Partition of a Graph into Cycles and Isolated Vertices

Shinya Fujita
Department of Applied Mathematics
Science University of Tokyo
1-3 Kagurazaka, Shinjuku-ku, Tokyo, 162-8601 Japan

Abstract

Let $k, r, n$ be integers with $k \geq 2$, $0 \leq r \leq k-1$ and $n \geq 10k+3$. We prove that if $G$ is a graph of order $n$ such that the degree sum of any pair of nonadjacent vertices is at least $n-r$, then $G$ contains $k$ vertex-disjoint subgraphs $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ and such that $H_i$ is a cycle or isomorphic to $K_1$ for each $i$ with $1 \leq i \leq r$, and $H_i$ is a cycle for each $i$ with $r+1 \leq i \leq k$.

1 Introduction

In this paper, we consider only finite, simple, undirected graphs with no loops and no multiple edges. For a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For a vertex $x$ of a graph $G$, the neighborhood of $x$ in $G$ is denoted by $N_G(x)$, and we let $d_G(x) := |N_G(x)|$. For a noncomplete graph $G$, let $\sigma_2(G) := \min \{d_G(x) + d_G(y) | xy \notin E(G)\}$; if $G$ is a complete graph, let $\sigma_2(G) := \infty$. For an integer $n \geq 1$, we let $K_n$ denote the complete graph of order $n$. In this paper, “disjoint” means “vertex-disjoint”.

A sufficient condition for the existence of a specified number of disjoint cycles covering all vertices was given by Brandt et al. in [1]:

**Theorem A**([1]) Let $k, n$ be integers with $n \geq 4k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n$. Then $G$ contains $k$ disjoint cycles $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$.

In [4], Enomoto and Li showed that if we regard $K_1$ and $K_2$ as cycles, then the condition on $\sigma_2(G)$ in Theorem A can be weakened:

**Theorem B**([4]) Let $k, n$ be positive integers with $n \geq k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n-k+1$. Then unless $k = 2$ and $G$ is a cycle of length 5, $G$ contains $k$ disjoint subgraphs $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ and such that for each $1 \leq i \leq k$, $H_i$ is either
a cycle or isomorphic to $K_1$ or $K_2$.

Also, in [7], Hu and Li showed that if the order of $G$ is sufficiently large, then we do not need $K_2$ in Theorem B:

**Theorem C**([7]) Let $k,n$ be positive integers with $n \geq 10k + 3$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n - k + 1$. Then $G$ contains $k$ disjoint subgraphs $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ and such that for each $1 \leq i \leq k$, $H_i$ is either a cycle or isomorphic to $K_1$.

Along a slightly different line, Kawarabayashi [8] proved the following refinement of Theorem A:

**Theorem D**([8]) Let $k,n$ be integers with $k \geq 2$ and $n \geq 4k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n - 1$. Then one of the following holds:

(i) $G$ contains $k$ disjoint cycles $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$;

(ii) $G$ has a vertex set $S \subset V(G)$ with $|V(S)| = \frac{n - 1}{2}$ such that $G - S$ is independent; or

(iii) $G$ is isomorphic to the graph obtained from $K_{n-1}$ by adding a vertex and join it to precisely one vertex of $K_{n-1}$ (i.e., $G \cong (K_{n-2} \cup K_1) + K_1$).

The purpose of this paper is to "interpolate" Theorem C and Theorems D and A by proving the following theorem, which was conjectured by Enomoto [5]:

**Theorem 1** Let $k,r,n$ be integers with $2 \leq r \leq k - 2$ and $n \geq 7k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n - r$. Then $G$ contains $k$ disjoint subgraphs $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ and such that $H_i$ is a cycle or isomorphic to $K_1$ for each $i$ with $1 \leq i \leq r$, and $H_i$ is a cycle for each $i$ with $r + 1 \leq i \leq k$.

Combining Theorems A,C and D and Theorem 1, we obtain the following corollary:

**Corollary 2** Let $k,r,n$ be integers with $k \geq 2$, $0 \leq r \leq k - 1$ and $n \geq 10k + 3$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq n - r$. Then $G$ contains $k$ disjoint subgraphs $H_i$, $1 \leq i \leq k$, such that $V(H_1) \cup \ldots \cup V(H_k) = V(G)$ and such that $H_i$ is a cycle or isomorphic to $K_1$ for each $i$ with $1 \leq i \leq r$, and $H_i$ is a cycle for each $i$ with $r + 1 \leq i \leq k$. 
Our notation is standard except possibly for the following. Let $G$ be a graph. For a subset $L$ of $V(G)$, the subgraph induced by $L$ is denoted by $\langle L \rangle$. For a subset $M$ of $V(G)$, we let $G - M = \langle V(G) - M \rangle$ and, for a subgraph $H$ of $G$, we let $G - H = \langle V(G) - V(H) \rangle$. For subsets $L$ and $M$ of $V(G)$, we let $E(L, M)$ denote the set of edges of $G$ joining a vertex in $L$ and a vertex in $M$. A vertex $x$ is often identified with the set $\{x\}$. Thus if $x \in V(G)$, then $\langle \{x\} \rangle$, $G - x$ means $G - \{x\}$, and $E(x, M)$ means $E(\{x\}, M)$ for $M \subset V(G)$. We say that $G$ is pancyclic if $|V(G)| \geq 3$ and $G$ contains a cycle of length $l$ for each $l$ with $3 \leq l \leq |V(G)|$. For a cycle $C = x_1x_2 \ldots x_{|V(C)|}x_1$ and for a vertex $x = x_i \in V(C)$, we define $x^{i+j} = x_{i+j}$ and $x^{-j} = x_{i-j}$ (indices are to be read modulo $|V(C)|$). Also, we let $x^+ = x^{i+1}$, $x^- = x^{-1}$.

We conclude this section by listing known results which we use in the proof of Theorem 1.

**Theorem E([6])** Let $n \geq 3$ be an integer. Let $G$ be a 2-connected graph of order $n$, and suppose that $\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}$ for any $x, y \in V(G)$ such that $x$ and $y$ are at distance 2 apart. Then $G$ has a hamiltonian cycle.

**Theorem F([2])** Let $k, d, n$ be integers with $k \geq 3, d \geq 4k - 1$ and $n \geq 3k$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq d$. Then $G$ contains $k$ disjoint cycles covering at least $\min\{d, n\}$ vertices of $G$.

The following theorem, announced in [2], asserts that Theorem F holds for $k = 2$ as well.

**Theorem G([3])** Let $d, n$ be integers with $d \geq 7$ and $n \geq 6$. Let $G$ be a graph of order $n$, and suppose that $\sigma_2(G) \geq d$. Then $G$ contains two disjoint cycles covering at least $\min\{d, n\}$ vertices of $G$.

## 2 Preparation for the proof of Theorem 1

We start with three lemmas related to Theorem E.

**Lemma 2.1.** Let $\alpha \geq 3$ be an integer. Let $F$ be a 2-connected graph of order $\alpha$, and suppose that $\max\{d_F(x), d_F(y)\} > \frac{\alpha}{2}$ for any $x, y \in V(F)$ with $x \neq y$ and $xy \notin E(F)$. Then $F$ is pancyclic.

**Proof.** If $\alpha = 3$ or 4, then the assumption of the Lemma implies that $F \cong K_{\alpha}$. Thus we may assume $\alpha \geq 5$. We first prove that the following claim.

**Claim.** There exists $x \in V(F)$ with $d_F(x) > \frac{\alpha}{2}$ such that $F - x$ contains a
cycle \( D \) of length \( \alpha - 1 \) or \( \alpha - 2 \).

**Proof.** By Theorem E, \( F \) contains a hamiltonian cycle \( C \). Take \( x \in V(C) = V(G) \) with \( d_F(x) > \frac{\alpha}{2} \). If \( d_F(x^-) \leq \frac{\alpha}{2} \) and \( d_F(x^+) \leq \frac{\alpha}{2} \), then \( x^-x^+ \in E(F) \), and hence \( F - x \) contains a cycle of length \( \alpha - 1 \); if \( d_F(x^-) > \frac{\alpha}{2} \) and \( d_F(x^+) > \frac{\alpha}{2} \), then there exists \( y \in V(C) \) such that \( y \in N_F(x^-) \) and \( y^+ \in N_F(x^+) \) (it is possible that \( y = x^+ \) or \( y^+ = x^- \)), and hence \( F - x \) contains a cycle of length \( \alpha - 1 \). Thus we may assume \( d_F(x^-) \leq \frac{\alpha}{2} \) and \( d_F(x^+) > \frac{\alpha}{2} \). Arguing similarly with \( x \) replaced by \( x^+ \), we may also assume \( d_F(x^+) \leq \frac{\alpha}{2} \). But then \( x^-x^+ \in E(F) \), and hence \( F - \{x, x^+\} \) contains a cycle of length \( \alpha - 2 \). \( \square \)

Returning to the proof of the lemma, let \( x,D \) be as in the Claim. If \( |V(D)| = \alpha - 2 \), then \( |E(x, V(D))| > \frac{\alpha}{2} - 1 = \frac{|V(D)|}{2} \); if \( |V(D)| = \alpha - 1 \), then \( |E(x, V(D))| > \frac{\alpha}{2} > \frac{|V(D)|}{2} \). In either case, \( |E(x, V(D))| > \frac{|V(D)|}{2} \). Now let \( 3 \leq l \leq \alpha - 1 \). Then there exists \( z \in V(D) \) such that \( z \in N_F(x) \) and \( z^{+l-2} \in N_F(x) \). Thus \( \{x\} \cup \{z, z^+, \ldots, z^{+l-2}\} \) contains a cycle of length \( l \). \( \square \)

**Lemma 2.2.** Let \( r, \alpha \) be integers with \( \alpha \geq r + 2 \geq 4 \). Let \( F \) be a graph of order \( \alpha \), and suppose that \( F \) is not 2-connected, and \( \max\{d_F(x), d_F(y)\} \geq \frac{\alpha}{2} \) for any \( x, y \in V(F) \) with \( x \neq y \) and \( xy \notin E(F) \). Then one of the following holds:

1. \( F \) contains \( r \) disjoint subgraphs \( A_1, \ldots, A_r \) such that \( V(A_1) \cup \ldots \cup V(A_r) = V(F) \) and such that for each \( 1 \leq j \leq r \), \( A_j \) is either a cycle or isomorphic to \( K_1 \);

2. \( r = 2 \), \( F \) is disconnected, and one of the components of \( F \) has order 2; or

3. \( r = 2 \), and there exists \( e \in E(F) \) such that one of the components of \( F - e \) has order 2.

**Proof.** If \( F \) is connected, then let \( B \) be an endblock of \( F \) such that \( B - c \) contains a vertex \( a \) with \( d_F(a) \geq \frac{\alpha}{2} \), where \( c \) is the cut vertex of \( F \) contained in \( B \); if \( F \) is disconnected, then let \( B \) be a component of \( F \) such that \( B \) contains a vertex \( a \) with \( d_F(a) \geq \frac{\alpha}{2} \), and take \( c \in V(B) \). Then \( |V(B)| \geq d_B(a) + 1 = d_F(a) + 1 \geq \frac{\alpha}{2} + 1 \). Hence for each \( z \in V(F - B) \), \( d_F(z) \leq |(V(F - B) \cup \{c\}) - \{z\}| \leq \frac{\alpha}{2} - 1 \). This implies that \( F - B \) is a complete graph, and that

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Lemma 2.4. We first prove the following lemma. \( \alpha \) contradicts the assumption that we have \( G \) be a counterexample to Theorem 1. Let \( \alpha \) be an integer with \( \alpha - (r - 1) \), and hence \( \{C\} \cup \{v\} v \in V(F - C) \} \) forms a collection of subgraphs having the properties required in (1). Thus we may assume \( |V(F - B)| \geq r \). Then \( |V(B)| \geq \frac{\alpha}{2} + 1 \geq |V(F - B)| + 2 > r + \frac{\alpha}{2} \). If \( |V(F - B)| \geq 3 \), then \( F - B \) contains a cycle \( C \) of length \( |V(F - B)| \) and \( B \) contains a cycle \( D \) of length \( |V(B)| - (r - 2) \), and hence \( \{C, D\} \cup \{v\} v \in V(F - C - D) \} \) forms a collection of subgraphs with the desired properties. Thus we may assume \( |V(F - B)| = 2 \), which forces \( r = 2 \). By (2.1), \( d_{B - c}(x) \geq \frac{\alpha}{2} - 1 = \frac{V(B - c) + 1}{2} \) for every \( x \in V(B - c) \). This in particular implies that \( B - c \) is 2-connected. Hence by Theorem E, \( B - c \) contains a cycle \( C \) of length \( |V(B)| - 1 = \alpha - 3 \). Now if \( |E(c, V(F - B))| = 2 \), then \( C \) and \( \langle V(F - B) \cup \{c\} \rangle \) satisfy the properties required in (1). Thus we may assume \( |E(c, V(F - B))| \leq 1 \), which implies that (2) or (3) holds. \( \square \)

Lemma 2.3. Let \( r, \alpha \) be integers with \( \alpha \geq r + 2 \geq 4 \). Let \( F \) be a graph of order \( \alpha \), and suppose that \( \max \{d_F(x), d_F(y)\} \geq \frac{\alpha}{2} \) for any \( x, y \in V(F) \) with \( x \neq y \) and \( xy \notin E(F) \). In the case where \( r = 2 \), suppose further that \( |V(F)| \leq 6 \). Then \( F \) contains \( r \) disjoint subgraphs \( A_1, \ldots, A_r \) such that \( V(A_1) \cup \ldots \cup V(A_r) = V(F) \) and such that for each \( 1 \leq j \leq r \), \( A_j \) is either a cycle or isomorphic to \( K_1 \).

Proof. If \( F \) is 2-connected, then by Lemma 2.1, \( F \) contains a cycle \( C \) of length \( \alpha - (r - 1) \), and hence \( \{C\} \cup \{v\} v \in V(F - C) \} \) forms a collection of desired subgraphs. Thus we may assume \( F \) is not 2-connected. In view of Lemma 2.2, we may also assume that (2) or (3) of Lemma 2.2 holds. Then \( r = 2 \) and, with \( B \) and \( a \) as in the proof of Lemma 2.2, we have \( d_F(a) > \frac{\alpha}{2} \), and hence \( \alpha = |V(F)| = |V(B)| + 2 \geq (d_F(a) + 1) + 2 > \frac{\alpha}{2} + 3 \). This contradicts the assumption that we have \( \alpha \leq 6 \) when \( r = 2 \). \( \square \)

Throughout the rest of this paper, let \( n, k, r \) be as in Theorem 1, and let \( G \) be a counterexample to Theorem 1. Let \( L = \{v \in V(G)\} d_G(v) < \frac{\alpha + r}{2} \} \). Note that \( xy \in E(G) \) for any \( x, y \in L \) by the assumption that \( \sigma_2(G) \geq n - r \). We first prove the following lemma.

Lemma 2.4. In \( G \), there exist \( k - r \) disjoint cycles \( H_1, \ldots, H_{k-r} \) such that \( n - 3r \leq |\bigcup_{i=1}^{k-r} V(H_i)| \leq n - r \).

Proof. Take \( v_1, \ldots, v_r \in V(G) \), and let \( G' = G - \{v_1, \ldots, v_r\} \). Then \( \sigma_2(G') \geq n - 3r \). Since \( k - r \geq 2 \) and \( n - r > n - 3r > 4(k - r) \), it follows
from Theorems F and G that $G'$ contains $k - r$ disjoint cycles $H_1, \ldots, H_{k-r}$ such that $|\bigcup_{i=1}^{k-r} V(H_i)| \geq n - 3r$. Since $|\bigcup_{i=1}^{k-r} V(H_i)| \leq |V(G')| = n - r$, $H_1, \ldots, H_{k-r}$ are cycles with the desired properties.

Let $H_1, \ldots, H_{k-r}$ be as in Lemma 2.4. We choose $H_1, \ldots, H_{k-r}$ so that

(a) $|\bigcup_{i=1}^{k-r} V(H_i)|$ is maximum (subject to the condition that $|\bigcup_{i=1}^{k-r} V(H_i)| \leq n - r$) and,

subject to condition (a), so that

(b) $|\left(\bigcup_{i=1}^{k-r} V(H_i)\right) \cap L|$ is maximum

(we make use of (b) only in the proof of Lemma 2.15).

Let $H = \langle \bigcup_{i=1}^{k-r} V(H_i) \rangle$ and let $\alpha = |V(G - H)|$. If $\alpha = r$, then $\{H_1, \ldots, H_{k-r}\} \cup \{\langle v \rangle | v \in V(G - H)\}$ forms a collection of subgraphs having the properties required in Theorem 1. Thus we may assume $\alpha \geq r + 1$.

We now prove several lemmas which we use in estimating the degree of various vertices.

**Lemma 2.5.** Let $P = v_1v_2\ldots v_l (l \geq 1)$ be a path in $G - H$ and let $1 \leq i \leq k - r$, and suppose that $|V(H_i)| \geq l + 1$. Suppose that $N_G(v_1) \cap V(H_i) \neq \emptyset$, and let $x \in N_G(v_1) \cap V(H_i)$. Then $E(v_l, \{x^{-l}, x^{l+1}\}) = \emptyset$.

**Proof.** Suppose not. By symmetry, we may assume $v_lx^{l+1} \in E(G)$. Then $\langle V(H_i) \cup V(P) - \{x^{l+1}, \ldots, x^l\} \rangle$ contains a cycle $C$ of length $|V(H_i)| + 1$. Hence by replacing $H_i$ by $C$, we get a contradiction to the maximality of $|\bigcup_{i=1}^{k-r} V(H_i)|$. \hfill \Box

**Lemma 2.6.** Let $v \in V(G - H)$, and let $1 \leq i \leq k - r$. Then the following hold.

(i) No two vertices in $N_G(v) \cap V(H_i)$ are consecutive on $H_i$.

(ii) $|E(v, V(H_i))| \leq |V(H_i)|/2$.

**Proof.** Applying Lemma 2.5 with $l = 1$, we see that (i) holds, and (ii) follows from (i). \hfill \Box
Lemma 2.7. Let \( v \in V(G - H) \). Then \( |E(v, V(H))| \leq (n - \alpha)/2 \).

Proof. By Lemma 2.6(ii), \( |E(v, V(H))| \leq \sum_{i=1}^{\alpha} |V(H_i)|/2 = (n - \alpha)/2 \). □

Lemma 2.8. Suppose that \( \alpha = r + 1 \). Let \( v, v' \in V(G - H) \) with \( v \neq v' \), and let \( 1 \leq i \leq k - r \). Let \( a, b \in V(H_i) \) with \( a \neq b \), and suppose that \( a, b^+ \in N_G(v) \) and \( a^+, b \in N_G(v') \). Then \( \{a, a^+\} \cap \{b, b^+\} \neq \emptyset \).

Proof. Suppose that \( \{a, a^+\} \cap \{b, b^+\} = \emptyset \). Then \( \langle V(H_i) \cup \{v, v'\}\rangle \) contains disjoint cycles \( C, D \) such that \( V(C) \cup V(D) = V(H_i) \cup \{v, v'\}\). Since \( \alpha = r + 1 \), this means that \( \{H_1, \ldots, H_{i-1}, C, D, H_{i+1}, \ldots, H_{k-r}\} \cup \{\langle u \rangle \mid u \in V(G - H) - \{v, v'\}\} \) forms a collection of subgraphs with the desired properties. □

Lemma 2.9. Let \( vv' \in E(G - H) \), and let \( 1 \leq i \leq k - r \). Then the following statements hold:

(i) If \( v \) is adjacent to a vertex \( x \in V(H_i) \) and \( E(v', \{x^-, x^+\}) \neq \emptyset \), then \( \alpha = r + 1 \).

(ii) \( |E(\{v, v'\}, V(H_i))| \leq (2|V(H_i)| + 4)/3 \).

(iii) If \( N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset \), then \( |E(\{v, v'\}, V(H_i))| \leq (|V(H_i)| + 1)/2 \).

Proof. If \( vx \in E(G) \), \( E(v', \{x^-, x^+\}) \neq \emptyset \) and \( \alpha \geq r + 2 \), then \( \langle V(H_i) \cup \{v, v'\} \rangle \) contains a cycle \( C \) of length \( |V(H_i)| + 2 \) and, by replacing \( H_i \) by \( C \), we get a contradiction to the maximality of \( \bigcup_{i=1}^{\alpha} V(H_i) \). Thus (i) holds. We proceed to the proof of (ii) and (iii). If \( |V(H_i)| = 3 \), then by Lemma 2.6(ii), \( |E(\{v, v'\}, V(H_i))| \leq 1 + 1 = 2 \). Thus we assume that \( |V(H_i)| \geq 4 \), and define \( f(x) = |E(\{v, v'\}, \{x^-, x, x^+\})| \) for each \( x \in V(H_i) \) and, if \( N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset \), then we also define \( g(x) = |E(\{v, v'\}, \{x^-, x, x^+, x^+2\})| \) for each \( x \in V(H_i) \).

We first prove (ii). We start with the following claim.

Claim 1. Let \( z \in V(H_i) \). Then \( f(z) \leq 3 \). Further if equality holds, then \( \alpha = r + 1 \), and one of the following holds:

1. \( E(v, \{z^-, z, z^+\}) = \{vz, vz^+\} \) and \( E(v', \{z^-, z, z^+\}) = \{v'z\} \); or
2. \( E(v, \{z^-, z, z^+\}) = \{vz\} \) and \( E(v', \{z^-, z, z^+\}) = \{v'z^-, v'z^+\} \).

Proof. Suppose that \( f(z) \geq 3 \). Then \( |E(v, \{z^-, z, z^+\})| \geq 2 \) or \( |E(v', \{z^-, z, z^+\})| \geq 2 \). We may assume \( |E(v, \{z^-, z, z^+\})| \geq 2 \). Then by Lemma 2.6(i), \( E(v, \{z^-, z, z^+\}) = \{vz-, vz^+\} \). Therefore applying Lemma 2.5 with \( l = 2 \),
we obtain $E(v', \{z^-, z, z^+\}) = \{v'z\}$, and hence $\alpha = r + 1$ by (i). \hfill \Box

Now by way of contradiction, suppose that $|E(\{v, v'\}, V(H_i))| > (2|V(H_i)| + 4)/3$. Then since $|E(\{v, v'\}, V(H_i))| = (\sum_{z \in V(H_i)} f(z))/3$, it follows from Claim 1 that $\alpha = r + 1$, $|V(H_i)| \geq 5$, and the number of those vertices $z$ of $H_i$ for which $f(z) = 3$ is at least 5. Hence there exist $x, y \in V(H_i)$ with $f(x) = f(y) = 3$ such that $|\{x^-, x, x^+\} \cap \{y^-, y, y^+\}| \leq 1$. By the symmetry of $x$ and $y$, we may assume $\{x^-, x\} \cap \{y^-, y, y^+\} = \emptyset$. By the symmetry of $v$ and $v'$, we may assume (1) of Claim 1 holds for $x$. Now if (1) holds for $y$, we get a contradiction by applying Lemma 2.8 with $a = x^-$ and $b = y$; similarly if (2) holds for $y$, we get a contradiction by applying Lemma 2.8 with $a = x^-$ and $b = y^-$. Thus (ii) is proved.

To prove (iii), suppose that $N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset$.

**Claim 2.** Let $z \in V(H_i)$. Then $g(z) \leq 3$. Further if equality holds, then $\alpha = r + 1$, and one of the following holds:

1. $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^+\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z\}$;
2. $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z^-, v'z^+\}$;
3. $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz, vz^{+2}\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z^+\}$; or
4. $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^+\}$ and $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z, v'z^{+2}\}$.

**Proof.** Suppose that $g(z) \geq 3$. Then $|E(v, \{z^-, z, z^+, z^{+2}\})| \geq 2$ or $|E(v', \{z^-, z, z^+, z^{+2}\})| \geq 2$. Then by Lemma 2.6(i), $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^{+2}\} \cup \{vz^-, vz^+\}$ or $\{vz, vz^{+2}\}$. If $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^{+2}\}$, then applying Lemma 2.5 with $l = 2, 3$, we get $E(v', \{z^-, z, z^+, z^{+2}\}) = \emptyset$, which contradicts the assumption that $g(z) \geq 3$. Thus $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^+\}$ or $\{vz, vz^{+2}\}$. We may assume $E(v, \{z^-, z, z^+, z^{+2}\}) = \{vz^-, vz^+\}$. Then applying Lemma 2.5 again with $l = 2, 3$, we obtain $E(v', \{z^-, z, z^+, z^{+2}\}) = \{v'z\}$, and hence $\alpha = r + 1$ by (i). \hfill \Box

Returning to the proof of (iii), suppose that $|E(\{v, v'\}, V(H_i))| > (|V(H_i)| + 1)/2$. Then since $|E(\{v, v'\}, V(H_i))| = (\sum_{z \in V(H_i)} g(z))/4$, it follows from Claim 2 that $\alpha = r + 1$ and the number of those vertices $z$ of $H_i$ for which $g(z) = 3$ is at least 3. Take $x \in V(H_i)$ with $g(x) = 3$. By symmetry, we may assume (1) of Claim 2 holds for $x$. Then $E(\{v, v'\}, x^{+2}) = \emptyset$. Applying Claim 2 with $z = x^+$, we also see that $E(\{v, v'\}, x^{+3}) = \emptyset$. Similarly applying Claim 2 with $z = x^-$ and $z = x^{-2}$, we get $E(\{v, v'\}, x^{-2}) = \emptyset$ and $E(\{v, v'\}, x^{-3}) = \emptyset$. Hence again by Claim 2, $g(z) \leq 2$ for each $z \in
\{x^{-4}, x^{-3}, x^{-2}, x^+, x^{+2}, x^{+3}\}. Consequently \(|V(H_i)| \geq 9\) and there exists \(y \in V(H_i) - \{x^{-4}, x^{-3}, x^{-2}, x^+, x^{+2}, x^{+3}\}\) such that \(g(y) = 3\). Then \(\{x^-, x^+, x^{+2}\} \cap \{y^-, y^+, y^{+2}\} = \emptyset\). Therefore we get a contradiction by applying Lemma 2.8 with \(a = x^-\) and \(b = y^-, y^+\) or \(y^{+2}\), which proves (iii). □

**Lemma 2.10.** Let \(vv' \in E(G - H)\). Then the following hold.

(i) \(|E(\{v, v'\}, V(H))| \leq (2(n - \alpha) + 4(k - r))/3.\)

(ii) If \(N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset\), then \(|E(\{v, v'\}, V(H))| \leq ((n - \alpha) + (k - r))/2.\)

**Proof.** By Lemma 2.9(ii), \(|E(\{v, v'\}, V(H))| \leq \sum_{i=1}^{k-r}(2|V(H_i)| + 4)/3 = (2(n - \alpha) + 4(k - r))/3\) and, if \(N_{G-H}(v) \cap N_{G-H}(v) \neq \emptyset\), then by Lemma 2.9(iii), \(|E(\{v, v'\}, V(H))| \leq \sum_{i=1}^{k-r}(|V(H_i)| + 1)/2 = ((n - \alpha) + (k - r))/2.\)

□

**Lemma 2.11.** Let \(v \in V(G - H)\), and let \(1 \leq i \leq k - r\). Let \(x \in V(H_i)\), and suppose that \(N_G(v) \supset \{x, x^{+2}\}\). Then \(d_H(x^+) \leq (n - \alpha)/2.\)

**Proof.** By the assumption that \(N_G(v) \supset \{x, x^{+2}\}\), there exists a cycle \(C\) of length \(|V(H_i)|\) in \(\{(V(H_i) - \{x^+\}) \cup \{v\}\}\). Thus arguing similarly as in the proof of Lemma 2.6, we see from the maximality of \(|\cup_{j=1}^{k-r} V(H_j)|\) that \(|E(x^+, V(H_i))| \leq |H_j|/2\) for each \(j\) with \(1 \leq j \leq k - r\) and \(j \neq i\), and \(|E(x^+, V(C))| \leq |V(C)|/2\), and hence \(|E(x^+, V(H_i) - \{x^+\})| \leq |V(C)|/2 = |V(H_i)|/2\). Consequently, \(d_H(x^+) \leq \frac{1}{2} \sum_{j=1}^{k-r} |V(H_j)| = (n - \alpha)/2.\)

□

The following two lemmas are used when we choose an appropriate vertex in \(H\) where degree is to be estimated.

**Lemma 2.12.** Let \(v \in V(G - H) - L\). Suppose that either \(d_{G-H}(v) \leq \frac{1}{2} \alpha\) or \(\alpha \leq r + 2\). Then for some \(i\) with \(1 \leq i \leq k - r\), there exist three distinct vertices \(x, y, z \in V(H_i)\) such that \(N_G(v) \supset \{x, x^{+2}, y, y^{+2}, z, z^{+2}\}\)(it is possible that \(\{x, y, z\} \cap \{x^{+2}, y^{+2}, z^{+2}\} \neq \emptyset\)).

**Proof.** Suppose not. Then it follows from Lemma 2.6(i) that for each \(1 \leq i \leq k - r\), we have \(|E(v, \{x, x^+, x^{+2}\})| \leq 1\) for every vertex \(x \in V(H_i)\) possibly except two. Hence \(|E(v, V(H))| = \frac{1}{2} \sum_{i=1}^{k-r} \sum_{x \in V(H_i)} |E(v, \{x^-, x, x^+\})| \leq \frac{1}{3}(n - \alpha) + \frac{2}{3}(k - r).\) Since \(v \notin L\), this implies \(n - \alpha + \frac{2}{3}(k - r) + d_{G-H}(v) \geq d_G(v) \geq \frac{4-\alpha}{2}\), and hence \(n \leq 4k - r - 2\alpha + 6d_{G-H}(v)\). Now if \(d_{G-H}(v) \leq \frac{\alpha}{2}\), then from \(\alpha \leq 3r\) and \(r \leq k - 2\), we obtain \(n \leq 4k - r + \alpha \leq 4k + 2r < 6k\),
Lemma 2.13. Let \( v \in V(G - H) - L \) and \( v' \in N_{G-H}(v) \), and suppose that either \( d_{G-H}(v) \leq \frac{\alpha}{2} \) or \( \alpha \leq r + 2 \). Then for some \( i \) with \( 1 \leq i \leq k-r \), there exists \( x \in V(H_i) \) such that \( x, x^+ \in N_G(v), v, v' \notin N_G(x^+) \) and \( |E(x^+, V(G - H))| \leq \frac{\alpha-2}{2} \).

**Proof.** Let \( i, x, y, z \) be as in Lemma 2.12. Then by Lemma 2.6(ii), \( |V(H_i)| \geq 6 \). Suppose that some two of \( x^+, y^+ \) and \( z^+ \), say \( x^+ \) and \( y^+ \), have a common neighbor \( u \) in \( V(G-H) - \{v\} \). Then \( V(H_i) \cup \{v, u\} \) contains a cycle of length \( |V(H_i)| + 2 \). In view of the maximality of \( |\cup_{i=1}^k V(H_i)| \), this implies \( \alpha = r + 1 \). On the other hand, since \( |V(H_i)| \geq 6 \), it follows from Lemma 2.6(i) that we have \( \{x, x^+\} \cap \{y^+, y^{+2}\} = \emptyset \) or \( \{x^+, x^{+2}\} \cap \{y, y^+\} = \emptyset \). Consequently, we get a contradiction by applying Lemma 2.8 with \( a = x \) and \( b = y^+ \) or \( a = y \) and \( b = x^+ \). Thus no two of \( x^+, y^+ \) and \( z^+ \) have a common neighbor in \( V(G - H) - \{v\} \). In particular, at most one of \( x^+, y^+ \) and \( z^+ \) is adjacent to \( v' \). We may assume \( x^+v', y^+v' \notin E(G) \). We may also assume \( |E(x^+, V(H_i))| \leq |E(y^+, V(H_i))| \). Then since \( x^+v, y^+v \notin E(G) \) by Lemma 2.6(i), \( E(x^+, V(G - H)) \leq \frac{|V(G-H-\{v,v'\})|}{2} = \frac{\alpha-2}{2} \). Thus \( x \) has the desired properties. \( \square \)

Finally we prove two lemmas which we need in considering the case where \( V(G - H) \subset L \).

**Lemma 2.14.** Suppose that \( \alpha = r + 1 \) and there exists a triangle \( T \) in \( G - H \). Let \( 1 \leq i \leq k-r \) with \( |V(H_i)| \geq 4 \), and let \( x \in V(H_i) \). Then \( d_H(x) + d_H(x^+) \leq n - \alpha \).

**Proof.** Suppose that \( d_H(x) + d_H(x^+) > n - \alpha \). Then there exists \( j \) such that \( |E(x, V(H_j))| + |E(x^+, V(H_j))| > |V(H_j)| \). Assume for the moment that \( j = i \). Then there exists \( y \in V(H_i) \) such that \( xy, x^+y^+ \in E(G) \) (it is possible that \( y = x^+ \) or \( y^+ = x \)). Since \( |V(H_i)| \geq 4 \), this implies that \( \langle V(H_i) \rangle \} \) contains a cycle \( C \) of length \( |V(H_i)| - 1 \), and hence \( \{H_1, \ldots, H_{i-1}, C, \{y^+\}, H_{i+1}, \ldots, H_{k-r}, T\} \cup \{v\} \in V(G - H - T) \) forms a collection of subgraphs with the desired properties. Thus we may assume \( j \neq i \). Then there exists \( y \in V(H_j) \) such that \( xy, x^+y^+ \in E(G) \). (it is possible that \( y = y^+ \)), which implies that \( \langle V(H_i) \cup (V(H_j) - \{y^+, y^{+2}\}) \rangle \)
contains a cycle $C$ of length $|V(H_i)| + |V(H_j)| - 2$. Hence replacing $H_i$ and $H_j$ by $C$ and $T$, we get a contradiction to the maximality of $|\bigcup_{h=1}^{k-r} V(H_h)|$.

\[ \square \]

**Lemma 2.15.** Suppose that $V(G - H) \subset L$, and let $1 \leq i \leq k - r$.

(i) If $z \in V(H_i)$ and $E(z, V(G - H)) \neq \emptyset$, then $E(z^{+2}, V(G - H)) = \emptyset$.

(ii) There exists $x \in V(H_i)$ such that $E(x, V(G - H)) = \emptyset$ and $E(x^{+}, V(G - H)) = \emptyset$.

**Proof.** Suppose that there exists $z \in V(H_i)$ such that $E(z, V(G - H)) \neq \emptyset$ and $E(z^{+2}, V(G - H)) \neq \emptyset$, and take $v \in N_G(z) \cap V(G - H)$ and $v' \in N_G(z^{+2}) \cap V(G - H)$. If $v \neq v'$, then $(\{V(H_i) \cup \{v, v'\}\} - \{z\})$ contains a cycle $C$ of length $|V(H_i)| + 1$, and hence we get a contradiction to the maximality of $|\bigcup_{j=1}^{k-r} V(H_j)|$ by replacing $H_i$ by $C$. Thus $v = v'$. Then $(\{V(H_i) \cup \{v\}\} - \{z\})$ contains a cycle $C$ of length $|V(H_i)|$. Since $vz \notin E(G)$ by Lemma 2.6(i) and since $v \in L$ by the assumption of the lemma, $z^{+} \notin L$ by the assumption that $\sigma_2(G) \geq n - r$. Consequently, replacing $H_i$ by $C$, we get a contradiction to the maximality of $|\bigcup_{j=1}^{k-r} V(H_j)) \cap L|$. This proves (i). We now prove (ii). We may assume $E(V(H_i), V(G - H)) \neq \emptyset$. Take $y \in V(H_i)$ with $E(y, V(G - H)) \neq \emptyset$. Then $E(y^{+2}, V(G - H)) = \emptyset$ by (i). If $E(y^{+}, V(G - H)) = \emptyset$, then $y^{+}$ has the desired properties. Thus we may assume $E(y^{+}, V(G - H)) \neq \emptyset$. Then $E(y^{+3}, V(G - H)) = \emptyset$ by (i)( so $|V(H_i)| \geq 4$), and hence $y^{+2}$ has the desired properties.

\[ \square \]

### 3 Proof of Theorem 1

We continue with the notation of the preceding section, and complete the proof of Theorem 1. We divide the proof into two cases.

**Case 1:** $V(G - H) \not\subset L$

**Subcase 1.1.** $r + 3 \leq \alpha \leq 3r$.

If $d_{G-H}(z) > \alpha/2$ for all $z \in V(G - H) - L$, then by Lemma 2.3, $G - H$ contains $r$ disjoint subgraphs $A_1, \ldots, A_r$ such that $V(A_1) \cup \ldots \cup V(A_r) = V(G - H)$ and $A_j$ is either a cycle or isomorphic to $K_1$ for each $j$ (note that we have $|V(G - H)| \geq 3r = 6$ in the case where $r = 2$), and they together with $H_1, \ldots, H_{k-r}$ yield subgraphs with the desired properties. Thus we may assume there exists $v \in V(G - H) - L$ such that $d_{G-H}(v) \leq \alpha/2$. We
first consider the case where there exists \( v' \in N_{G-H}(v) \) such that \( N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset \). By Lemma 2.13, there exists a cycle \( H_1 \) and there exists \( x \in V(H_1) \) such that \( x, x^2 \in N_G(v) \) and \( v, v' \notin N_G(x^+) \). Since \( \alpha \geq r + 3 \), we see from the maximality of \( |\sum_{j=1}^k V(H_j)| \) that \( N_G(x^+) \cap N_G(v) \cap V(G-H) = \emptyset \) and \( N_G(x^+) \cap N_G(v') \cap V(G-H) = \emptyset \), and hence \( |N_G(x^+) \cap V(G-H)| + |N_G(v) \cap V(G-H)| \leq \alpha \) and \( |N_G(x^+) \cap V(G-H)| + |N_G(v') \cap V(G-H)| \leq \alpha \). Since \( |N_G(x^+) \cap V(H)| \leq (n-\alpha)/2 \) by Lemma 2.11 and \( |N_G(v) \cap V(H)| + |N_G(v') \cap V(H)| \leq ((n-\alpha)+(k-r))/2 \) by Lemma 2.10(ii), this implies \( 2d_G(x^+) + d_G(v) + d_G(v') \leq 2\alpha + (n-\alpha) + ((n-\alpha)+(k-r))/2 = 3n/2 + k/2 - r/2 + \alpha/2 \). On the other hand, since \( v, v' \notin N_G(x^+) \), \( 2d_G(x^+) + d_G(v) + d_G(v') \geq 2n - 2r \) by the assumption that \( \sigma_2(G) \geq n-r \). Consequently \( 2n - 2r \leq 3n/2 + k/2 - r/2 + \alpha/2 \), which implies \( n \leq k + 3r + \alpha \leq k + 6r < 7k \), a contradiction. We now consider the case where \( N_{G-H}(v) \cap N_{G-H}(z) = \emptyset \) for every \( z \in N_{G-H}(v) \). In this case, we have \( |N_G(v) \cap (L-V(H))| \leq 1 \) by the fact that \( (L-V(H)) \) is a complete graph. Since \( d_G(v) = d_G(v') - |N_G(v) \cap V(H)| \geq (n-r)/2 - (n-\alpha)/2 > 1 \) by Lemma 2.7 and the assumption of Subcase 1.1, this implies \( N_{G-H-L}(v) \neq \emptyset \). Take \( v' \in N_{G-H-L}(v) \). Since \( N_{G-H}(v) \cap N_{G-H}(v') = \emptyset \), \( |N_G(v) \cap V(G-H)| + |N_G(v') \cap V(G-H)| \leq \alpha \). Since \( |N_G(v) \cap V(H)| + |N_G(v') \cap V(H)| \leq (2(n-\alpha) + 4(k-r))/3 \) by Lemma 2.10(i), this implies \( d_G(v) + d_G(v') \leq \alpha + (2(n-\alpha) + 4(k-r))/3 \). On the other hand, we get \( d_G(v) + d_G(v') \geq n - r \) from \( v, v' \notin L \). Consequently \( n - r \leq 2n/3 + 4k/3 - 4r/3 + \alpha/3 \), which implies \( n \leq 4k - r + \alpha \leq 4k + 2r < 6k \), a contradiction.

Subcase 1.2. \( r + 1 \leq \alpha \leq r + 2 \).

Let \( v \in V(G-H) - L \). By Lemma 2.7, \( d_G(v) - |N_G(v) \cap V(H)| \geq \frac{\alpha-r}{2} > 0 \). Take \( v' \in N_{G-H}(v) \). By Lemma 2.13, we can find a cycle \( H_1 \) for which there exists \( x \in V(H_1) \) such that \( x, x^2 \in N_G(v) \), \( v, v' \notin N_G(x^+) \), and \( |N_G(x^+) \cap V(G-H)| \leq \frac{\alpha-r}{2} \). If \( N_{G-H}(v) \cap N_{G-H}(v') \neq \emptyset \), then by Lemma 2.10(ii) and Lemma 2.11, \( 2n - 2r \leq 2d_G(x^+) + d_G(v) + d_G(v') \leq 2(\frac{n-\alpha}{2} + \frac{\alpha-r}{2}) + (\frac{n-\alpha}{2} + \frac{\alpha-r}{2} + 2(\alpha-1)) \), which implies \( n \leq k + 3r + 3\alpha - 8 \leq k + 6r - 2 < 7k \), a contradiction. Thus we may assume \( N_{G-H}(v) \cap N_{G-H}(v') = \emptyset \).

Then \( |N_G(v) \cap V(G-H)| + |N_G(v') \cap V(G-H)| \leq \alpha \). Hence by Lemma 2.10(ii) and Lemma 2.11, \( 2n - 2r \leq 2d_G(x^+) + d_G(v) + d_G(v') \leq 2(\frac{n-\alpha}{2} + \frac{\alpha-r}{2}) + \frac{2(\alpha-r)}{3} + \alpha \). This is a contradiction, which completes the discussion for Case 1.

Case 2: \( V(G-H) \subset L \)

In this case, \( G-H \) is a complete graph by the definition of \( L \). If \( \alpha \geq r + 2 \), then \( G-H \) contains a cycle \( C \) of length \( \alpha - (r-1) \geq 3 \), and hence
\{H_1, \ldots, H_{k-r}, C\} \cup \{\langle v \rangle \mid v \in V(G - H - C)\} forms a collection of desired subgraphs of \(G\). Thus we may assume \(\alpha = r + 1\). Since \(|V(H)| = n - (r + 1) > 3k\), there exists \(H_i\) with \(|V(H_i)| \geq 4\). By Lemma 2.15(ii), there exists \(x \in V(H_i)\) such that \(N_G(x) \subseteq V(H)\) and \(N_G(x^+) \subseteq V(H)\). Take \(v, v' \in V(G - H)\). Note that \(\{v, v'\}\) is contained in a triangle of \(G - H\) because \(|V(G - H)| = r + 1 \geq 3\). Hence by Lemma 2.10(i) and Lemma 2.14,
\[
2n - 2r \leq d_G(v) + d_G(v') + d_G(x) + d_G(x^+) = (|N_G(v) \cap V(H)| + |N_G(v') \cap V(H)| + |N_G(v') \cap V(G - H)| + |N_G(x) \cap V(H)| + |N_G(x^+) \cap V(H)|) \leq \frac{(n-r-1)(k-r)}{2} + 2r + (n-r-1).
\]
Therefore \(n \leq k + 4r - 3 < 5k\), which is a contradiction.

This completes the proof of Theorem 1.

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References


[8] K.Kawarabayashi, Degree sum conditions and graphs which are not covered by \(k\) cycles, preprint.