Robust Output Feedback Control of a Magnetic Levitation System via High-Gain Observer

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Abstract—This paper proposes a novel robust output feedback controller for an electromechanical system in the presence of external disturbance and uncertainties of physical parameters. By exploiting the cascade features of backstepping design, a simple disturbance observer is proposed to suppress the effects of the uncertainties, and a high-gain observer is applied to estimate the unmeasurable states of the system. Strict analysis of the nonlinear control system is given. Experimental results are provided to support the theoretical results.

I. INTRODUCTION

In this paper, we consider the position-tracking control problem of a single-input-single-output (SISO) electromechanical system in the strict feedback form[1]:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)u \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)u
\end{align*}
\]

where, \(x_1\): position, \(x_2\): velocity, \(x_3\): current (or equivalent driving force, torque, acceleration), \(u\): voltage control input; \(f_2(x_1, x_2), g_2(x_1, x_2), f_3(x_1, x_2, x_3), g_3(x_1, x_2, x_3)\): nonlinear functions which may include uncertainties.

If the electromechanical system model (1) can be transformed into the following normal form

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= F(\xi) + G(\xi)u
\end{align*}
\]

where \(\xi = [\xi_1, \xi_2, \xi_3]^T = [x_1, x_2, x_3(x_1, x_2, x_3)]^T\), the controller design becomes much easier. However, in practice, \(F(\xi)\) and \(G(\xi)\) may be uncertain. Moreover, \(\xi_2\) and \(\xi_3\) may not be available. The high-gain observer (HGOB) output feedback approach has been considered as a powerful tool to tackle the problems mentioned here [3]. Typical control techniques combined with the HGOB are adaptive control[4], sliding mode control [5] and robust control by nonlinear damping terms [6].

In this paper, a robust output feedback controller using a disturbance observer (DOB) is proposed for the position-tracking control problem of an electromechanical system (magnetic levitation system). Seeing (2c) where the nonlinear functions may be uncertain, one may likely introduce an adaptive mechanism or a DOB directly to (2c) to compensate the uncertainties. However, by exploiting the cascade features of backstepping design, our idea is to introduce a DOB to the subsystem (2b), so that the controller design is much simpler. Then by using a HGOB that estimates the state variables, the designed controller is implementable as an output feedback controller. It is shown that the stability of the overall nonlinear control system is established and the DOB helps to make \(\xi_1\) track the reference trajectory with small tracking error. Experimental results are given to support the theoretical results.

II. MODEL OF THE MAGNETIC LEVITATION SYSTEM

For the sake of physical transparency, the controller is designed directly on the model of a magnetic levitation system shown in Fig. 1, governed by the following equations.

\[
\begin{align*}
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} &=
\begin{bmatrix}
x_2 \\
\alpha(x) \\
\beta(x)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
\gamma(x)
\end{bmatrix} (u + d) +
\begin{bmatrix}
0 \\
g \\
0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\alpha(x) &= -\frac{Qx_2^2}{2M(X_\infty + x_1)^2} \\
\beta(x) &= \frac{x_3(Qx_2 - R(X_\infty + x_1)^2)}{Q(X_\infty + x_1) + L_\infty(X_\infty + x_1)^2} \\
\gamma(x) &= \frac{x_3}{Q(X_\infty + x_1) + L_\infty(X_\infty + x_1)}
\end{align*}
\]

where \(x = [x_1, x_2, x_3]^T = [x, \dot{x}, i]^T\) is the state variable vector. And, \(x\): air gap (vertical position) of the steel ball; \(i\): coil current; \(g\): gravity acceleration; \(M\): mass of the steel ball; \(R\): electrical resistance; \(u\): voltage control input; \(L_\infty, Q\) and \(X_\infty\): positive constants; \(d\): bounded external disturbance.

III. COORDINATE TRANSFORMATION

By using the coordinate transformation \(\xi = [\xi_1, \xi_2, \xi_3]^T = [x_1, x_2, \alpha(x) + g]^T\), the system model (3) is transformed into

\[
\begin{align*}
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= \xi_3 \\
\dot{\xi}_3 &= F_1(\xi) + F_2(\xi) + G(\xi)(u + d)
\end{align*}
\]

where

\[
\begin{align*}
F_1(\xi) &= \frac{2(g - \xi_3)}{(X_\infty + \xi_1)} \xi_2 \\
F_2(\xi) &= -2(g - \xi_3) \frac{Q\xi_2 - R(X_\infty + \xi_1)^2}{(X_\infty + \xi_1)(Q + L_\infty(X_\infty + \xi_1))} \\
G(\xi) &= -\frac{\sqrt{2MQ(g - \xi_3)}}{M(Q + L_\infty(X_\infty + \xi_1))}
\end{align*}
\]
Denoting the nominal physical parameters as \( g_0, M_0, R_0, L_0, Q_0 \) and \( X_0 \), we can rewrite (5) as follows.
\[
\begin{align*}
\xi_1 &= \xi_2 \\
\xi_2 &= \xi_3 \\
\xi_3 &= F_{10}(\xi) + \Delta F_1(\xi) + F_{20}(\xi) + \Delta F_2(\xi) \\
&+ \left( G_0(\xi) + \Delta G(\xi) \right)(u + d)
\end{align*}
\]
where the error functions are expressed as
\[
\Delta F_i(\xi) = F_i(\xi) - F_{i0}(\xi), \quad i = 1, 2
\]
\[
\Delta G(\xi) = G(\xi) - G_0(\xi)
\]
and the nominal functions \( F_{10}(\xi), F_{20}(\xi), G_0(\xi) \) are obtained by replacing the physical parameters in \( F_1(\xi), F_2(\xi), G(\xi) \) by their nominal values. Notice that according to (6), \( (g - \xi_3) > 0 \) and \( (g_0 - \xi_3) > 0 \).

IV. ROBUST NONLINEAR STATE FEEDBACK CONTROLLER
We first show the backstepping design of the state feedback controller, assuming that \( \xi_1, \xi_2, \xi_3 \) are known.

Step 1:
Define the error signals of position \( \xi_1 \) and velocity \( \xi_2 \) as
\[
z_1 = \xi_1 - v_r, \quad z_2 = \xi_2 - \alpha_1
\]
where \( \alpha_1 \) is a virtual input to stabilize \( z_1 \). Then we have subsystem \( \mathcal{S}1 \) as the following.
\[
\mathcal{S}1: \quad \dot{z}_1 = \dot{\alpha}_1 + z_2 - \dot{v}_r
\]
The virtual input \( \alpha_1 \) is designed as
\[
\alpha_1 = -c_1 z_1 + \dot{v}_r
\]
where \( c_1 > 0 \). Our next step is to stabilize \( z_2 \).

Step 2:
Define the error signal of the acceleration as
\[
z_3 = \xi_3 - \alpha_2
\]
where \( \alpha_2 \) is a virtual input to stabilize \( z_2 \). Then we have the dynamical equation of subsystem \( \mathcal{S}2 \) as
\[
\mathcal{S}2: \quad \dot{z}_2 = -\alpha_1 + \alpha_2 + z_3
\]
Viewing \( z_3 \) as a disturbance to the subsystem \( \mathcal{S}2 \), we have
\[
z_3 = \dot{z}_2 - (-\dot{\alpha}_1 + \alpha_2)
\]
Since \( z_3 \) itself can only be used explicitly at the next step of backstepping design [1], we have to use its estimate \( \tilde{z}_3 \). To this end, we pass \( z_3 \) through a low-pass filter \( P(s) \):
\[
\tilde{z}_3 = P(s)z_3 = P(s)(z_{20} + \alpha_1 - \alpha_2)
\]
This is similar to the idea of DOB in the literature. Here, we adopt a simple low-pass filter as the following.
\[
P(s) = \frac{1}{1 + \lambda s^n} = \frac{1}{s^n + p_{n-1}s^{n-1} + \cdots + p_1s + p_0}
\]
where \( \lambda > 0 \) is a time-constant that determines its pass-band.

We design the virtual control \( \alpha_2 \) as
\[
\alpha_2 = \alpha_{20} + \alpha_{2d}
\]
\[
\alpha_{20} = -c_{2z2} + \alpha_1 - \dot{\tilde{z}}_3
\]
\[
\alpha_{2d} = -\kappa_{21}\sqrt{\dot{\tilde{z}}^2_3 + v\tilde{z}_2}
\]
where \( c_2, \kappa_{21} > 0; v \) is a small positive number, e.g., \( v = 0.01 \);
\( \alpha_{2d} \) is a nonlinear damping term to enhance the damping effect when \( \tilde{z}_3 \) is relatively large.

Step 3:
The dynamics of subsystem \( \mathcal{S}3 \) can be obtained as
\[
\mathcal{S}3: \quad \dot{z}_3 = -c_3 z_3 + \left[ c_3 z_3 + F_{10}(\xi) + \Delta F_1(\xi) + F_{20}(\xi) + \Delta F_2(\xi) + \left( G_0(\xi) + \Delta G(\xi) \right) (u + d) - \alpha_1 \right]
\]
where \( c_3 > 0 \). To stabilize the subsystem \( \mathcal{S}3 \), we design the following controller.
\[
\begin{align*}
\alpha_{30} &= -c_3 z_3 + \alpha_2 - F_{10}(\xi) - F_{20}(\xi) - \Delta F_1(\xi) - \Delta F_2(\xi) - \left( G_0(\xi) + \Delta G(\xi) \right) (u + d) - \alpha_1 \\
u_{d1} &= \kappa_{31} F_{d1}(\xi) \\
u_{d2} &= \kappa_{32} F_{d2}(\xi) \\
u_{d3} &= \kappa_{33} |\alpha_{30}|, \quad u_{d4} = \kappa_{34} G(\xi)|\dot{\xi}_3|
\end{align*}
\]
\[
\begin{align*}
F_{d1}(\xi) &= \frac{2(g_0 - \xi_3)\xi_3}{(X_0 + \xi_3)} \\
F_{d2}(\xi) &= \frac{2(g_0 - \xi_3)(Q_0|\dot{\xi}_3| + R_0(X_0 + \xi_3)^2)}{(X_0 + \xi_3)(Q_0 + L_0(X_0 + \xi_3))}
\end{align*}
\]
where \( \kappa_{31}, \kappa_{32}, \kappa_{33} > 0; \alpha_{30} \) is a feedback controller with model compensation. \( u_{d1}, u_{d2}, u_{d3}, u_{d4} \) are nonlinear damping terms to suppress respectively \( \Delta F_1(\xi), \Delta F_2(\xi), \Delta G(\xi)u, G(\xi)d \).

V. STABILITY ANALYSIS OF THE STATE FEEDBACK CONTROL SYSTEM

Step 1:
The subsystem \( \mathcal{S}1 \) can be expressed as
\[
\dot{z}_1 = -c_1 z_1 + z_2
\]
Assuming $z_2$ is made uniformly bounded at the next step, we can derive

$$
\frac{d}{dt} \left( \frac{z_2^2}{2} \right) \leq - \frac{c_1}{2} z_2^2 - \frac{c_1}{2} |z_1| \left( |z_1| - \frac{2}{c_1} |z_2| \right)
$$

and hence

$$
|z_1(t)| \leq |z_1(0)| e^{-c_1 t/2} + \sup_{0 \leq \tau \leq t} \left[ \frac{2}{c_1} z_2(\tau) \right]
$$

(24)

Furthermore, we have

$$
\frac{d}{dt} \left( \frac{z_1^2}{2} \right) = -c_1 z_1^2 + z_1 z_2 \leq - \frac{c_1}{2} z_1^2 + \frac{z_2^2}{2c_1}
$$

and hence

$$
\lim_{t \to \infty} \frac{1}{T} \int_0^T z_1^2 dt \leq \frac{1}{c_1} \lim_{T \to \infty} \frac{1}{T} \int_0^T z_2^2 dt
$$

(26)

Equation (24) characterizes the input-to-state stability (ISS) property [1] of $z_1$, and (26) characterizes the mean-square.

**Step 2:**

Applying $\alpha_2$ to the subsystem $\mathcal{S} 2$, we have

$$
\dot{z}_2 = -c_2 z_2 - \kappa_2 \sqrt{\frac{z_2^3}{z_2^2 + \nu} + \nu z_2 + z_3 - z_3}
$$

(27)

Assuming $z_3$ is made uniformly bounded at the next step, we can derive

$$
\frac{d}{dt} \left( \frac{z_3^2}{2} \right) \leq - \frac{c_2}{2} z_3^2 - \left[ \frac{c_2}{2} + \kappa_2 \sqrt{\frac{z_3^2}{z_2^2 + \nu} + \nu} \right] |z_2| (|z_2| - \mu_2(t))
$$

(28)

$$
\mu_2(t) = \frac{|z_3 - z_2|}{\frac{c_2}{2} + \kappa_2 \sqrt{\frac{z_3^2}{z_2^2 + \nu} + \nu}}
$$

(29)

Usually, we can only expect $\dot{z}_2 \approx z_3$ at low-frequencies. When the estimate $\dot{z}_3$ grows, the nonlinear damping term in the denominator grows accordingly such that the damping effect is enhanced. Then we have

$$
|z_2(t)| \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_2(\tau)
$$

(30)

Provided the ISS property of $z_2$, we furthermore have

$$
\frac{d}{dt} \left( \frac{z_2^2}{2} \right) \leq - \frac{c_2}{2} z_2^2 + \frac{|z_3 - z_2|^2}{2c_2}
$$

(31)

and hence

$$
\lim_{t \to \infty} \frac{1}{T} \int_0^T z_2^2 dt \leq \frac{1}{c_2} \lim_{T \to \infty} \frac{1}{T} \int_0^T \left( [1 - P(s)] z_3 \right)^2 dt
$$

(32)

**Step 3:**

Applying $u$ to the subsystem $\mathcal{S} 3$ (18), we have

$$
\dot{z}_3 = -c_3 z_3 - (u_{d1} + u_{d2} + u_{d3} + u_{d4}) \frac{G(\xi)}{G_0(\xi)} z_3
$$

$$
+ \Delta F_1(\xi) + \Delta F_2(\xi) + \alpha_{30} \frac{\Delta G(\xi)}{G_0(\xi)} + G(\xi) d
$$

(33)

Similar to the previous analysis, we can derive

$$
\frac{d}{dt} \left( \frac{z_3^2}{2} \right) \leq - \frac{c_3}{2} z_3^2 - \left[ \frac{c_3}{2} + D_3 \right] |z_3| (|z_3| - \mu_3(t))
$$

(34)

where

$$
\mu_3(t) = \frac{|w|}{c_3^2 + D_3}
$$

$$
w = \Delta F_1(\xi) + \Delta F_2(\xi) + \alpha_{30} \frac{\Delta G(\xi)}{G_0(\xi)} + G(\xi) d
$$

$$
D_3 = \frac{G(\xi)}{G_0(\xi)} \left( \kappa_{31} F_{d1}(\xi) + \kappa_{32} F_{d2}(\xi) + \kappa_{33} |\alpha_{30}| + \kappa_{34} G_0(\xi) \right)
$$

(35)

It can be verified that each uncertain term in the numerator of $\mu_3(t)$ is suppressed by the corresponding nonlinear damping term, i.e., when a term in the numerator grows, the corresponding term in the denominator grows at the same order such that $\mu_3(t)$ is uniformly bounded. Then we have

$$
|z_3(t)| \leq |z_3(0)| e^{-c_3 t/2} + \sup_{0 \leq \tau \leq t} \mu_3(\tau)
$$

(36)

Provided the ISS property of $z_3$, we furthermore have

$$
\frac{d}{dt} \left( \frac{z_3^2}{2} \right) \leq - \frac{c_3}{2} z_3^2 + \frac{|z_3|^2}{2c_3}
$$

(37)

and hence

$$
\lim_{t \to \infty} \frac{1}{T} \int_0^T z_3^2 dt \leq \frac{1}{c_3} \lim_{T \to \infty} \frac{1}{T} \int_0^T w^2 dt
$$

(38)

**Overall error system:**

The aforementioned ISS properties of the three subsystems can be summarized as follows.

$$
|z_1(t)| \leq |z_1(0)| e^{-c_1 t/2} + \sup_{0 \leq \tau \leq t} \left[ \frac{2}{c_1} z_2(\tau) \right]
$$

(39a)

$$
|z_2(t)| \leq |z_2(0)| e^{-c_2 t/2} + \sup_{0 \leq \tau \leq t} \mu_2(\tau)
$$

(39b)

$$
|z_3(t)| \leq |z_3(0)| e^{-c_3 t/2} + \sup_{0 \leq \tau \leq t} \mu_3(\tau)
$$

(39c)

These results imply that the overall error system can be expressed as the cascade of three ISS systems. Based on Lemma C.4 in [1], the overall error system is also ISS:

$$
\| z(t) \| \leq \sqrt{2\lambda_2} \| z(0) \| e^{-\rho_2 t} + \frac{\gamma_2}{c_1} \sup_{0 \leq \tau \leq t} \mu_3(\tau)
$$

(40)

where $z = [z_1, z_2, z_3]^T$, and

$$
\lambda_2 = \lambda_1^2 + (\lambda_1 + 1) \gamma_1 + 1, \quad \rho_2 = \min(c_1/8, c_2/8, c_3/4)
$$

$$
\gamma_1 = (\lambda_1 + 1) \gamma_1 + 1, \quad \lambda_1 = \frac{4}{c_1 + 2}
$$

$$
\gamma_2 = \left( \frac{4}{c_1 + 2} \right) \beta_2, \quad \beta_2 = \frac{2}{c_2} (1 + \| p(t) \|_1)
$$

(41)

where $p(t)$ is the impulse response of $P(s)$.

The results are valid only in the domain of interest

$$
\Omega_\xi = \{ \xi | 0 \leq \xi \leq x_{IM}, (g - \xi) > 0, (g_0 - \xi) > 0 \} \subset R^3
$$

(42)

due to the constraint of the setup. We should verify if there is a compact set $\mathcal{D}_\xi$ such that $\xi \in \mathcal{D}_\xi \subset \Omega_\xi$ by the designed controller. To this end, we impose the following assumption.
Assumption 1: The smooth reference trajectory is appropriately chosen such that
\[
\Omega_{\gamma} = \{y_r = [y_r, y_r, y_r, y_r, y_r(y_r)]^T | 0 < y_r < x_{1M}, |y_r| \leq \bar{y}_r, \\
|y_r| \leq \bar{y}_r, |y_r| \leq \bar{y}_r, |y_r| \leq \bar{y}_r, |y_r| \leq \bar{y}_r, |y_r| \leq \bar{y}_r \} \subset R^4
\] (43)

Define a compact set \( \Omega_z = \{z(t) \| 0 \leq z(t) \| c_z, \| z(t) \| c_z > 0 \} \subset R^3 \).

As long as the error vector \( z \) is relatively small, the steel ball may track a smooth reference trajectory with acceptable accuracy and does not escape from the domain of interest \( \Omega_z \). Thus we impose the following assumption:

Assumption 2: If \( z \in \Omega_z \), then \( \xi \in \Omega_z \subset \Omega_z \).

According to (26), (32) and (38), the mean-squares of the error signals of the three subsystems are summarized as follows.

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{z}^2 dt \leq \frac{1}{c_z} \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{z}^2 dt
\] (45a)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{z}^2 dt \leq \frac{1}{c_z} \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{z}^2 dt
\] (45b)

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{z}^2 dt \leq \frac{1}{c_z} \lim_{T \to \infty} \frac{1}{T} \int_0^T \dot{z}^2 dt
\] (45c)

The results are summarized in the following theorem.

Theorem 1: Let Assumptions 1 and 2 hold. And let the initial conditions \( z(0) \in \Omega_{\gamma} \subset \Omega_z \). If the state feedback robust nonlinear controller is applied to the magnetic levitation system under study, the following results hold:

1) There exists a compact set \( \Omega_z \) such that \( \xi \in \Omega_z \subset \Omega_{\gamma} \), where \( \xi \) is in the domain of interest \( [3, 4, 5] \). The physically allowable operating region of the steel ball shown in Fig. 1 is limited to \( 0 \leq x_1 \leq x_{1M} = 0.013 \). The output of the controllable voltage source is limited to \(-60[V] \leq u \leq 60[V]\).

2) The overall error system is ISS such that
\[
\| z(t) \| \leq \sqrt{2} \| \dot{z}(0) \| + \gamma_3 \sup_{0 \leq \tau \leq T} \| z(t) \| e^{-\rho_3 t} + \gamma_3 \sup_{0 \leq \tau \leq T} \| z(t) \| e^{-\rho_2 t}
\]
with \( \gamma_3, \gamma_2, \gamma_1 > 0 \).

3) The mean-squares of the error signals satisfy (45).

VI. CONSTRUCTION OF THE OUTPUT FEEDBACK CONTROLLER

A. Construction of the HGOb

To implement the state feedback controller by output feedback, we use the following HGOb [3], [4], [5]:

\[
\hat{\xi} = \hat{\xi} + \left( 1/\tau \right) (\hat{\xi} - \hat{\xi})
\]
(46)

where \( \tau \) is a small positive constant. The roots of \( s^2 + l_2 s + l_3 = 0 \) are in the open left-half \( s \)-plane. In this paper, \( l_1, l_2, l_3 \) are chosen such that a polynomial \( (s + 1)^3 \) is obtained.

B. Saturation of the controller

Since the output of the HGOb may introduce incorrect peaking signals, we have to saturate the signals outside of the domain of interest [3], [4], [5]. The physically allowable operating region of the steel ball shown in Fig. 1 is limited to \( 0 \leq x_1 \leq x_{1M} = 0.013 \). The output of the controllable voltage source is limited to \(-60[V] \leq u \leq 60[V]\). Also, according to (42), \( \hat{\xi}_1 \) and \( \hat{\xi}_2 \) should be saturated:

1. \( \hat{\xi}_3 \) should be saturated such that \( \sqrt{g_0 - \hat{\xi}_3} \) is well defined.

2. \( \hat{\xi}_1 \) should be saturated such that \( 0 \leq \hat{\xi}_1 \leq x_{1M} \).

3. \( u(\hat{\xi}^2) \) should be saturated such that \(-60[V] \leq u(\hat{\xi}^2) \leq 60[V] \).

where \( (\cdot)^s \) denotes a saturated signal.

C. Singularly perturbed form

Equations (22), (27) and (18) lead to the closed-loop system of the three subsystems:

\[
\dot{z} = A_z z + d_z
\] (47)

where

\[
d_z = \left[ \begin{array}{c} 0 \\ -\kappa_1 \sqrt{\frac{\gamma_2}{\gamma_1}} + \nu \gamma_2 - \hat{\xi}_3 \\ c_3 z_3 + F_1(\hat{\xi}) + F_2(\hat{\xi}) + G(\hat{\xi}) [u^2(\hat{\xi}^2) + d] - \alpha_2 \end{array} \right]
\]
(48)

\[
A_z = \left[ \begin{array}{ccc} -c_1 & 1 & 0 \\ 0 & -c_2 & 1 \\ 0 & 0 & -c_3 \end{array} \right]
\] (49)

Define the estimation error of HGOb as \( \tilde{\xi} = \xi - \hat{\xi} \). Then (7) and (46) lead to the dynamics of the estimation error:

\[
\dot{\tilde{\xi}} = \left[ \begin{array}{ccc} -l_1/\epsilon & 1 & 0 \\ -l_2/\epsilon & 0 & 1 \\ -l_3/\epsilon & 0 & 1 \end{array} \right] \tilde{\xi} + \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right] \eta_d
\] (50)

Scale the observer estimation error

\[
\eta_1 = \frac{\tilde{\xi}_1}{\epsilon}, \quad \eta_2 = \frac{\tilde{\xi}_2}{\epsilon}, \quad \eta_3 = \tilde{\xi}_3
\] (51)

and define \( \eta = [\eta_1, \eta_2, \eta_3]^T \). Then (46) and (47) lead to the standard singularly perturbed form:

\[
\dot{z} = A_z z + d_z
\] (52a)

\[
e \dot{\eta} = A \eta + e b d_\eta
\] (52b)

where

\[
A \eta = \left[ \begin{array}{ccc} -l_1 & 1 & 0 \\ -l_2 & 0 & 1 \\ -l_3 & 0 & 1 \end{array} \right], \quad b = \left[ \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right]
\] (53)
VII. STABILITY ANALYSIS OF THE OUTPUT FEEDBACK CONTROL SYSTEM

The stability analysis follows the same lines in [3], [4], [5]. However, due to the limit of paper length, we only mention the analysis results briefly.

We can establish that there is a short transient period during which the fast signal η decays to $O(\varepsilon)$ level, while, the slow signal $z$ still remains within the domain of interest.

For the slow system (52a), we can show that if the initial conditions are chosen such that $\|z(0)\| \leq c_0 < c_2$ for some $c_0 > 0$, then according to the definition of $\Omega_x$, there exists a time $T_2 > 0$, independent of $\varepsilon$, such that $z \in \Omega_x$ over the time interval $[0, T_2]$.

We can next investigate the behavior of the fast system (52b) over the interval $[0, T_2]$. For a $T_1(\varepsilon)$ that tends to zero as $\varepsilon \to 0$, there is a small $\varepsilon^*$ such that $T_1 < (0, T_2/2)$ for all $\varepsilon \in (0, \varepsilon^*)$. Hence we deduce that there exists a time $T_1 \in (0, T_2/2)$ such that for all $t \in [T_1, T_2]$, we have $\eta$ decays to $O(\varepsilon)$ level, where $T_2$ is the first time that $z$ escapes from the set $\Omega_2$, and $T_3$ can be made to be $\infty$.

We now go back to studying the behavior of $z(t)$ included in the slow system (52a) over $[T_1, T_2]$. Notice that over this time interval, $\eta$ and hence $\hat{\xi}$ are of $O(\varepsilon)$. Considering the locally Lipschitz property of the $\xi$ dependent functions in the control input $u(\hat{\xi})$, we have $\|u^*(\hat{\xi}) - u(\hat{\xi})\| \leq k_4 \varepsilon$ for some $k_4 > 0$. Substituting $u^*(\hat{\xi}) = u(\hat{\xi}) + [u^*(\hat{\xi}) - u(\hat{\xi})]$ into the subsystem $\mathcal{F}_3$ (18), we can rewrite (33) as

$$\dot{z}_3 = -c_3 z_3 - (u_{d1} + u_{d2} + u_{d3} + u_{d4}) + [G(\hat{\xi})/G_0(\hat{\xi})]z_3 + \Delta F_1(\hat{\xi}) + \Delta F_2(\hat{\xi}) + \alpha_{00} \Delta G(\hat{\xi}) + G(\hat{\xi}) + G(\hat{\xi}) [u^*(\hat{\xi}) - u(\hat{\xi})]$$

And (34) is rewritten as

$$\frac{d}{dt} \left( \frac{z_3^2}{2} \right) = -\frac{c_3}{2} z_3^2 - \left[ \frac{c_3}{2} + D_3 \right] |z_3| - \mu_{3e}(t)$$

where, $\mu_3(t)$ in (35) is rewritten as

$$\mu_{3e}(t) = \frac{1}{c_3} \frac{|w + G(\hat{\xi}) [u^*(\hat{\xi}) - u(\hat{\xi})]|}{|z_3| + D_3}$$

for some $k_3 > 0$. And hence (36) is rewritten as

$$|z_3(t)| \leq |z_3(T_1)| e^{-c_3(t-T_1)/2} + \sup_{T_1 \leq t \leq T} \mu_3(t) + k_3 \varepsilon$$

For any $\varepsilon \in (0, \varepsilon^*)$, (57) characterizes the input-to-state practical stability property (IsPS which is an extension of the concept of ISS) of the third subsystem. Notice that the discussions are for $t \in [T_1, T_2]$ and thus the starting time of the IsPS property is $T_1$. Finally, (40) is rewritten as

$$\|z(t)\| \leq \sqrt{2} \lambda_2 \|z(T_1)\| e^{-\rho_2(t-T_1)} + \rho_3 \sup_{T_1 \leq t \leq T} \mu_3(t) + k_6 \varepsilon$$

for some $k_6 > 0$. And we can find an appropriate set of initial conditions $z(0) \in \Omega_{3e} \subset \Omega_x$, so that we can make $z \in \Omega_x$ provided a sufficiently small $\varepsilon \in (0, \varepsilon^*)$. Here, $\Omega_{3e}$ is defined as the following.

$$\Omega_{3e} = \left\{ z(0) \mid \|z(0)\| \leq c_0 < c_2, \|z(t)\| \leq c_2, \right\}$$

Therefore, we can make the escape time $T_3 = \infty$.

Provided $z \in \Omega_x$, we can rewrite (37) as

$$\frac{d}{dt} \left( \frac{z_3^2}{2} \right) \leq -\frac{c_3}{2} z_3^2 + \frac{\rho_3^2}{2c_3} + k_7 \varepsilon$$

for some $k_7 > 0$. And hence (38) can be rewritten as

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T z_3^2 dt \leq \frac{1}{c_3} \lim_{T \to \infty} \frac{1}{T} \int_0^T \rho_3^2 dt + \frac{2k_7}{c_3} \varepsilon$$

Finally, we have

Theorem 2: Let Assumptions 1 and 2 hold. And let the initial conditions $z(0) \in \Omega_{3e} \subset \Omega_x$. If the output feedback robust nonlinear controller is applied to the magnetic levitation system under study, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$, the following results hold.

1) There exists a compact set $\mathcal{P}_2$ such that $\xi \in \mathcal{P}_2 \subset \Omega_x = \{ \xi \mid 0 \leq \xi_1 \leq x_{1,M}, (g - \xi_3) > 0, (g_0 - \xi_3) > 0 \} \subset \mathbb{R}^3$.

2) The overall error system is IsPS such that

$$\|z(t)\| \leq \sqrt{2} \lambda_2 \|z(T_1)\| e^{-\rho_2(t-T_1)} + \rho_3 \sup_{T_1 \leq t \leq T} \mu_3(t) + k_6 \varepsilon$$

for $\forall \rho_3, \lambda_2, \rho_2, k_6 > 0$.

3) The mean-squares of the error signals satisfy

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T z_3^2 dt \leq \frac{1}{c_3} \lim_{T \to \infty} \frac{1}{T} \int_0^T z_3^2 dt + \frac{2k_7}{c_3} \varepsilon$$

for $\forall k_7 > 0$.

VIII. EXPERIMENTAL RESULTS

A. Experimental setup

The physical parameters of the setup shown in Fig. 1 are given in Table 1. The vertical position $\xi_1$ of the steel ball is measured by a laser distance sensor.

<table>
<thead>
<tr>
<th>TABLE 1</th>
</tr>
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<tbody>
<tr>
<td>Parameters of the magnetic levitation system.</td>
</tr>
<tr>
<td>$M$</td>
</tr>
<tr>
<td>$g$</td>
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<tr>
<td>$x_0$</td>
</tr>
<tr>
<td>$Q$</td>
</tr>
<tr>
<td>$L_m$</td>
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<tr>
<td>$R$</td>
</tr>
</tbody>
</table>
B. Design of the controllers

The controller parameters were given as follows.

\[ c_1 = 40, \ c_2 = 40, \ c_3 = 10 \]
\[ \kappa_{21} = 1, \ \kappa_{31} = \kappa_{32} = \kappa_{33} = \kappa_{34} = 0.1 \] (62)

The external disturbance \( d \) in (3) was given as \( d = 5 \sin(\pi/2t) \), and the following nominal system parameters with considerable errors were used for experimental studies, to verify the robust performance of our controllers.

\[ M_0 = 0.30[\text{kg}], \ g_0 = 9.0[\text{m/s}^2] \]
\[ X_{\infty} = 0.0020[\text{m}], \ Q_0 = 0.0003[\text{Hm}] \]
\[ L_{\infty} = 0.50[\text{H}], \ R_0 = 10.0[\Omega] \] (63)

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The experimental results are shown in Figs. 2 and 3, where from the top to the bottom are the measured position \( \xi_1 \), position error \( z_1 \), velocity error \( z_2 \), acceleration error \( z_3 \), estimated position \( \hat{\xi}_1 \), estimated velocity \( \hat{\xi}_2 \), estimated acceleration \( \hat{\xi}_3 \), and the control input \( u \). Notice that \( z_2 \) and \( z_3 \) are calculated based on the estimated states \( \hat{\xi} \), not on the true \( \xi \). It can be found in Fig. 2 that in the absence of the DOB, the position-tracking performance is not acceptable, and we can even see the oscillations due to the sinusoidal disturbance \( d \). The results in Fig. 3 indicate the high performance of the proposed controller. As discussed previously, we need not pay great efforts to suppress \( z_3 \), and thus it is allowable that the amplitude of \( z_3 \) is not so small. However, owing to the

\[ \lambda = 0.02, \ \varepsilon = 0.0020 \]

\[ \text{DOB} \]

DOB employed at step 2, we can see that \( z_2 \) is suppressed significantly and consequently \( z_1 \) is suppressed to be very small. Therefore, it is not necessary to suppress \( z_3 \) by paying great efforts. Additionally, it should be mentioned here that we have verified that the HGOB works for some different values of \( \varepsilon \).

C. Comments on the experimental results

The experimental results are shown in Figs. 2 and 3, where from the top to the bottom are the measured position \( \xi_1 \), position error \( z_1 \), velocity error \( z_2 \), acceleration error \( z_3 \), estimated position \( \hat{\xi}_1 \), estimated velocity \( \hat{\xi}_2 \), estimated acceleration \( \hat{\xi}_3 \), and the control input \( u \). Notice that \( z_2 \) and \( z_3 \) are calculated based on the estimated states \( \hat{\xi} \), not on the true \( \xi \). It can be found in Fig. 2 that in the absence of the DOB, the position-tracking performance is not acceptable, and we can even see the oscillations due to the sinusoidal disturbance \( d \). The results in Fig. 3 indicate the high performance of the proposed controller. As discussed previously, we need not pay great efforts to suppress \( z_3 \), and thus it is allowable that the amplitude of \( z_3 \) is not so small. However, owing to the

IX. Conclusions

In this paper, we have treated the position-tracking control problem of an electromechanical system in the presence of external disturbance and modeling errors due to uncertainties of physical parameters. A robust output feedback nonlinear controller with DOB was designed to achieve excellent position-tracking performance. Strict analysis was performed based on the singularly perturbed system. Experimental results on a real magnetic levitation system were provided to support the theoretical results and to verify the advantage of the proposed controller.

REFERENCES