Robust Global Exponential Synchronization of Uncertain Chaotic Delayed Neural Networks via Dual-Stage Impulsive Control

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Abstract—This paper is concerned with the robust exponential synchronization problem of a class of chaotic delayed neural networks with different parametric uncertainties. A novel impulsive control scheme (so-called dual-stage impulsive control) is proposed. Based on the theory of impulsive functional differential equations, a global exponential synchronization error bound together with some new sufficient conditions expressed in the form of linear matrix inequalities (LMIs) is derived in order to guarantee that the synchronization error dynamics can converge to a predetermined level. Furthermore, to estimate the stable region, a novel optimization control algorithm is established, which can deal with the minimum problem with two nonlinear terms coexisting in LMIs effectively. The idea and approach developed in this paper can provide a more practical framework for the synchronization of multiperturbation delayed chaotic systems. Simulation results finally demonstrate the effectiveness of the proposed method.

Index Terms—Chaos synchronization, chaotic delayed neural networks (DNNs), dual-stage impulsive control, impulsive functional differential equations (FDEs), linear matrix inequality (LMI), parametric uncertainty.

I. INTRODUCTION

In the past two decades, chaos synchronization has attracted considerable attention due to its great potential applications in secure communication, chemical reactions, and biological systems [1], [2]. After the pioneering work of Ott et al. [3] and Pecora and Carroll [4], many different methods have been applied theoretically and experimentally to synchronize the chaotic systems, such as linear and nonlinear feedback control [5]–[8], backstepping control [9], [10], variable structure control [11], [12], adaptive control [13]–[19], impulsive control [20]–[26], active control [27], [28], nonlinear observer approach [29]–[32], etc. Among these methods, impulsive control is an efficient method to deal with the dynamical systems which cannot be controlled by continuous control methods. In addition, in synchronization processes, the response system receives the information from the drive system only at the discrete time instants, which dramatically reduces the amount of synchronization information transmitted from the drive system to the response system and makes the method more efficient in a large number of real-life applications [33].

Recently, it has been revealed that some type of delayed neural networks (DNNs) can exhibit some complicated dynamics and even chaotic behavior if the parameters and time delays are appropriately chosen [34]–[39]. Thus, the synchronization problem of (chaotic) DNNs has received extensive consideration [6], [7], [14], [40]–[46].

It is well known that the effects of parametric uncertainties cannot be ignored in many applications; otherwise, the stability of the controlled system would be destroyed. In the past few years, the analysis of synchronization for chaotic systems with parametric uncertainties has gained much research attention [23], [47], [48], but most of the researches just assume that the parametric uncertainties of two chaotic systems are the same. In fact, when two identical chaotic systems are placed in a synchronization scheme, due to the inevitable perturbation in operations, the parametric uncertainties of the drive and response systems are always different and time varying, which will lead to parameter mismatch between the chaotic systems. In the case of small uncertainties or mismatches, the synchronization error does not decay to zero but fluctuates around zero. In particular, in the case of large ones, it can even result in desynchronization [25], [49]. Therefore, it is significant to investigate a more practical synchronization scheme for keeping the synchronization error within a predetermined region when the different time-varying parametric uncertainties exist. In [50]–[52], in order to estimate the synchronization error bound, two balls and one ellipsoid are given as follows:

\[ B_1(a_1) = \{ e : e^T e \leq a_1 \} \]

\[ E(b_1) = \{ e : e^T P e \leq b_1 \} \supset B_1(a_1) \quad \text{and is smallest} \]

\[ B_2(a_2) = \{ e : e^T e \leq a_2 \} \supset E(b_1) \quad \text{and is smallest} \]

Based on a quadratic Lyapunov function and convex optimization procedures, the error bound can be obtained. Unfortunately, the robust synchronization schemes in [50]–[52] only consider the nodelayed chaotic systems via continuous feedback control (corresponding to ordinary differential equations). For
delayed chaotic systems (e.g., chaotic DNNs studied in this paper), the relation between the error bound and parameter mismatches is quite intricate and complex. Therefore, the aforementioned method is no longer suitable, and a new analytical method needs to be put forward and studied. Theoretically, robust synchronization for delayed chaotic systems is more challenging due to the importance of time delay.

Generally speaking, to achieve the synchronization between different chaotic systems or identical ones with different parametric uncertainties, complicated continuous controllers are often required to compensate for the structure mismatch [13], [14], [27], [28], [51]. Compared with continuous control, the impulsive control to be studied in this paper is just composed of the linear feedback of the error signal, and only the state information at discrete time instants is needed, which is more efficient, secure, and, thus, useful in a great number of applications, particularly for secure communication.

In [20] and [21], the impulsive synchronization scheme for different chaotic systems with parametric uncertainties was proposed. Based on the new definition of synchronization, the derived criteria can guarantee that the error dynamics converge to below a given bound \( \xi > 0 \). It is worth pointing out that, once the error magnitude \( \|e\| \) is less than or equal to \( \xi \), the criteria in [20] and [21] will become unsuppressed, and during the time interval between any two successive impulsive instants, the error may likely exceed \( \xi \). Motivated by this, to overcome this possible instability and keep the error below \( \xi \) all along, in this paper, a novel impulsive control scheme (so-called dual-stage impulsive control) is proposed.

Inspired by the earlier discussions, this paper addresses a practical issue of using the dual-stage impulsive control method [corresponding to impulsive functional differential equations (FDEs)] to synchronize a class of chaotic DNNs with different time-varying parametric uncertainties, which has not been investigated in the literature to the best of our knowledge. By establishing an effective impulsive delay differential inequality, some sufficient conditions in the form of linear matrix inequalities (LMIs) are derived, which can be conducted readily using the Matlab LMI Toolbox. Furthermore, the designed dual-stage impulsive controller can not only realize the exponential synchronization with error bound but also estimate the exponential convergence rate. It is worth pointing out that, to estimate the stable region of synchronization error dynamics (i.e., the maximum upper bound of impulsive distance under given control gain), a novel and practical optimization control algorithm is proposed, which can deal with the minimum problem with two nonlinear terms coexisting in LMIs effectively.

The organization of this paper is as follows. In Section II, we give the problem formulation and preliminaries. In Section III, main results for robust exponential synchronization via dual-stage impulsive control are derived. In Section IV, two illustrative examples are presented to demonstrate the effectiveness of the proposed method. Finally, in Section V, concluding remarks and discussions on future research topics are given.

Throughout this paper, \( \mathbb{R} \) and \( \mathbb{R}^n \) denote the real number and \( n \)-dimensional Euclidean space, respectively. \( \mathbb{R}^{n \times m} \) is the set of all \( n \times m \) real matrices. \( \mathbb{R}^+ \doteq [0, \infty) \). \( N \doteq \{1, 2, \ldots\} \). For matrix \( Z, Z > 0(\leq 0) \) means that \( Z \) is a positive (negative) definite symmetric matrix. \( I \) denotes the identity matrix with appropriate dimensions. \( \| \cdot \| \) refers to the Euclidean vector norm or the induced matrix 2-norm. \( \lambda_{\max}(A) [\lambda_{\min}(A)] \) represents the maximum (minimum) eigenvalue of matrix \( A \). The notation * always denotes the symmetric block in one symmetric matrix.

II. PROBLEM FORMULATION AND PRELIMINARIES

Consider the following chaotic DNNs with time-varying parametric uncertainties:

\[
\dot{x}(t) = -D [C(x(t)) - (A + \Delta A(t)) f(x(t))]
- (B + \Delta B(t)) f(x(t - \tau(t))) + U
\]

where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \) is the neuron state vector, \( C(x(t)) = [c_1(x_1), c_2(x_2), c_3(x_3), \ldots, c_n(x_n)]^T \) is an appropriate function such that the solution of the model (1) remains bounded, the activation function \( f(\cdot) \) for all \( \hat{x}, \tilde{x} \) in \( \mathbb{R} \), satisfies

\[
\|f(\hat{x}) - f(\tilde{x})\| \leq L_1 \|\hat{x} - \tilde{x}\|, \quad \forall \hat{x}, \tilde{x} \in \mathbb{R}
\]

and

\[
\|f(\hat{x})\| \leq L_2, \quad \forall \hat{x} \in \mathbb{R}
\]

and

\[
\Delta A(t) = E_1 F_1(t) H_1, \quad \Delta B(t) = E_2 F_2(t) H_2
\]

where \( E_1 \) and \( H_i (i = 1, 2) \) are known real constant matrices with appropriate dimensions and the uncertain matrix \( F_i(t) \) satisfies

\[
\|F_i(t)\| \leq 1
\]
Remark 3: Formula (1) (with $\Delta A(t) = \Delta B(t) = 0$) unifies several well-known chaotic DNNs. In particular, if $d_1 = 1$, $c_i(x_i) = c_i x_i$ ($c_i \in \mathbb{R}$), and the activation function $f_i(x_i)$ is sigmoid, then (1) describes the dynamics of chaotic delayed Hopfield neural networks. Similarly, if $d_1 = 1$, $c_i(x_i) = c_i x_i$, and the activation function $f_i(x_i) = (|x_i + 1| - |x_i - 1|)/2$, then (1) describes the dynamics of chaotic delayed cellular neural networks (DCNNs).

We take system (1) as the drive system, and the response system is given by
\[
\dot{x}(t) = -D \left[ C(x(t)) - \left( A + \Delta \bar{A}(t) \right) f(x(t)) \right. \\
- \left. \left( B + \Delta \bar{B}(t) \right) f(\bar{x}(t - \tau(t))) + U \right]
\] (4)
where $\bar{x}(t)$ is the state vector of the response system and $\Delta \bar{A}(t) = \bar{E}_1 \bar{F}_1(t) \bar{H}_1$ and $\Delta \bar{B}(t) = \bar{E}_2 \bar{F}_2(t) \bar{H}_2$ have the same assumptions with $\Delta A(t)$ and $\Delta B(t)$.

At discrete time $t_k$, the state variables of the drive system are transmitted to the response system as the control input such that the state variables of the response system are suddenly changed at these instants. Therefore, the impulsive controlled response system can be written as (5), shown at the bottom of the page, where $W_k \in \mathbb{R}^{n \times n}$ is the control gain matrix and $e(t) = \bar{x}(t) - x(t)$ is the synchronization error. Suppose that the discrete instant set $\{t_k\}$ satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_k < t_{k+1} < \cdots$, $\lim_{k \to +\infty} t_k = \infty$, $\lim_{k \to 0^+} \bar{x}(t_k - h) = \bar{x}(t_k^+)$ and $\lim_{h \to 0^+} x(t_k - h) = x(t_k^+)$ imply that $\bar{x}(t)$ is right continuous at $t = t_k$. Thus, from (1), (5), and the fact that $x(t_k) = x(t_k^+)$, the error system on (6), shown at the bottom of the page can be obtained, where
\[
\begin{align*}
C(e(t)) &= C(e(t) + x(t)) - C(x(t)) = C(\bar{x}(t)) - C(x(t)) \\
\dot{h}(e(t)) &= f(e(t) + x(t)) - f(x(t)) = f(\bar{x}(t)) - f(x(t)).
\end{align*}
\]

Using Assumption 2, we have
\[
\|h(e(t))\|^2 = \sum_{i=1}^{n} h_i^2(c_i) \leq \sum_{i=1}^{n} \sigma_i^2 c_i^2(t) \leq \sigma^2 \|e(t)\|^2 (7)
\]
\[
\|h(e(t - \tau(t)))\|^2 \leq \sigma^2 \|e(t - \tau(t))\|^2 (8)
\]
where $\sigma = \max_{1 \leq i \leq n} (\sigma_i)$.

Note that the origin $e(t) = 0$ is not the equilibrium point of the error system (6), which implies that it is impossible to achieve complete synchronization [1] between systems (1) and (5). Therefore, we can employ the following definition of synchronization with error bound $\xi_1$, which is less restrictive than complete synchronization but more realistic and practical.

Definition 1: The robust global exponential synchronization between (1) and (5) is said to be achieved with error bound $\xi_1 > 0$ if, for arbitrary initial conditions $x_0$ and $\bar{x}_0$, the zero solution of the error system (6) globally exponentially converges to a predetermined neighborhood of the origin for any admissible parametric uncertainty that satisfies Assumptions 3 and 4, i.e., there exist $\xi_1 > 0$ and $T_1 > 0$ such that synchronization error $e(t)$ exponentially converges to the region $\Upsilon_1 \equiv \{e \in \mathbb{R}^n : \|e\| \leq \xi_1 \}$ for all $t \geq T_1$.

Our goal is to set the control gain $W_k$ and the impulse distance $\Delta_k = t_k - t_{k-1}$ ($k \in \mathbb{N}$) such that the error magnitude $\|e\|$ exponentially converges to a neighborhood of the origin with a given radius $\xi_1$ (i.e., the region $\Upsilon_1$) if the error starts with $\|e\| > \xi_1$, which implies that the impulsive controlled response system (5) exponentially synchronizes with the drive system (1) with error bound $\xi_1$ for arbitrary initial conditions.

For the sake of analytical simplification, we can choose a constant control gain and same impulse distance, which does not cause any loss of generality in the sense of synchronization analysis, i.e.,
\[
W_k = wI, \quad \Delta_k = \delta.
\] (9)

In [20] and [21], the robust synchronization with error bound for different nondelayed chaotic systems were considered. Under the impulsive controller $(w, \delta)$ described in (9) (i.e., the impulsive controller $(B_k, \Delta_k)$ in [20] and [21] are chosen as constants correspondingly), the synchronization error $e(t)$ can enter the region $\Upsilon_1$ if the error starts with $\|e\| > \xi_1$. However, theoretically speaking, after synchronization error $e(t)$ enters the region $\Upsilon_1$, the designed impulsive controller $(w, \delta)$ will cut no ice. During the time interval $\delta$ between any two successive impulsive instants, $e(t)$ may likely go away from $\Upsilon_1$. To overcome this possible unstability and keep $e(t)$ within $\Upsilon_1$ all along, we introduce another region $\Upsilon_2 \equiv \{e \in \mathbb{R}^n : \|e\| \leq \xi_2 \}$, where $0 < \xi_2 < \xi_1$. As shown in Fig. 1, under the fixed control gain coefficient $w$, if error $e(t)$ starts with $\|e\| > \xi_1$, we choose

\[
\begin{align*}
\dot{x}(t) &= -D \left[ C(x(t)) - \left( A + \Delta \bar{A}(t) \right) f(x(t)) \right. \\
&\left. - \left( B + \Delta \bar{B}(t) \right) f(\bar{x}(t - \tau(t))) + U \right] , \\
\dot{\bar{x}}(t_k) &= \bar{x}(t_k^+) + (W_k - I)e(t_k^+) , \\
e(t) &= \phi(t),
\end{align*}
\] (5)

\[
\begin{align*}
\dot{e}(t) &= -D \left[ C(e(t)) - \left( A + \Delta \bar{A}(t) \right) h(e(t)) - \left( B + \Delta \bar{B}(t) \right) h(\bar{x}(t - \tau(t))) \right. \\
&\left. - \left( 2 \Delta \bar{A}(t) - \Delta A(t) \right) f(x(t)) - \left( \Delta \bar{B}(t) - \Delta B(t) \right) f(x(t - \tau(t))) \right] , \\
e(t_k) &= W_ke(t_k^+) , \\
e(t) &= \phi(t),
\end{align*}
\] (6)

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impulsive distance $\delta_1$ such that $e(t)$ enters $\mathcal{Y}_1$. If error $e(t)$ enters $\mathcal{Y}_1$, we choose impulsive distance $\delta_2$ such that $e(t)$ enters $\mathcal{Y}_2$. After that, during the time interval $\delta_2$ between any two successive impulsive instants, $e(t)$ may likely go away from $\mathcal{Y}_2$; if we can ensure that the maximum value of $\|e\|$ is not larger than $\xi_1$ by choosing appropriate $w$, $\delta_1$, and $\delta_2$ values, then our goal is achieved. This control scheme is the so-called dual-stage impulsive control and can be described as follows:

$$ W_k = w I \quad \Delta_k = \begin{cases} \delta_1, & \text{if } \|e\| > \xi_1 \\ \delta_2, & \text{if } \|e\| \leq \xi_1. \end{cases} \tag{10} $$

Before giving our main results, we present several lemmas which are employed for future derivations.

**Lemma 1 (see [53]):** Given any real matrices $X_1 \in \mathbb{R}^{n \times k}$, $X_2 \in \mathbb{R}^{n \times k}$, $\Lambda \in \mathbb{R}^{n \times n}$, and $\Lambda = \Lambda^T > 0$, the following inequality holds:

$$ X_1^T X_2 + X_2^T X_1 \leq X_1^T \Lambda X_1 + X_2^T \Lambda^{-1} X_2. \tag{11} $$

**Lemma 2:** Let $0 \leq \tau(t) \leq \bar{\tau}$. If there exist constants $\rho$, $\bar{\rho} > 0$, and $\omega > 0$ such that

$$ \begin{aligned} & D^u(t) \leq \bar{\rho} u(t) + \rho u(t - \tau(t)), & t \neq t_k; t \geq 0 \\ & u(t_k) \leq \omega u(t_k), & k \in \mathbb{N} \\ & D^v(t) \leq \bar{\rho} v(t) + \rho v(t - \tau(t)), & t \neq t_k; t \geq 0 \\ & v(t_k) \leq \omega v(t_k), & k \in \mathbb{N} \end{aligned} \tag{12} \tag{13} $$

then $u(t) \leq v(t)$ for $-\bar{\tau} \leq t \leq 0$ implies that

$$ u(t) \leq v(t), \quad \text{for } t \geq 0 \tag{14} $$

where functions $u(t)$ and $v(t)$ belong to the set $PC(1)$. $PC(t) \triangleq \{ \varphi : [-\bar{\tau}, \infty) \to \mathbb{R}, I \in \mathbb{N}, \varphi(t) \text{ is continuous everywhere except for the finite number of points } t_k \text{ at which } \varphi(t_k^+) = \varphi(t_k^-) \text{ and } \varphi(t_k) \text{ exist}, \text{ and } D^+ \{ \} \text{ denotes the upper and right (Dini) derivative (the corresponding definition can be seen in [25]).} \$

The proof of Lemma 2 can be found in Appendix A.

### III. MAIN RESULTS

#### A. Robust Synchronization via Dual-Stage Impulsive Control

In this section, we let us deal with the robust global exponential synchronization with error bound $\xi_1$ for the drive system (1) and response system (5).

**Theorem 1:** If the control gain and impulsive distance are chosen as (10) and there exist positive diagonal matrices $P_m > 0$, $Q_m > 0$, and $S_m > 0$ and a real matrix $Z_m$ with appropriate dimensions and scalars $\xi_2 > 0$, $v_m$, $\tilde{v}_m > 0$, $\zeta_m > 0$, and $\bar{\rho}_m > 0$ ($m = 1, 2$) such that the following conditions hold:

$$ 0 < \|w\| < 1 \tag{15} $$

$$ \begin{bmatrix} \Xi_m & P_m D A & P_m D B & P_m D \tilde{E}_1 & P_m D \tilde{E}_2 \\ * & -Q_m & 0 & 0 & 0 \\ * & * & -S_m & 0 & 0 \\ * & * & * & -\zeta_m I & 0 \\ * & * & * & * & -\bar{\rho}_m I \end{bmatrix} < 0 \tag{16} $$

$$ Z_m < v_m P_m \theta_m < \tilde{v}_m P_m \bar{\rho}_m < 2 \ln \|w\| + \frac{1}{\delta_m} \left( v_m + \frac{\tilde{v}_m}{w^2} \right) < 0 \tag{17} \tag{18} \tag{19} \tag{20} \tag{21} $$

where $\Xi_m = -2 P_m D L + \Sigma_m Q_m \Sigma_m + \xi_m \Sigma_m \Sigma_m + \zeta_m \rho_m \zeta_m I - Z_m$, $\Theta_m = \Sigma_m \Sigma_m \Sigma_m + \xi_m \Sigma_m \Sigma_m$, $\zeta_m = 2d \sigma \chi(||E_1|| + ||E_1|| + ||E_1|| + ||E_2|| + ||E_1|| + ||E_2|| + ||E_1|| + ||E_2|| + ||E_1|| + ||E_2||)$, $L = \text{diag}(l_1, l_2, \ldots, l_n)$, $\Sigma_m = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\gamma_1 = ||H_1||^2$, $\gamma_2 = ||H_2||^2$, $\alpha = \xi_2 + d \bar{\rho}_m \xi_m (||B|| + ||E_2|| + ||H_2||) + \chi(||E_1|| + ||E_1|| + ||E_1|| + ||E_2|| + ||E_2|| + ||E_2|| + ||E_1|| + ||E_2||)$, $\beta = d \theta + \sigma(||A|| + ||E_2|| + ||H_1||)$, $\theta = \max_{1 \leq i \leq n}(\theta_i)$, and $d = \max_{1 \leq i \leq n}(d_i)$, then the response system (5) exponentially synchronizes with the drive system (1) with error bound $\xi_1 > 0$, and the exponential convergence rate $\lambda_m > 0$ is a unique solution of the following:

$$ 2 \lambda_m - a_m + b_m e^{2 \lambda_m \bar{\tau}} = 0, \quad m = 1, 2 \tag{22} $$

in which $a_m = -2 \ln \|w\|/\delta_m > 0$ and $b_m = \tilde{v}_m/w^2 > 0$.

The proof of Theorem 1 can be found in Appendix B.

**Remark 4:** Under successive impulsive signals, after the synchronization error $e(t)$ enters the region $\mathcal{Y}_2$, it is almost impossible for $e(t)$ to go away from $\mathcal{Y}_2$ in a short time. Therefore, in Appendix B, the assumption $T_{21} - T_1 \geq \bar{\tau}$ is usually reasonable. In case of $T_{21} - T_1 < \bar{\tau}$, the estimate of $\|e(s - \tau(s))\|$ for $s \in [T_{21}, T_{31}]$ will change to $\|e(s - \tau(s))\| \leq \xi_1 + \varepsilon_1 (t > 1)$, and then, (21) becomes $\alpha^I \exp(\beta \delta_2) \leq \xi_1$, where $\alpha^I = \xi_2 + d \bar{\rho}_m \xi_m (||B|| + ||E_2|| + ||H_2||) + \chi(||E_1|| + ||E_1|| + ||E_1|| + ||E_2|| + ||E_2|| + ||E_2|| + ||E_2||)$. By Theorem 1, a smaller impulsive distance $\delta_2$ can be obtained. In order to ensure $\|e(t)\| \leq \xi_1$, we can choose $\delta_2$ for $t \in [T_1, T_1 + \bar{\tau}]$ and choose $\delta_3$ for $t \geq T_1 + \bar{\tau}$ (note that the estimate of $\|e(s - \tau(s))\|$ in (69) is not larger than $\xi_1$ for $s \geq T_1 + \bar{\tau}$; therefore, the switch of the impulsive distance is reasonable).
Remark 5: In [25] and [49], the so-called quasi-synchronization schemes of a class of scalar delayed chaotic systems and the three-variable autocatalator model with single-parameter mismatch via impulsive control were investigated, respectively. For $t > N$ ($N$ is a large positive number), the quasi-synchronization can be achieved with the given error bound $\varepsilon > 0$. It is worth noting that the parameter mismatch studied in [25] and [49] just occurs for $t > N$, and in practice, the mismatch often exists all along. Therefore, the synchronization scheme studied in this paper, where the mismatch exists in the whole synchronization process, is more reasonable, and the designed method is more practical. Furthermore, [25] only considers the synchronization scheme of scalar delayed chaotic systems, and [49] only considers that of non-delayed chaotic systems with single-parameter mismatch. By comparison, this paper studies the robust synchronization of a class of multidimensional chaotic DNNs, and the parameter mismatch caused by the different parametric uncertainties of the drive and response systems exists in both the connection weight matrix and the delayed connection weight matrix. Thus, our results provide a more general method for the synchronization of multidimensional delayed chaotic systems with multiparameter mismatch.

Theorem 1 uses the control scheme (10), which has fixed control gain and switched impulsive distance. Alternatively, we can also use another version of dual-stage impulsive control scheme with fixed impulsive distance and switched control gain, which can be described as follows:

$$\Delta_k = \delta \quad W_k = \begin{cases} w_1 I, & \text{if } \|e\| > \xi_1 \\ w_2 I, & \text{if } \|e\| \leq \xi_1 \end{cases}$$

(23)

where $w_1 > 0$ and $w_2 > 0$ are the control gain coefficients. The analytical procedure is similar to that of Theorem 1. To avoid unnecessary duplication, in the sequel, only the synchronization criterion is given, and the details of proof are omitted.

Theorem 2: If the control gain and impulsive distance are chosen as (23) and there exist positive diagonal matrices $P_m > 0$, $Q_m > 0$, and $S_m > 0$ and a real matrix $Z_m$ with appropriate dimensions and scalars $\xi_2 > 0$, $v_m$, $\bar{v}_m > 0$, $\varepsilon_m > 0$, $\varepsilon_{m2} > 0$, and $\rho_m > 0$ ($m = 1, 2$) such that the following conditions hold:

$$0 < \|w_m\| < 1$$

(24)

$$\begin{bmatrix} Z_m & P_m DA \\ * & -Q_m \\ * & * & -S_m \\ * & * & * & -\frac{\varepsilon_{m1}}{\gamma_1} I \\ * & * & * & * & -\frac{\varepsilon_{m2}}{\gamma_2} I \end{bmatrix} < 0$$

(25)

$$Z_m < v_m P_m$$

(26)

$$\Theta_m < \bar{v}_m P_m$$

(27)

$$P_m < \rho_m I$$

(28)

$$\frac{2 \ln \|w_m\|}{\delta} + v_m + \frac{\bar{v}_m}{w_m^2} < 0$$

(29)

$$\hat{\alpha} \exp(\beta \delta) \leq \xi_1$$

(30)

where $\hat{\alpha} = \xi_2 + d\sigma \delta [\xi_1 (\|B\| + \|\dot{E}_2\| + \|\dot{H}_2\|) + \chi (\|E_1\| + \|H_1\| + \|E_2\| + \|H_2\|)]$ and $\Xi_m$, $\Theta_m$, $\xi$, $L$, $\Sigma_\sigma$, $\gamma_1$, $\gamma_2$, $\beta$, $\theta$, and $d$ are defined as in Theorem 1, then the response system (5) exponentially synchronizes with the drive system (1) with error bound $\xi_1 > 0$, and the exponential convergence rate $\lambda_m > 0$ is a unique solution of the following:

$$2\lambda_m - a_m + b_m e^{2\lambda_m \tau} = 0, \quad m = 1, 2$$

(31)

in which $a_m = -2 \ln \|w_m\|/\delta - v_m > 0$ and $b_m = \bar{v}_m/w_m^2 > 0$.

Remark 6: It is worth pointing out that, the results in Theorems 1 and 2 can also extend to the robust synchronization scheme for nonchaotic systems as long as the state boundedness of nonchaotic systems is satisfied.

B. Optimization Control Algorithm to Estimate the Stable Region

From (15) and (20), if $v_1 + w^{-2}\bar{v}_1 \leq 0$, then we can take $\delta_1 < \infty$ (i.e., $\delta_1$ can be taken as any finite positive number). Otherwise, we can estimate the stable region of the error system (6) (i.e., obtain the maximum impulsive distance $\delta_1$ under the given error bound $\xi_1$ and fixed control gain coefficient $w$), i.e., $\delta_1 < -2 \ln |w|/(v_1 + w^{-2}\bar{v}_1)$, by solving the following optimization problem:

$$\text{minimize } v_1 + w^{-2}\bar{v}_1 > 0,$$

s.t. $v_1 + w^{-2}\bar{v}_1 > 0$, with $m = 1$. (32)

For the aforementioned optimization problem (32), note that two nonlinear terms, i.e., $v_1 P_1$ and $\bar{v}_1 P_1$, coexist in the LMIIs (17) and (18); thus, the conventional LMI solvers (feasp, mincx, and GEVP) are not applicable directly. In view of this fact, we present the following procedure (see Fig. 2) to solve the optimization problem.

1) Initialize the system parameters, including $D$, $A$, $B$, $\Sigma_\sigma$, $L$, $\theta$, $\bar{\tau}$, the control gain $w$ that satisfies (15), and two given coefficients of accuracy $\gamma > 0$ and $\xi > 0$.

2) To obtain the theoretic minimum value of $v_1$ that satisfies the LMIIs (15)–(19), we can solve the following generalized eigenvalue minimization problem (GEVP) (see [54]):

$$\text{minimize } v_1$$

s.t. $\left\{ \begin{array}{l} \text{inequalities (15)} - (19) \text{ with } m = 1 \\ Z_1 < v_1 P_1. \end{array} \right.$$

(33)

We denote the minimum value obtained by $\mu$. Meanwhile, we choose a constant $\bar{\mu} > 0$ which satisfies $\bar{\mu} > \mu$.

3) Let $v_1 = \bar{\mu}$ (i.e., choose $\bar{\mu}$ as the first value of $v_1$ in the beginning of the new iterative operation). Choose $s_0$ and $s_1$ as temporary flag numbers, and let $s_0 = \bar{\chi}$, where positive constant $\bar{\chi}$ is large enough to guarantee the output of $\min \{v_1 + w^{-2}\bar{v}_1\}$.

4) Fix $v_1$ and use the following GEVP technique to obtain the minimum value of $\bar{v}_1$:

$$\text{minimize } \bar{v}_1 > 0$$

s.t. $\left\{ \begin{array}{l} \text{inequalities (15)} - (17), (19) \text{ with } m = 1 \\ \Theta_1 < \bar{v}_1 P_1. \end{array} \right.$$

(34)
Next, we give the following directions, which imply the rules for choosing $\varsigma$ and $\mu$, and the feasibility of the algorithm.

- From the iterative operation steps 4)–6), all the possible values of $s_0$ are tested, and the minimum one can be obtained. However, if the first value of $s_0$ is too small (i.e., $\varsigma$ is smaller than the actual minimum value of $v_1 + w^{-2}v_1$), then the aforementioned iterative operation will be disabled. Therefore, in this algorithm, constant $\varsigma$ is often chosen as a large-enough number to guarantee the output of $\min\{v_1 + w^{-2}v_1\}$.

- In step 2), the derived number $\mu$ can be regarded as the minimum value $v_1$ that satisfies LMIs (15)–(19). By solving the GEVP (33), we can get the minimum value $\mu$ that satisfies LMIs (15)–(17) and (19). If we add another LMI constraint, i.e., (18), the $v_1$ that satisfies LMIs (15)–(19) must be no less than $\mu$. Thus, we can take $\mu$ as the minimum value of $v_1$.

- We choose $v_1 = \bar{\mu}$ in step 3), and for the first operation in step 4), by the GEVP (34), we obtain a minimum value of $\bar{v}_1$, which is denoted by $\bar{v}$. Let $s_0 = \bar{\mu} + w^{-2}v$. Obviously, $\bar{\mu} < s_0$. If the constant $\bar{\mu}$ satisfies $s_0 < \bar{\mu}$, where $s_0$ is the final output in step 8) of the algorithm (32), then it implies that $s_0 < \bar{s}_0$, and in this case, $s_0$ can be regarded as the minimum value (i.e., $\min\{v_1 + w^{-2}v_1\}$). Accordingly, the constant $\bar{\mu}$ can be regarded as the theoretic maximum value of $v_1$. Otherwise, we choose another larger $\bar{\mu}$ (i.e., $\bar{\mu} + \varsigma$) to repeat the aforementioned procedure until $\bar{\mu}$ satisfies $s_0 \leq \bar{\mu}$.

- The interval $[\underline{\mu}, \bar{\mu}]$ contains the whole numerical range that $v_1$ should be, which guarantees the reasonable estimation of the minimum $s_0$. Furthermore, the minimum value obtained in step 8) will be more precise as $\varsigma$ decreases.

Subsequently, for estimating the maximum impulsive distance $\delta_{2\max}$, we can solve the following optimization problem:

$$
\begin{align*}
\text{maximize} & \quad \delta_2 > 0 \\
\text{s.t.} & \quad \begin{cases} 
\text{inequalities (15)–(20)} \\
\exp(\beta\delta_2) \leq \xi_1
\end{cases}
\end{align*}
$$

in which we only have to add condition (21) on the base of the algorithm described in Fig. 2, and the details are omitted for unnecessary repetition.

**C. Some Other Special Cases**

If the drive system (1) and response system (5) have the same parametric uncertainties, i.e., $\Delta A(t) = \Delta \dot{A}(t) = E_1 F_1(t) H_1$ and $\Delta B(t) = \Delta \dot{B}(t) = E_2 F_2(t) H_2$, correspondingly, we can easily obtain the following corollary.

**Corollary 1:** If the control gain and impulsive distance are chosen as (9) and there exist positive diagonal matrices $P > 0$, $Q > 0$, and $S > 0$ and a real matrix $Z$ with appropriate
dimensions and scalars $r, \tilde{r} > 0, \varepsilon_1 > 0,$ and $\varepsilon_2 > 0$ such that the following conditions hold:

$$0 < |w| < 1$$

$$\begin{bmatrix} \Xi & PDA & PDB & PDE_1 & PDE_2 \\ * & -Q & 0 & 0 & 0 \\ * & * & -S & 0 & 0 \\ * & * & * & -\frac{\Delta 1}{2} I & 0 \\ * & * & * & * & -\frac{\Delta 2}{2} I \end{bmatrix} < 0$$

$$Z < rP$$

$$\Sigma_\sigma S \Sigma_\sigma + \varepsilon_2 \Sigma_\sigma \Sigma_\sigma - Z - \frac{\Delta 1}{2} A \frac{\Delta 1}{2} = 0$$

$$\frac{2 \ln |w|}{\delta} + r + \frac{\tilde{r}}{w^2} < 0$$

where $\Xi = -2 PDL + \Sigma_\sigma Q \Sigma_\sigma + \varepsilon_1 \Sigma_\sigma \Sigma_\sigma - Z$, $\gamma_1 = \|H_1\|^2$, $\gamma_2 = \|H_2\|^2$, and $L$ and $\Sigma_\sigma$ are defined as in Theorem 1, then the error system (6) is globally exponentially stable, which implies that the response system (5) globally exponentially synchronizes with drive system (1), and the exponential convergence rate $\lambda > 0$ is a unique solution of

$$2 \lambda - a + be^{2\lambda \tilde{r}} = 0$$

in which $a = -2 \ln |w|/\delta - r$ and $b = \tilde{r}/w^2$.

**Proof:** Letting $\zeta = 0$ in Theorem 1 and in a similar manner to the first half part of the proof of Theorem 1, we can easily obtain the corollary. The details are omitted.

From (36) and (40), if $r + w^{-2} \tilde{r} \leq 0$, then we can take $\delta < \infty$. Otherwise, we can estimate the stable region of system (6) with the fixed control gain coefficient $w$, i.e., $\delta < -2 \ln |w|/(r + w^{-2} \tilde{r})$, by solving the following optimization problem:

$$\text{minimize } r + w^{-2} \tilde{r} > 0$$

$$\text{s.t. inequalities (36)–(39)}$$

which can be solved by a similar algorithm described in Fig. 2.

Similarly, if $\Delta A = \Delta A = \Delta B = \Delta \tilde{B} = 0$ in Corollary 1, it is straightforward to get the corresponding synchronization results. For the sake of simplicity, the detailed criteria are omitted.

**Remark 7:** The impulsive control scheme in Corollary 1 (with $\Delta A = \Delta A = \Delta B = \Delta \tilde{B} = 0$) can reduce to the ones discussed in [25] and [33]. Note that the result in Corollary 1 guarantees the exponential stability of error system instead of asymptotical stability studied in [25], which has better convergence characteristics. Furthermore, by solving the optimization problem (42), we can estimate the maximum value of impulsive distance $\delta$, in which the conditions can be solved readily through the LMI Toolbox. By comparison, some adjustable matrices of [33, Th. 3.1] need to be predetermined (e.g., in [33, Proposition 4.1], to obtain the estimate of maximum impulsive distance $\rho$, it needs $Q = E$, which will inevitably lead to conservatism). Therefore, the results in Corollary 1 have wider application fields than those of [25] and [33].

**Remark 8:** The sufficient conditions in Theorems 1–2 and Corollary 1 are all independent of the delay parameter $\tilde{r}$ [note that the delay parameter just influences the size of exponential convergence rate that is described in (22), (31), and (41)] but rely on the inequalities (LMIs) of the system’s parameters, impulsive distance and control gain, which are significant when the size of the delay $\tilde{r}$ is unknown.

### IV. EXPERIMENTAL VERIFICATIONS

**A. Synchronization With Error Bound for Chaotic DNNs With Different Parametric Uncertainties**

In order to observe the synchronization behavior of two chaotic DNNs, we define the drive and response systems as follows (see [34]):

$$\dot{x}(t) = -D \left[ C(x(t)) - (A + \Delta A(t)) f(x(t)) \right.$$

$$\left. - (B + \Delta B(t)) f(x(t - \tau(t))) \right]$$

$$\dot{\tilde{x}}(t) = -D \left[ C(\tilde{x}(t)) - (A + \Delta \tilde{A}(t)) f(\tilde{x}(t)) \right.$$

$$\left. - (B + \Delta \tilde{B}(t)) f(\tilde{x}(t - \tau(t))) \right]$$

where $d_i = 1$, $c_i(x_i) = x_i$, $f_i(x_i) = (|x_i + 1| - |x_i - 1|)/2$, $i = 1, 2$, $\tau(t) \equiv \tau = 1$, and the connection weight matrix and the delayed connection weight matrix are specified as

$$A = \begin{bmatrix} 1 + \pi/4 & 0.1 \\ 0.1 & 1 + \pi/4 \end{bmatrix}$$

$$B = \begin{bmatrix} -1.3\sqrt{\pi}/4 & 0.1 \\ 0.1 & -1.3\sqrt{\pi}/4 \end{bmatrix}.$$
the stable region of system (6) for different $\xi_1$. Furthermore, as shown in Fig. 4, we can obtain the estimates of the stable region of system (6) with different $\xi_1$ values based on Corollary 1.

Therefore, from (20), the estimate of the stable region of system (6) is given by

$$\delta_1 < -\frac{2}{s_0} \ln |w| = 0.0204.$$ 

Furthermore, as shown in Fig. 4, we can obtain the estimates of the stable region of system (6) for different $\xi_1$ values based on Theorem 1.

From (22), the exponential convergence rate with $\delta_1 = 0.02$ can be obtained as $\lambda_1 = 0.1628$. Synchronization errors and their magnitude with $(w, \delta_1, \delta_2) = (0.2, 0.02, 0.009)$ are shown in Fig. 5. We can see that the synchronization has been achieved practically and that $\|e\|$ is smaller than $\xi_1 = 0.05$. The initial conditions of the drive and response systems are taken as $x(s) = [0.1, 0.1]^T$ and $\hat{x}(s) = [0.2, -0.2]^T$ for $-1 \leq s \leq 0$, respectively.

B. Complete Synchronization for Chaotic DNNs With Same Parametric Uncertainties

If the drive system (43) and response system (44) have the same parametric uncertainties, i.e., $\Delta A(t) = \Delta A(t) = E_1 F_1(t) H_1$ and $\Delta B(t) = \Delta B(t) = E_2 F_2(t) H_2$, then the parameter mismatches (i.e., $\Delta A(t) - \Delta A(t)$ and $\Delta B(t) - \Delta B(t)$) vanish and the global exponential (complete) synchronization can be achieved by Corollary 1.

The parametric uncertainties are specified as

$$\Delta A(t) = \begin{bmatrix} \sin t & 0 \\ 0 & \sin t \end{bmatrix} \quad \Delta B(t) = \begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix}$$

i.e., $F_1(t) = \begin{bmatrix} \sin t & 0 \\ 0 & \sin t \end{bmatrix}$ and $F_2(t) = \begin{bmatrix} \cos t & 0 \\ 0 & \cos t \end{bmatrix}$

where $E_i = H_i = I$, $i = 1, 2$, and other parameters are the same as in systems (43) and (44).

If $w = 0.5$, $\gamma_1 = 1$, and $\gamma_2 = 1$, based on Corollary 1 and a similar algorithm described in Fig. 2, we get the minimum
value \( \min \{r + w^{-2} \hat{r}\} = 45 + 0.5^{-2} \times 1.2628 = 50.0512 \) and a set of feasible solutions as follows:

\[
P = \begin{bmatrix} 10.3673 & 0 \\ 0 & 10.3673 \end{bmatrix},
\]

\[
Q = \begin{bmatrix} 208.9483 & 0 \\ 0 & 208.9483 \end{bmatrix},
\]

\[
S = \begin{bmatrix} 7.7133 & 0 \\ 0 & 7.7133 \end{bmatrix},
\]

\[
Z = \begin{bmatrix} 465.9830 & 7.7884 \\ 7.7884 & 354.9798 \end{bmatrix},
\]

\[
\epsilon_1 = 10.3716, \quad \epsilon_2 = 5.3783.
\]

Therefore, from (40), system (6) is globally exponentially stable if \( \delta < -2 \ln |w|/(r + w^{-2} \hat{r}) = 0.0277 \).

Fig. 6 shows the time response curves of system (6) with \( w = 0.5 \) and \( \delta = 0.025 \).

![Time response curves of system (6) with w = 0.5 and δ = 0.025.](image)

**V. CONCLUSION AND OUTLOOK**

In this paper, we have investigated the issue of robust global exponential synchronization for a class of chaotic DNNs with different parametric uncertainties by using the dual-stage impulsive control method. Based on the definition of robust global exponential synchronization with error bound and the stability theory of impulsive FDEs, some new sufficient conditions are established. In addition, by using a novel and effective optimization control algorithm, the estimation of the stable region of synchronization error dynamics has also been given. Finally, some numerical simulations have been given to demonstrate the effectiveness of the method. Since parameter mismatch caused by different parametric uncertainties and other internal and external perturbations are ubiquitous in both nature and man-made systems, complete synchronization is always impossible and unnecessary. Thus, for the synchronization of multiperturbation delayed chaotic systems, the idea and approach developed in this paper can provide a more practical framework by keeping the synchronization error within a reasonable region, which is significant for applications of creating secure communication systems. It should be pointed out that future research topics include various potential application as well as further generalization and improvement, among which we would like to mention the following aspects.

1. The method of impulsive synchronization between two different chaotic DNNs is far from being straightforward due to their different structures and full parameter mismatch.

2. We can use the impulsive control scheme to investigate other synchronization problems, such as lag synchronization, anticipating synchronization, projective synchronization, phase synchronization, general synchronization, etc.

3. For the stability, stabilization, and synchronization of nonchaotic systems, the condition of time-varying delay can be extended to a more general case, such as \( 0 \leq \hat{\tau}(t) \leq h < 1 \) (see [40]) or \( 0 \leq \hat{\tau}(t) \leq h \) and \( 0 \leq \tau(t) \leq \hat{\tau} \) (see [55]).

4. The conditions we give for impulsive synchronization are only sufficient ones; less conservative conditions still need to be developed.

These issues are interesting and important, and continued research is desirable.

**APPENDIX A**

**Proof of Lemma 2:** We first prove that

\[
u(t) \leq v(t), \quad t \in [0, t_1).
\]

If (45) is not true, by using the continuity of \( u(t), v(t) \) for \( t \in [0, t_1) \) and \( u(t) \leq v(t) \) for \( t \in [-\bar{\tau}, 0] \), there must exist a \( \hat{t} \in [0, t_1) \) such that

\[
u(\hat{t}) = v(\hat{t})
\]

\[
D^+ u(\hat{t}) \geq D^+ v(\hat{t})
\]

\[
u(t) \leq v(t), \quad t \leq \hat{t}.
\]

In view of (12), (13), (46), (48), and the fact that \( \bar{\vartheta} > 0 \), we have

\[
D^+ u(\hat{t}) \leq \partial u(\hat{t}) + \vartheta u(\hat{t} - \tau(\hat{t}))
\]

\[
\leq \partial v(\hat{t}) + \bar{\vartheta} v(\hat{t} - \tau(\hat{t}))
\]

\[
< D^+ v(\hat{t})
\]

which contradicts with (47), and therefore, (45) holds. Next, assume that \( u(t) \leq v(t) \) for \( t \in [t_{m-1}, t_k), k \leq m, \) and \( m \in \mathbb{N} \). Then, we have \( u(t_m) \leq w v(t_m) \leq v(t_m) = v(t_m) \). Thus, \( u(t) \leq v(t) \) for \( t \in [t_m - \tau(t), t_m] \). In a similar manner to the proof of (45), we can have \( u(t) \leq v(t) \) for \( t \in [t_m, t_{m+1}) \). By mathematical induction, it is easy to conclude that \( u(t) \leq v(t) \) for \( t \geq 0 \). This completes the proof.

\[\square\]
APPENDIX B

Proof of Theorem 1: Denote \( y(\lambda_m) = 2\lambda_m - a_m + b_m\epsilon^{2\lambda_m} \). From (20), we have \( a_m - b_m > 0 \); thus, \( y(0) < 0 \), \( y(\infty) > 0 \), and \( y'(\lambda_m) > 0 \). Using the continuity and the monotonicity of \( y(\lambda_m) \), (22) has a unique positive solution \( \lambda_m > 0 \).

Now, we construct a Lyapunov-like function in the form of

\[
V(t, e(t)) = e^T(t)P_1e(t).
\]

(49)

For \( t \in [t_{k-1}, t_k) \) and \( k \in \mathbb{N} \), the Dini derivative of \( V(t, e(t)) \) along the trajectory of (6) is obtained as follows:

\[
D^+V(t, e(t)) = 2e^T(t)P_1D \left[ -C(e(t)) + (A^T + \Delta \hat{A}(t))h(e(t)) + \left(B + \Delta \hat{B}(t)\right)h(e(t) - \tau(t)) + \left(\Delta \hat{A}(t) - \Delta A(t)\right)f(x(t)) + \left(\Delta \hat{B}(t) - \Delta B(t)\right)f(x(t) - \tau(t))\right]
\]

\[
= -2e^T(t)P_1DC(e(t)) + 2e^T(t)P_1DAb(e(t)) + 2e^T(t)P_1DBh(e(t) - \tau(t)) + 2e^T(t)P_1DE\hat{E}_1(t)\hat{H}_1h(e(t)) + 2e^T(t)P_1\Delta \hat{A}(t) f(x(t)) + \left(\Delta \hat{B}(t) - \Delta B(t)\right)f(x(t) - \tau(t))
\]

(50)

Because \( c_i'(x_i) \geq 1 \), \( i = 1, 2, \ldots, n \), we have

\[
-e^T(t)P_1DC(e(t)) \leq -e^T(t)P_1DLc(t).
\]

(51)

To make the error enter the region \( \mathcal{Y}_1 \), i.e., \( \|e\| \leq \xi_1 \), here, we only have to consider the case of \( \|e\| > \xi_1 \), which implies that \( \|e\| \leq \xi_1^2 \|e\|^2 \leq \xi_1^2 e^T(t)E(t) \). From (17)–(19), (51), and Lemma 1, (50) becomes

\[
D^+V(t, e(t)) \leq -2e^T(t)P_1DLc(t) + e^T(t)P_1DAQ_1^{-1}A^TD P_1 e(t) + h^T(e(t))Q_1h(e(t)) + e^T(t)P_1DBS_1^{-1}B^TD P_1 e(t) + h^T(e(t) - \tau(t))S_1h(e(t) - \tau(t)) + \xi_{11}h^T(e(t))h(e(t)) + \xi_{11}e^T(t)P_1DE\hat{E}_1(t)\hat{H}_1\hat{H}_1^T(e(t))\hat{E}_1^T(t)D P_1 e(t)
\]

\[
+ \xi_{12}e^T(t)P_1DE\hat{E}_1(t)\hat{H}_1\hat{H}_1^T(e(t))\hat{E}_1^T(t)D P_1 e(t) + \xi_{12}e^T(t)P_1DE\hat{E}_2(t)\hat{H}_2\hat{H}_2^T(e(t))\hat{E}_2^T(t)D P_1 e(t) + 2\sigma_1P_{1}D\|D\|\left(\|E_1\|_{\mathcal{H}_1} + \|\hat{E}_1\|_{\mathcal{H}_1}\right)\|e\| + 2\sigma_1P_{1}D\|D\|\left(\|E_2\|_{\mathcal{H}_2} + \|\hat{E}_2\|_{\mathcal{H}_2}\right)\|e\|
\]

(52)

where

\[
\Pi_1 = \Xi_1 + P_1DAQ_1^{-1}A^TD P_1 + P_1DBS_1^{-1}B^TD P_1 + \gamma_1e_{11}P_1DE\hat{E}_1\hat{E}_1^TD P_1 + \gamma_2e_{12}P_1DE\hat{E}_2\hat{E}_2^TD P_1.
\]

By the Schur complement [56], one can easily derive that (16) is equivalent to \( \Pi_1 \geq 0 \), which implies that

\[
D^+V(t, e(t)) \leq e^T(t)\Pi_1 e(t) + e^T(t) - \tau(t)\{\Theta_1 - \hat{v}_1P_1)e(t - \tau(t)) + v_1e^T(t)P_1 e(t) + \hat{v}_1e^T(t - \tau(t))P_1 e(t - \tau(t)) \}
\]

(53)

On the other hand, for \( t = t_k, k \in \mathbb{N} \), we have

\[
V(t_k, e(t_k)) = e^T(t_k)W_k^T P_kw_k e(t_k) = \|w\|^2 e^T(t_k) P_k e(t_k) = \|w\|^2 V(t_k, e(t_k)).
\]

(54)

For any \( \varepsilon > 0 \), let \( v(t) \) be a unique solution of the following impulsive FDE:

\[
\begin{cases}
\dot{v}(t) = v_1(t) + \hat{v}_1(t) - \tau(t) + \varepsilon, & t \neq t_k; t \geq 0 \\
v(t_k) = \frac{v(t_k)}{k}, & k \in \mathbb{N} \\
v(t) = \rho_1\|\phi(t)\|^2, & -\bar{\tau} \leq t \leq 0.
\end{cases}
\]

(55)

From Lemma 2 and the fact that \( V(t) \leq \rho_1\|\phi(t)\|^2 \) for \( -\bar{\tau} \leq t \leq 0 \), we can conclude that

\[
V(t) \leq v(t), \quad \text{for} \quad \|e\| > \xi_1; t \geq 0.
\]

(56)

By the formula for the variation of parameters [57], the solution \( v(t) \) of (52) can be represented as

\[
v(t) = W(t, 0)v(0) + \int_0^t W(t, s)(\hat{v}_1v(s - \tau(s)) + \varepsilon)ds.
\]

(57)
where \( W(t, s), 0 \leq s \leq t, \) is the Cauchy matrix of the following linear system:

\[
\begin{align*}
\dot{q}(t) &= v_1 \eta(t), \\
q(t_k) &= w_2 \eta(t_k), \quad k \in \mathbb{N}.
\end{align*}
\]

By the representation of the Cauchy matrix \([57],\) we have the following estimate by \(|w| \in (0, 1)\) and \(\delta_1 \geq t_k - t_{k-1}:
\]

\[
W(t, s) = e^{a_1(t-s)} \prod_{s \leq t \leq t_k} w^2 \\
\leq e^{(-a_1 - 2\ln|w|)(t-s)} w^2 \left( \frac{w}{\pi} - 1 \right) \\
= w^{-2} e^{a_1(t-s)}, \quad 0 \leq s \leq t.
\]

Let \( \gamma = w^{-2} \sup_{-\bar{s} \leq s \leq 0} \{ v(s) \} \). Accordingly, it follows from (54) and (56) that

\[
\begin{align*}
v(t) &\leq w^{-2} e^{-a_1 t} v(0) + \int_0^t w^{-2} e^{-a_1 (t-s)} \\
&\quad \times \left( \bar{v}_1 v(s - \tau(s)) + \varepsilon \right) \, ds \\
&\leq \gamma e^{-a_1 t} + \int_0^t e^{-a_1 (t-s)} \\
&\quad \times \left( b_1 v(s - \tau(s)) + \frac{\varepsilon}{w^2} \right) \, ds, \quad t \geq 0.
\end{align*}
\]

Since \( \lambda_1 > 0, \varepsilon > 0, 0 < |w| < 1, \) and \( a_1 - b_1 > 0 \) [the inequality \( a_1 - b_1 > 0 \) is easy to get from (20)], (57) yields

\[
\begin{align*}
v(t) &\leq \frac{v(t)}{w^2} e^{-2\lambda_1 t} \\
&< \gamma e^{-2\lambda_1 t} + \frac{\varepsilon}{w^2(a_1 - b_1)}, \quad -\bar{s} \leq t \leq 0.
\end{align*}
\]

In the following, we will prove that the following inequality holds:

\[
v(t) < \gamma e^{-2\lambda_1 t} + \frac{\varepsilon}{w^2(a_1 - b_1)}, \quad t \geq 0.
\]

If this is not true, by the estimate (58) and \( v(t) \in PC(1), \) then there exists a \( t^* > 0 \) such that

\[
\begin{align*}
v(t^*) &\geq \gamma e^{-2\lambda_1 t^*} + \frac{\varepsilon}{w^2(a_1 - b_1)} \\
v(t) &< \gamma e^{-2\lambda_1 t} + \frac{\varepsilon}{w^2(a_1 - b_1)}, \quad t < t^*.
\end{align*}
\]

By (22), (57), and (61), we have

\[
\begin{align*}
v(t^*) &\leq \gamma e^{-a_1 t^*} + \int_{t^*}^t e^{-a_1 (t^*-s)} (b_1 v(s - \tau(s)) + \varepsilon) \, ds \\
&< \gamma e^{-a_1 t^*} + \int_0^{t^*} e^{a_1 s} \left[ b_1 \gamma e^{-2\lambda_1 (s - \tau(s))} + \frac{b_1 \varepsilon}{w^2(a_1 - b_1)} + \frac{\varepsilon}{w^2} \right] ds
\end{align*}
\]

\[
\leq e^{-a_1 t^*} \left\{ \gamma + \frac{b_1 \varepsilon}{w^2(a_1 - b_1)} + \frac{\varepsilon}{w^2} \right\} e^{(a_1 - 2\lambda_1)t^*} \int_0^{t^*} e^{(a_1 - 2\lambda_1)s} \, ds \\
+ \frac{a_1 \varepsilon}{w^2(a_1 - b_1)} \int_0^{t^*} e^{a_1 s} \, ds
\]

\[
= \gamma e^{-2\lambda_1 t^*} + \frac{\varepsilon}{w^2(a_1 - b_1)} - \frac{\varepsilon e^{-a_1 t^*}}{w^2(a_1 - b_1)} \\
< \gamma e^{-2\lambda_1 t^*} + \frac{\varepsilon}{w^2(a_1 - b_1)}.
\]

This contradicts with (60), and therefore, the estimate (59) holds. Next, we will further prove that the following inequality holds:

\[
v(t) \leq \gamma e^{-2\lambda_1 t}, \quad \text{for } t \geq 0.
\]

For the sake of simplicity, let \( \kappa(t) = v(t) - \gamma e^{-2\lambda_1 t} \) and \( \eta_0 = \varepsilon/(w^2(a_1 - b_1)) \); then, (59) changes to

\[
\kappa(t) < \eta_0, \quad t \geq 0
\]

and (63) changes to

\[
\kappa(t) \leq 0, \quad t \geq 0.
\]

If (65) is not true, there must exist \( \bar{t} \geq 0 \) such that \( \kappa(\bar{t}) = \eta_1 > 0 \), where \( \eta_1 \) is some positive constant. Since \( w^2 \in (0, 1) \) and \( a_1 - b_1 > 0 \), for arbitrary \( \varepsilon > 0 \), constant \( \eta_0 \) should also be an arbitrary positive number. We can let \( \eta_0 < \eta_1 \), and thus, \( \kappa(\bar{t}) = \eta_1 \geq \eta_0 \) holds. However, this contradicts with (64). Therefore, (65) holds, which implies that (63) holds.

Summarizing (53) and (63), we can conclude that

\[
V(t) \leq v(t) \leq \gamma e^{-2\lambda_1 t}, \quad \text{for } ||e|| > \xi_1; t \geq 0
\]

i.e.,

\[
||e(t)|| \leq M_1 e^{-\lambda_1 t} \sup_{-\bar{s} \leq s \leq 0} \{ ||\phi(s)|| \}, \quad \text{for } ||e|| > \xi_1; t \geq 0
\]

where

\[
M_1 = (1/|w|) \sqrt{P_1 / P_2} > 1 \quad \text{and} \quad \gamma = w^{-2} \sup_{-\bar{s} \leq s \leq 0} \{ v(s) \} = w^{-2} \sqrt{P_1} \sup_{-\bar{s} \leq s \leq 0} \{ ||\phi(s)||^2 \}.
\]

From (67), it is clear that the synchronization error \( e(t) \) can exponentially enter \( \Upsilon_1 \) with the exponential convergence rate \( \lambda_1 \) if the error starts with \( ||e|| > \xi_1 \). By the similar analytical procedure [just let \( m = 2 \) in (16)–(20) and (22)], it is easy to conclude that the synchronization error \( e(t) \) can exponentially enter \( \Upsilon_2 \) with the exponential convergence rate \( \lambda_2 \) if the error magnitude satisfies \( \xi_2 < ||e|| \leq \xi_1 \), and the relative proof is omitted for avoiding unnecessary repetition.

To keep the synchronization error \( e(t) \) within \( \Upsilon_1 \) in case of an error going away from \( \Upsilon_2 \), the appropriate control gain coefficient \( w \) and impulsive distance \( \delta_2 \) have to be set. In the following, we will give the sufficient condition to ensure that the maximum value of \( ||e|| \) does not exceed \( \xi_1 \) even if the synchronization error goes away from \( \Upsilon_2 \).
First, in the process where the synchronization error enters Υ₁, we define three special instants (cf. Fig. 1).

1) $T_1$, which satisfies $\|e(t)\| \leq \xi_1$ for $t \geq T_1$, particularly, $\|e(T_1)\| = \xi_1$. From (67), we can estimate its value as $T_1 = -\ln(\xi_1/(M_1 \sup_{s \leq 0}(\|\phi(s)\|))/\lambda_1$.

2) $T_{2j}$, the previous impulsive instant before $e(t)$ goes away from $\Upsilon_j$ for the $j$th time, which satisfies $\|e(T_{2j})\| \leq \xi_2$, $j \in \mathbb{N}$.

3) $T_{3j}$, the next impulsive instant after $e(t)$ goes away from $\Upsilon_j$ for the $j$th time, which satisfies $\xi_2 < \|e(T_{3j})\| \leq \xi_1$, $j \in \mathbb{N}$.

Obviously, we can obtain $T_{3j} - T_{2j} = \delta_2$. Furthermore, we assume that $T_{21} - T_1 \geq \bar{\tau}$, i.e., $T_{2j} - T_1 \geq \bar{\tau}, j \in \mathbb{N}$ (the reasonableness of this assumption is described in Remark 4).

Integrating the first equation of (6) from $T_{2j}$ to $t$, where $t \in [T_{2j}, T_{3j})$ yields

$$e(t) = e(T_{2j}) - \int_{T_{2j}}^{t} D \left[ C(e(s)) - \left( A + \Delta \tilde{A}(s) \right) \dot{h}(e(s)) \right] ds - \left( B + \Delta \tilde{B}(s) \right) \dot{h}(e(s) - \tau(s)) - \left( \Delta \tilde{A}(s) - \Delta A(s) \right) f(x(s)) - \left( \Delta \tilde{B}(s) - \Delta B(s) \right) f(x(s) - \tau(s)) ds. \tag{68}$$

By using the Gronwall–Bellman inequality [58], it follows from (21) and (70) that

$$\|e(t)\| \leq \alpha \exp[\beta(T - T_{2j})] \leq \alpha \exp[\beta(T_{3j} - T_{2j})] = \alpha \exp(3\delta_2) \leq \xi_1, \quad j \in \mathbb{N}. \tag{71}$$

For $t \in [T_{3j}, T_{2(j+1)}), j \in \mathbb{N}$, by the first half part of this proof, the synchronization error will exponentially enter $\Upsilon_j$ again. Thus, it is clear that the aforementioned dual-stage impulsive control method can guarantee $\|e\| \leq \xi_1$ for all $t \geq T_1$.

Therefore, we can conclude that the synchronization error $e(t)$ can exponentially converge to the region $\Upsilon_j$, if $e(t)$ starts with $\|e(0)\| > \xi_1$, which implies that the response system (5) exponentially synchronizes with the drive system (1) with an error bound $\xi_1$. This completes the proof. □

**REFERENCES**


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