3 = -c_3 z_3 + \Delta_\Psi - \alpha_{31} - \alpha_{32} - \alpha_{33} + \Delta_G(x)u \]
\[ = -c_3 z_3 + \Delta_F(x) + F_3 \Delta_\alpha(x) + F_3 \Delta_g \]
\[ + \Delta_G(x) \frac{\alpha_{30}}{G_0(x)} - G(x) \frac{\alpha_{31} + \alpha_{32} + \alpha_{33}}{G_0(x)} \]
(42)

and
\[ \frac{d}{dt} \left( \frac{z_3^2}{2} \right) = -c_3 z_3^3 + \Delta_F(x) z_3 + F_3 \Delta_\alpha(x) + F_3 \Delta_g \]
\[ + \Delta_G(x) \frac{\alpha_{30}}{G_0(x)} - G(x) \frac{\alpha_{31} + \alpha_{32} + \alpha_{33}}{G_0(x)} \]
\[ \leq -c_3 z_3^3 + c_3 z_3^3 - D_3 z_3^2 \]
(39b)
\[ \leq -c_3 z_3^3 + \Delta_F(x) z_3 + F_3 (\Delta_\alpha(x) + \Delta_g) z_3 + \Delta_G(x) \frac{\alpha_{30}}{G_0(x)} z_3 \]
\[ \leq -c_3 z_3^3 + \Delta_F(x) z_3 + F_3 (\Delta_\alpha(x) + \Delta_g) z_3 + \Delta_G(x) \frac{\alpha_{30}}{G_0(x)} z_3 \]
(39c)
\[ \leq -c_3 z_3^3 + \Delta_F(x) z_3 + F_3 (\Delta_\alpha(x) + \Delta_g) z_3 + \Delta_G(x) \frac{\alpha_{30}}{G_0(x)} z_3 \]
(39d)
where
\[
\mu_3(t) = \left| \Delta F(x) + F_3(\Delta \alpha(x) + \Delta g) + \Delta G(x) \alpha_0 + G_0(x) \right| \frac{c_3^2 + D_3}{2} = 0.5 G(x) (\kappa_3 \alpha_0 + \ldots)
\]

Just as the case of step 2, it can be verified that the nonlinear damping terms in the denominator of \( \mu_3(t) \) grow at least as the same order as the uncertain terms in the numerator grow. Therefore we can conclude that \( \mu_3(t) \)

**Lemma 3**: If the control input \( u \) is applied to subsystem \( S_3 \), then \( S_3 \) is ISS:
\[
|z_3(t)| \leq |z_3(0)| e^{-c_3 t/2} + \sup_{0 \leq \tau \leq t} \mu_3(\tau)
\]

**Stability of the overall error system**: Since the overall error system is a cascade of the three subsystems characterized by Lemmas 1~3, along the same line of the proof of Lemma C.4 in [2], we can prove the cascaded system is also ISS. Define the error signal vector
\[
z(t) = [x_3^T(t), z_2(t), z_3(t)]^T
\]

Then based on Lemmas 1~3, we can prove the following results.
\[
|z(t)| \leq \sqrt{2} \lambda_2 |z(0)| e^{-\rho_2 t} + \gamma_3 \sup_{0 \leq \tau \leq t} \mu_3(\tau) + \beta_3 \sup_{0 \leq \tau \leq t} \mu_3(\tau)
\]

where
\[
|z(t)| = \sqrt{[z_1(t)]^2 + [z_2(t)]^2 + [z_3(t)]^2}
\]

\[
\gamma_3 = (\lambda_1 + 1) \gamma_2
\]

\[
\beta_3 = (\lambda_1 + 1) \gamma_2 \beta_2 + 1
\]

\[
\lambda_2 = \lambda_1^2 + (\lambda_1 + 1) \gamma_2 \beta_2 + 1
\]

\[
\rho_2 = \min(\rho_0/4, c_2/8, c_3/4)
\]

**Remark 3**: The result 1) of Theorem 1 characterizes the ISS property of the overall error system, which ensures the boundedness of the error signals. According to (40), \( \mu_2^{\text{wb}}(\tau) \) is small at low-frequencies, owing to the DOB. On the other hand, \( \mu_3(\tau) \) of the electrical subsystem may not be so small. Therefore, \( |z(t)| \) itself may not be so small. However, our final purpose is to make the position error \( z_1 \) small. According to Lemma 1 and (41a), we can conclude that \( z_1(\tau), z_2(\tau) \) can be made small at low-frequencies. The comments will be confirmed by experimental results.

**VI. Experimental results**

To verify the performance of the proposed robust nonlinear controller with DOB, experimental studies have been carried out on the magnetic levitation system shown in Fig. 1. The physically allowable operating region of the steel ball shown in Fig. 1 is limited to \( 0[m] < x_1 < 0.013[m] \). The output of the controllable voltage source is limited to \( -60.0[V] \leq u \leq 60.0[V] \). The velocity \( x_2 \) is measured by pseudo-differentiation of the measured position \( x_1 \) as \( x_1/(0.004s+1) \). The resolution of the laser distance sensor is \( \pm 0.00018[m] \), which is considered to be relatively noisy. The physical parameters are identified as follows.

\[
M = 0.54[kg], \quad g = 9.8[m/s^2]
\]

\[
X_{\infty} = 0.008114[m], \quad Q = 0.001624[Hm]
\]

\[
L_{\infty} = 0.8052[\Omega], \quad R = 11.88[\Omega]
\]

The following nominal system parameters with considerable errors are used for experimental studies.

\[
M_0 = 0.30[kg], \quad g_0 = 9.0[m/s^2]
\]

\[
X_{\infty 0} = 0.0020[m], \quad Q_0 = 0.0003[Hm]
\]

\[
L_{\infty 0} = 0.50[\Omega], \quad R_0 = 10.0[\Omega]
\]

The following two controllers are implemented:

1) **Robust nonlinear controller without DOB**:

\[
c_{1p} = 40, \quad c_{1i} = 20^2, \quad c_2 = 40, \quad c_3 = 20
\]

\[
\kappa_{21} = 1, \quad \kappa_{22} = 3
\]

\[
\kappa_{31} = 0.3, \quad \kappa_{32} = 0.3, \quad \kappa_{33} = 0.3
\]

2) **Robust nonlinear controller with DOB**:

\[
c_{1p} = 40, \quad c_{1i} = 20^2, \quad c_2 = 40, \quad c_3 = 20
\]

\[
\tau_2 = 0.02
\]

\[
\kappa_{21} = 1, \quad \kappa_{22} = 3, \quad \kappa_{23} = 1
\]

\[
\kappa_{31} = 0.3, \quad \kappa_{32} = 0.3, \quad \kappa_{33} = 0.3
\]

The results are shown Fig. 2 and Fig. 3. In each figure, from the top to the bottom are the position \( x_1 \), velocity \( x_2 \), coil current \( x_3 \), control voltage \( u \), error signals \( z_1, z_2, z_3 \). It can be found in Fig. 2 that owing to the nonlinear damping terms, the error signals are made bounded. However, empirically, we found it is difficult to further reduce the position tracking error \( z_1 \) by simply increasing the control gains.

A comparison between Fig. 2 and Fig. 3 can be made here. For the electrical subsystem, since the DOB is not used, the
For the sake of physical transparency, the controller was designed directly on the model of the magnetic levitation system. However, we should emphasize here that the basic idea and theory can be extended to a general electro-mechanical system governed by the following third-order strict feedback form [2]:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u
\end{align*}
\]

where \( x_1 \): position; \( x_2 \): velocity; \( x_3 \): current or equivalent driving force, torque, acceleration; \( u \): voltage control input; \( f_2(x_1, x_2), g_2(x_1, x_2), f_3(x_1, x_2, x_3), g_3(x_1, x_2, x_3) \) nonlinear functions which may include uncertainties.

In contrast to most DOB based controllers reported in the literature, our major academic contribution is to propose a theoretically guaranteed robust nonlinear controller with DOB for a strongly nonlinear and unstable system.

**REFERENCES**