Global exponential stability in Lagrange sense for periodic neural networks with various activation functions

Ailong Wu a,*, Zhigang Zeng a, Chaojin Fu b, Wenwen Shen a

a Department of Control Science and Engineering, Huazhong University of Science and Technology, Wuhan 430074, China
b College of Mathematics and Statistics, Huabei Normal University, Huangshi 435002, China

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A B S T R A C T

In this paper, global exponential stability in Lagrange sense for periodic neural networks with various activation functions is further studied. By constructing appropriate Lyapunov-like functions, we provide easily verifiable criteria for the boundedness and global exponential attractiveness of periodic neural networks. These theoretical analysis can narrow the search field of optimization computation, associative memories, chaos control and provide convenience for applications.

1. Introduction

Lyapunov stability is one of the important properties of dynamic systems. From a system-theoretic point of view, the global stability of neural networks is a very interesting issue for research because of the special nonlinear structure of neural networks. From a dynamical system point of view, globally stable neural networks in Lyapunov sense are monostable systems, which have a unique equilibrium attracting all trajectories asymptotically, more specific results are referred to [1–10,16–18]. In many other applications, however, monostable neural networks have been found to be computationally restrictive and multistable dynamics are essential to deal with the important neural computations desired. In these circumstances, neural networks are no longer globally stable and more appropriate notions of stability are need to deal with multistable systems. In this context, many researchers focus on the Lagrange stability. It is noted that unlike Lyapunov stability, Lagrange stability refers to the stability of the total system, not the stability of the equilibriums, because the Lagrange stability is considered on the basis of the boundedness of solutions and the existence of global attractive sets (see [11–15,19–21]). We also note that Lagrange stability has attracted phenomenal world-wide attention. In [11,12], Liao et al. apply Lyapunov functions to study Lagrange stability for recurrent neural networks. In [13], Yang and Cao consider stability in Lagrange sense of a class of feedback neural networks for optimization problems. In [14,15], Lagrange stability is discussed for Cohen–Grossberg neural networks. At present, although a series of results for periodic neural networks are obtained (see [1–10,16–18]), the stability analysis in the Lagrange sense for periodic neural networks does not appear. So from the theoretical and application views, it is necessary to study the stable properties in the Lagrange sense for periodic neural networks.

Moreover, in conducting stability analysis of a neural network, the conditions to be imposed on the neural network are determined by the characteristics of activation function as well as network parameters. As we know, when neural networks are designed for problem solving, it is desirable for their activation functions to be general. To facilitate the design of neural networks, it is important that the neural networks with general activation functions are studied. The generalization of activation functions will provide a wider scope for neural network designs and applications. Motivated by the above discussions, our objective in this paper is to study the global exponential stability (GES) in Lagrange sense for periodic neural networks with various activation functions, which include both bounded and unbounded activation functions. We provide verifiable criteria for the boundedness of the networks and the existence of globally exponentially attractive (GEA) sets by constructing appropriate Lyapunov-like functions. It is believed that the results are significant and useful for the design and applications of the periodic neural networks.

This paper is organized as follows. In Section 2, we define the notions of GEA sets and GES in Lagrange sense, and give two preliminary results that will be used in the proofs of the main results. In Section 3, we provide several sufficient conditions for the GES in Lagrange sense of periodic neural networks with bounded activation functions. GES in Lagrange sense of Lurie-type activation functions is studied in Section 4. In Section 5, a numerical example is given to
illustrate the applications of the results. Finally, in Section 6 we give the conclusion.

2. Preliminaries

Consider the following neural network model of the form:

$$x_i(t) = -d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)g_i(x_j(t) + b_i(t)g_j(x_i(t-t_j))), \quad t \geq 0,$$

where $$i = 1, 2, \ldots, n$$, $$d_i(t) > 0$$, $$x_i(t)$$ is the state variable of the $$i$$th neuron, $$t_i(t) \in C(R, R)$$ is an external input, $$a_{ij}(t)$$ and $$b_i(t)$$ are connection weights from neuron $$i$$ to neuron $$j$$, $$\tau_i$$ corresponds to the transmission delay and satisfies $$0 \leq \tau_i \leq T$$ ($$T$$ is a constant), $$g_i(t) \in C(R, R)$$ is the neuron activation function, the initial conditions associated with (1) is of the form $$x_i(s) = \psi_i(s)$$ for $$s \in [-\tau, 0]$$.

Throughout the paper, we assume that $$d_i(t), a_{ij}(t), b_i(t)$$ and $$t_i(t)$$ are continuous $$\omega$$-periodic functions, i.e., $$d_i(t+\omega) = d_i(t)$$, $$a_{ij}(t+\omega) = a_{ij}(t)$$, $$b_i(t+\omega) = b_i(t)$$ and $$t_i(t+\omega) = t_i(t)$$.

In this paper, we will consider two classes of activation functions for the neural network model (1). To this end, we define the vector function $$g \in C(R^n, R^n)$$ by $$g(y) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n))$$, where $$x = (x_1, x_2, \ldots, x_n) \in R^n$$.

Firstly, we consider the bounded activation functions, which can be given in the form
$$B \triangleq (g_i \in C(R, R) | \forall k_i > 0, \forall x_i \in R, |g_i(x_i)| \leq k_i, \forall x_i \in R, i = 1, 2, \ldots, n),$$
where the constants $$k_i$$, $$i = 1, 2, \ldots, n$$, are generally called to be the Dulire constants.

Secondly, we consider the Lurie-type activation functions, which can be given in the form
$$F \triangleq (\psi \in C(R, R) | \exists \psi \in C, \exists k > 0, \forall x \in R, |\psi(x)| \leq k, \forall x \in R, i = 1, 2, \ldots, n),$$
where $$K \triangleq (\psi \in C(R, R) | \psi(s) > 0 \text{ and } D^+ \psi(s) > 0, s \in R)$$, and the constants $$k_i$$, $$i = 1, 2, \ldots, n$$, are generally called to be the Dulire constants.

For convenience, we introduce the following notations:
Let $$C$$ be the Banach space of continuous functions $$\psi : [-\tau, 0] \rightarrow R^n$$ with the norm $$\|\psi\| = \sup_{t \in [-\tau, 0]} \|\psi(t)\|$$. For a given constant $$H > 0$$, $$G_H$$ is defined as the subset $$\{\psi \in C | \|\psi\| \leq H\}$$. Let $$CC^2$$ be the set of all non-negative continuous functions $$K : C \rightarrow [0, +\infty)$$, mapping bounded sets in $$C$$ into bounded sets in $$[0, +\infty)$$. For any initial condition $$\psi \in C$$, the solution of (1) that starts from the initial condition $$\psi$$ will be denoted by $$x(t; \psi)$$. If there is no need to emphasize the initial condition, any solution of (1) will also simply be denoted by $$x(t)$$. For any continuous bounded function $$h(t), t \geq 0$$, we write $$|h| = \sup_{t \geq 0} |h(t)|$$ and $$\overline{h} = \lim_{t \rightarrow +\infty} |h(t)|$$.

Definition 2.1 [Wang et al. [14]]. Network (1) is said to be uniformly stable in Lagrange sense (or uniformly bounded), if for any $$H > 0$$, there exists a constant $$K = K(H) > 0$$ such that $$x(t; \psi) < K$$ for all $$\psi \in C_H$$ and $$t \geq 0$$.

Definition 2.2 [Liao et al. [11]]. Let $$\Omega \subseteq R^n$$ be a compact set in $$R^n$$ and the complement of $$\Omega$$ by $$R^n \setminus \Omega$$. For any $$x \in R$$, $$\rho(x; \Omega) = \inf_{y \in \Omega} |x-y|$$ is the distance between $$x$$ and $$\Omega$$, a compact set $$\Omega \subseteq R^n$$ is said to be a global attractive set of network (1), if for every solution $$x(t)$$ such that $$x(t) \in R^n \setminus \Omega, t \geq 0$$, we have $$\lim_{t \rightarrow +\infty} \rho(x(t), \Omega) = 0$$.

Obviously, if network (1) has a global attractive set, it is ultimately bounded.

In many neural network applications, the rate of convergence or attractiveness is very important in improving computational performance. In the following, we introduce the notion of globally exponentially attractive (GEA) sets.

Definition 2.3 [Wang et al. [14]]. A compact set $$\Omega \subseteq R^n$$ is said to be a GEA set of (1) (in strong sense), if there exist a constant $$\Lambda > 0$$ and a continuous functional $$K \in CC^2$$ such that for every solution $$x(t)$$ with $$x(t) \in R^n \setminus \Omega, t \geq 0$$, we have $$\rho(x(t), \Omega) \leq K(t) \exp(-\Lambda t)$$ for all $$t \geq 0$$.

It is easy to see that the above definition for a GEA set is often inconvenient to use, and the main difficulty lies in the use of the distance function $$\rho(x, \Omega)$$, which seems hard to be linked to the exponential decay function. In the following, we give two definitions for a GEA set in a weaker form. And these definitions explicitly involve in positive definite functions, hence they are typically suitable in employing Lyapunov-like functions to analyze global exponential attractivity of neural network models.

Definition 2.4 [Liao et al. [11]]. If there exist a radially unbounded and positive definite function $$V(x)$$, a continuous functional $$K \in CC^2$$, positive constants $$\ell$$ and $$\alpha$$, such that for any solution $$x(t) = x(t; \psi)$$ of (1), $$V(x(t)) > \ell, t \geq 0$$, implies $$V(x(t)) - \ell \leq K(t) \exp(-\alpha t)$$, then network (1) is said to be globally exponentially attractive with respect to $$V$$, and the compact set $$\Omega := \{x \in R^n | V(x) \leq \ell\}$$ is called to be a GEA set of (1).

Definition 2.5 [Liao et al. [12]]. If there exist radially unbounded and positive definite functions $$V_i(x_i), x_i \in R$$, continuous functions $$K_i \in CC^2$$, positive constants $$\ell_i$$ and $$\alpha_i$$, such that for any solution $$x(t) = x(t; \psi)$$ of (1), $$V_i(x_i(t)) > \ell_i, t \geq 0$$, implies $$V_i(x_i(t)) - \ell_i \leq K_i(t) \exp(-\alpha_i t), \quad i = 1, 2, \ldots, n,$$
then network (1) is said to be globally exponentially attractive with respect to $$(V_1, V_2, \ldots, V_n)$$, and the compact set $$\Omega := \{x \in R^n | V_i(x_i) \leq \ell_i\}$$ is called to be a GEA set of (1).

Definition 2.6 [Wang et al. [14]]. Network (1) is called globally exponentially stable (GES) in Lagrange sense, if it is both uniformly stable in Lagrange sense and globally exponentially attractive. If there is a need to emphasize the Lyapunov-like functions, the network will be called globally exponentially stable in Lagrange sense with respect to $$V$$ or $$(V_1, V_2, \ldots, V_n)$$.

We end this section with two preliminary results, which will be used in the proofs of the main results.

Lemma 2.1 [Wang et al. [14]]. Let $$G \in C([t_0, \infty), R)$$, and there exist positive constants $$\alpha$$ and $$\beta$$ such that
$$D^+ G(t) \leq -\alpha G(t) + \beta, \quad t \geq t_0,$$
then
$$G(t) - \frac{\beta}{2} \leq \left( G(t_0) - \frac{\beta}{2} \right) \exp(-\alpha(t-t_0)), \quad t \geq t_0.$$

In particular, if $$G(t) \geq \beta/\alpha, t \geq t_0$$, then $$G(t)$$ exponentially approaches $$\beta/\alpha$$ as $$t$$ increases.

The following lemma provides a key step in proving global exponential attractivity for system (1).

Lemma 2.2. Let $$x(t) = x(t; \psi)$$ be a solution of (1) and $$g_i$$ be given in (1) with $$x \in R^n$$, $$x_i \in R, i = 1, 2, \ldots, n$$. Define the function
$$V(t) = \sum_{i=1}^{n} \int_{0}^{t} g_i(y) \, dy + \sum_{i=1}^{n} \int_{t-t_i}^{t} g_i^2(x_i(s)) \, ds, \quad t \geq 0,$$
where $0 \leq \tau_i \leq \tau$, $i = 1, 2, \ldots, n$, are constants. If there exist positive constants $\alpha$, $\beta$, and $\gamma$ such that
\[
\dot{V}(t) \leq -\alpha \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy - \beta \sum_{i=1}^{n} g_i^2(x_i(t)) + \gamma, \quad t \geq 0,
\]
then for any $\eta > 0$ with $\eta \leq \alpha$ and $\sum_{i=1}^{n} \eta \tau_i \exp(\eta \tau_i) \leq \beta$, we have
\[
\sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy \leq K(\eta) \exp(-\eta t), \quad t \geq 0,
\]
where $K \in C^\infty$ is given by
\[
K(\eta) = \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy + \sum_{i=1}^{n} \int_{0}^{\tau_i} (1 + \eta \tau_i \exp(\eta \tau_i)) g_i^2(\psi_i(s)) \, ds.
\]

**Proof.** Consider the following function:
\[
W(t) = \exp(\eta t) \left( V(t) - \frac{\eta}{\eta t} \right),
\]
then we have
\[
\frac{dW(t)}{dt} = \eta \exp(\eta t) V(t) - \exp(\eta t) \gamma + \exp(\eta t) \dot{V}(t)
\]
\[
\leq \eta \exp(\eta t) V(t) - \exp(\eta t) \gamma + \exp(\eta t) \left[ -\alpha \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy + \gamma \right]
\]
\[
- \beta \exp(\eta t) \sum_{i=1}^{n} g_i^2(x_i(t))
\]
\[
= \exp(\eta t) \left[ -\alpha + \eta \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy 
\right. 
\]
\[
+ \eta \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i^2(x_i(s)) \, ds - \beta \sum_{i=1}^{n} g_i^2(x_i(t)) \right].
\]
Let $t_1 > 0$ be arbitrarily given, integrating the above equation from 0 to $t_1$, then we can get
\[
\exp(\eta \tau_i) \left( V(t_1) - \frac{\eta}{\eta t_1} \right) \leq V(0) - \frac{\eta}{\eta t_1} + \int_{0}^{t_1} \eta \exp(\eta t) \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy \, dt
\]
\[
+ \int_{0}^{t_1} \beta \exp(\eta t) \sum_{i=1}^{n} g_i^2(x_i(t)) \, dt.
\]
Since $\eta \leq \alpha$ and $\alpha g_i(x_i) \geq 0$, this implies
\[
\exp(\eta \tau_i) \left( V(t_1) - \frac{\eta}{\eta t_1} \right) \leq V(0) - \frac{\eta}{\eta t_1} + \int_{0}^{t_1} \eta \exp(\eta t) \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i^2(x_i(s)) \, ds \, dt
\]
\[
- \int_{0}^{t_1} \beta \exp(\eta t) \sum_{i=1}^{n} g_i^2(x_i(t)) \, dt. \tag{2}
\]
We now estimate the double integral term in (2), it follows that
\[
\int_{0}^{t_1} \eta \exp(\eta t) \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i^2(x_i(s)) \, ds \, dt
\]
\[
= \eta \int_{0}^{t_1} \tau_i \exp(\eta (\eta + \tau_i)) \int_{0}^{\tau_i} g_i^2(x_i(s)) \, ds \, dt
\]
\[
\leq \eta \int_{0}^{t_1} \tau_i \exp(\eta (\eta + \tau_i)) g_i^2(x_i(s)) \, ds \, dt
\]
\[
= \eta \int_{0}^{t_1} \tau_i g_i^2(x_i(s)) \, ds \, dt + \eta \int_{0}^{t_1} \tau_i \exp(\eta (\eta + \tau_i)) g_i^2(x_i(s)) \, ds \, dt.
\]
Recall that $x(s) = \psi_i(s)$ for $s \in [-\tau_i, 0]$ and $\sum_{i=1}^{n} \eta \tau_i \exp(\eta \tau_i) \leq \beta$, from (2), we have
\[
\exp(\eta \tau_i) \left( V(t_1) - \frac{\eta}{\eta t_1} \right) \leq V(0) - \frac{\eta}{\eta t_1} + \sum_{i=1}^{n} \eta \tau_i \exp(\eta \tau_i) \int_{0}^{\tau_i} g_i^2(x_i(s)) \, ds
\]
\[
+ \sum_{i=1}^{n} (\eta \tau_i \exp(\eta \tau_i) - \beta) \int_{0}^{t_1} \exp(\eta t) g_i^2(x_i(s)) \, ds
\]
\[
\leq V(0) - \frac{\eta}{\eta t_1} + M,
\]
where
\[
M = \sum_{i=1}^{n} \eta \tau_i \exp(\eta \tau_i) \int_{0}^{\tau_i} g_i^2(\psi_i(s)) \, ds.
\]
Therefore,
\[
V(t_1) - \frac{\eta}{\eta t_1} \leq \left( V(0) - \frac{\eta}{\eta t_1} + M \right) \exp(-\eta \tau_i).
\]
Since $t_1$ is arbitrary and $x g_i(x_i) \geq 0$, it follows that
\[
\sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy - \frac{\eta}{\eta t_1} \leq \left( V(0) - \frac{\eta}{\eta t_1} + M \right) \exp(-\eta \tau_i)
\]
\[
\leq (V(0) + M) \exp(-\eta \tau_i). \tag{4}
\]
Note that
\[
V(0) = \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy + \sum_{i=1}^{n} \int_{0}^{\tau_i} (1 + \eta \tau_i \exp(\eta \tau_i)) g_i^2(\psi_i(s)) \, ds,
\]
```
\]
together with (3), we can obtain
\[
V(0) + M = \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i(y) \, dy + \sum_{i=1}^{n} \int_{0}^{\tau_i} (1 + \eta \tau_i \exp(\eta \tau_i)) g_i^2(\psi_i(s)) \, ds
\]
\[
\leq \sum_{i=1}^{n} \int_{0}^{\tau_i} g_i^2(x_i(s)) \, ds.
\]
Let $K(\psi) = V(0) + M$, since $\alpha g_i(x_i) \geq 0$ for all $x_i \in R$, $i = 1, 2, \ldots, n$, and all $g_i(\cdot)$ are continuous, then the functional $K(\psi)$ is continuous, non-negative and bounded on bounded sets. Therefore, $K \in C^\infty$, then Lemma 2.2 can be followed from (4) and (5). The proof is completed. \qed

**Remark 2.1.** The difficult point of Lemma 2.2 lies in how to choose desirable parameters $\alpha$, $\beta$, and $\gamma$. Generally speaking, once $\tau_i$ ($i = 1, 2, \ldots, n$) are given, from $\sum_{i=1}^{n} \eta \tau_i \exp(\eta \tau_i) \leq \beta$, we can choose desirable parameters $\beta$ and $\gamma$, then we select $\alpha$ and $\gamma$ by means of $\eta \leq \alpha$ and $V(t)$. In general, most researchers control the brief range of $V(t)$ in order to select appropriate parameters $\alpha$ and $\gamma$.

### 3. Bounded activation functions

In this section, we consider network (1) with the bounded activation functions $g_i$, $i = 1, 2, \ldots, n$, that is, $g \in B$. Let $k_i > 0$, $i = 1, 2, \ldots, n$, be the saturation constants of $g_i(\cdot)$, define
\[
M_i = \frac{1}{2} \left( \sum_{j=1}^{n} (B_i + B_{ij}) k_i + B_{ii} \right),
\]
and the compact sets in $R^n$,
\[
\Omega_i = \left\{ x \in R^n : \sum_{j=1}^{n} x_j^2 / 2 \leq \frac{\sum_{j=1}^{n} M_j^2 / \xi_i}{2 \min_{1 \leq s < n} \lambda s - \lambda i} \right\} \quad \text{where} \quad 0 < \xi_i < d.
\]
\[ \Omega_2 = \left\{ x \in \mathbb{R}^n \mid |x| \leq 2M_i/d_i, i = 1, 2, \ldots, n \right\}, \]
\[ \Omega_3 = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^{n} |x| \leq \min_{1 \leq j \leq n} (d_j) \right\}. \]

**Theorem 3.1.** Assume that \( g(x) \in B \), then network (1) is GES in Lagrange sense. Moreover, the compact sets \( \Omega_i (i = 1, 2, 3) \) are GEA sets of (1).

**Proof.** We first prove that network (1) is uniformly stable in Lagrange sense. To this end, we consider the radially unbounded and positive definite Lyapunov function \( V(x(t)) = \sum_{i=1}^{n} x_i^2(t)/2 \). Choose \( \epsilon_i, 0 < \epsilon_i < d_i \) \((i = 1, 2, \ldots, n)\). Computing \( dV(x(t))/dt \) along the positive half trajectory of (1), we have
\[
\frac{dV(x(t))}{dt} \bigg|_{(1)} = \sum_{i=1}^{n} x_i(t) \frac{dx_i(t)}{dt} \\
\leq \sum_{i=1}^{n} \left[ \sum_{j=1}^{n} \left( (\mathcal{P}_1 + |\mathcal{R}_1|)k_j |x_i(t)| \right) - d_i x_i^2(t) + \sum_{j=1}^{n} (|\mathcal{R}_1| + |\mathcal{P}_1|)k_j + |I_i| \right] \\
= \sum_{i=1}^{n} \left[ - \sum_{j=1}^{n} d_i x_i^2(t) + \sum_{j=1}^{n} (|\mathcal{R}_1| + |\mathcal{P}_1|)k_j + |I_i| \right] \\
\leq \sum_{i=1}^{n} \left[ - \sum_{j=1}^{n} d_i x_i^2(t) + \sum_{j=1}^{n} d_i x_i^2(t) + \sum_{j=1}^{n} M_i^2 \right] \\
\leq - \min_{1 \leq i \leq n} (d_i) \sum_{i=1}^{n} M_i^2/\epsilon_i. \\
\leq - 2 \min_{1 \leq i \leq n} (d_i) |V(x(t))| + \sum_{i=1}^{n} M_i^2/\epsilon_i.
\]

By Lemma 2.1, we have
\[ V(x(t)) - \bar{t} \leq V(x(0)) - \bar{t} \exp\{-\alpha t\}, \quad t \geq 0, \tag{6} \]
where
\[ \bar{t} = \frac{\sum_{i=1}^{n} M_i^2/\epsilon_i}{2 \min_{1 \leq i \leq n} (d_i - \epsilon_i)}. \]
This immediately implies the uniform boundedness of the solutions of (1). Hence, network (1) is uniformly stable in Lagrange sense. Observe that
\[ V(x(0)) - \bar{t} \leq V(x(0)) = \frac{1}{2} \sum_{i=1}^{n} x_i^2(0) = \frac{1}{2} \sum_{i=1}^{n} \psi_i(0) := K(\psi), \]
then \( K \in \mathcal{C} F^+ \), and from (6) it implies that
\[ V(x(t)) - \bar{t} \leq K(\psi) \exp\{-\alpha t\}, \quad t \geq 0. \]

By Definition 2.4, network (1) is globally exponentially attractive and \( \Omega_3 \) is a GEA set. This proves the GES in Lagrange sense of (1).

Now we prove that \( \Omega_2 \) also is a global exponentially attractive set. To this end, we employ \( n \) radially unbounded and positive definite Lyapunov functions \( V(x(t)) = \sum_{i=1}^{n} |x_i(t)|, i = 1, 2, \ldots, n \). Computing the right upper Dini derivative \( D^+ V(x(t)) \) along the positive half trajectory of (1), we have
\[ D^+ V(x(t)) \big|_{(1)} \leq -d_i |x_i(t)| + \sum_{j=1}^{n} (|\mathcal{R}_1| + |\mathcal{P}_1|)k_j + |I_i| = -d_i V(x(t)) + 2M_i. \]

Applying Lemma 2.1 again, we have
\[ V(x(t)) - \bar{t} \leq (V(x(0)) - \bar{t}) \exp\{-\alpha t\}, \quad t \geq 0, \tag{7} \]
where
\[ \bar{t} = \frac{2M_i}{d_i}, \quad \bar{t} = \frac{2M_i}{d_i}. \]

Let \( K_i(\psi) = V_i(\psi(0)) = |\psi_i(0)|, i = 1, 2, \ldots, n \), then \( K_i \in \mathcal{C} F^+ \), and from (7) it implies that
\[ V_i(\psi(t)) - \bar{t} \leq K_i(\psi) \exp\{-\alpha t\}, \quad t \geq 0. \]

By Definition 2.5, \( \Omega_2 \) is a GEA set.

Finally, in order to see that \( \Omega_3 \) is a GEA set, we use the radially unbounded and positive definite Lyapunov function \( V(x(t)) = \sum_{i=1}^{n} |x_i(t)| \). Computing the right upper Dini derivative \( D^+ V(x(t)) \) along the positive half trajectory of (1), we have
\[ D^+ V(x(t)) \big|_{(1)} \leq \sum_{i=1}^{n} \left( -d_i |x_i(t)| + \sum_{j=1}^{n} (|\mathcal{R}_1| + |\mathcal{P}_1|)k_j + |I_i| \right) \\
\leq - \sum_{i=1}^{n} d_i |x_i(t)| + \sum_{i=1}^{n} 2M_i. \]

Similarly, by Lemma 2.1 and Definition 2.4, we conclude that \( \Omega_3 \) is a GEA set. This completes the proof. \( \square \)

**Remark 3.1.** The global attractive sets \( \Omega_i (i = 1, 2, 3) \) derived in Theorem 3.1 may not be the best. However, in many applications, firstly one often need to understand whether the network designed has the desired properties such as the GES in Lagrange sense.

**Remark 3.2.** The GEA set \( \Omega_i \) involves some arbitrarily selected positive number \( \epsilon_i \). Smaller \( \epsilon_i \) will give larger attractive sets, and the convergence rate of the trajectories into \( \Omega_i \) will get greater.

**Remark 3.3.** When the coefficients of network (1) are fixed constants (see [11]), Theorem 3.1 still can be applied, so our result improves and extends some existing ones.

4. **Lurie-type activation functions**

In this section, we consider network (1) with Lurie-type activation functions \( g(x) \in \mathcal{F} \). Let \( k_i > 0, i = 1, 2, \ldots, n \), be the Lurie constants of \( g(x) \).

We first introduce some notation. Let
\[ Q^{(1)} = \begin{pmatrix} Q_{11}^{(1)} & Q_{12}^{(1)} \\ Q_{21}^{(1)} & Q_{22}^{(1)} \end{pmatrix}, \quad Q^{(2)}_{2n \times 2n}, \]
where
\[ Q_{11}^{(1)} = (A + A^T)/2 + E_{n \times n} + \text{diag}(-d_1/k_1, -d_2/k_2, \ldots, -d_n/k_n), \]
\[ Q_{12}^{(1)} = B/2, \]
\[ Q_{21}^{(1)} = -E_{n \times n}, \]
\[ A = (|\mathcal{R}_1|)_{n \times n}, \]
\[ B = (|\mathcal{P}_1|)_{n \times n}. \]
For any given \( \zeta \in R \), define
\[ Q^{(1)}_{11}(\zeta) = \frac{A + A^T}{2} + E_{n \times n} + \zeta \text{ diag} \left\{ -d_1/k_1, -d_2/k_2, \ldots, -d_n/k_n \right\} \]
and
\[ Q^{(1)}_{22}(\zeta) = \frac{A + A^T}{2}, \quad 2n \times 2n. \]

We will consider global exponential attractivity with respect to the following positive definite and radially unbounded function:
\[ W(x) = \sum_{i=1}^{n} \int_{0}^{\sqrt{k_i}} g_i(y) \ dy. \tag{8} \]
Theorem 4.1. Assume that \( g(\cdot) \in F \). If the matrix \( Q^{(1)} \) is negative definite, then network (1) is GES in Lagrange sense with respect to \( W \). Moreover, there exist positive constants \( \gamma \) and \( \eta \) such that the set \( \Omega(\gamma, \eta) \) defined in (9) is a GEA set of (1). 

Proof. Since \( Q^{(1)}(\xi_1) \) is negative definite, there exists \( 0 < \xi_1 < 1 \) such that \( Q^{(1)}(\xi_1) \) and \( Q^{(1)}(\xi_1) \) are also negative definite. Let \( \xi_2 = 1 - \xi_1 > 0 \) and \( -\mu_1 \) be the maximal eigenvalue of \( Q^{(1)}(\xi_1) \), where \( \mu_1 > 0 \). Choose \( 0 < \varepsilon < \mu_1 \), define \( \lambda = \min_{1 \leq i \leq n}(\xi_2+i) \) and \( \gamma = \sum_{i=1}^{n} |\xi_2|/(4\varepsilon) \). Let \( 0 < \eta < \lambda \) be such that \( \max_{1 \leq i \leq n}(\eta_2, \exp(\eta_1)) \leq \mu_1 - \varepsilon \). In the following, we will show that network (1) is uniformly stable in Lagrange sense and \( \Omega(\gamma, \eta) \) as defined in (9) is a GEA set.

For any given solution \( x(t) = x(t; \psi) \) of network (1), we consider the function

\[
\mathcal{V}(t) = \sum_{i=1}^{n} \int_{t}^{t-\tau_i} g_i(y) \, dy + \sum_{i=1}^{n} \int_{t-\tau_i}^{t-\tau_i+\tau_i} \nabla^2 g_i(x_i(t)) \, ds, \quad t \geq 0,
\]

then

\[
\frac{d\mathcal{V}(t)}{dt} \bigg|_{t=1} = \sum_{i=1}^{n} \left[ -\frac{d}{dt} x_i(t) \nabla g_i(x_i(t)) + \sum_{j=1}^{n} \frac{d}{dt} a_{ij}(t) \nabla g_i(x_i(t)) \nabla g_j(x_j(t)) \right] + \sum_{j=1}^{n} b_{ij}(t) \nabla g_i(x_i(t)) \nabla g_j(x_j(t)) + \sum_{i=1}^{n} \nabla^2 g_i(x_i(t)) - \sum_{i=1}^{n} \nabla^2 g_i(x_i(t-\tau_i)).
\]

By Lemma 2.2, there exists \( K \in \mathcal{C}F^+ \) such that

\[
\sum_{i=1}^{n} \int_{0}^{\lambda_i(t)} g_i(y) \, dy - \frac{\gamma}{\eta} \leq K(\psi) \exp(-\eta t), \quad t \geq 0.
\]

Since \( g(\cdot) \in F \), (10) implies that \( x(t) \) is uniformly bounded, hence network (1) is uniformly stable in Lagrange sense. Moreover, by Definition 2.4 and inequality (10), the set \( \Omega(\gamma, \eta) \) as defined in (9) is a GEA set. This proves the GES in Lagrange sense of network (1) and the proof is completed. \( \square \)

Remark 4.1. When the coefficients of network (1) are fixed constants (see [11]), Theorem 4.1 also can be applied, hence our result is to be of weaker conservatism.

5. Illustrative example

Example 5.1. Consider the following two-neuron network:

\[
\begin{align*}
\dot{x}_1(t) &= -\frac{1}{\tau_1} x_1(t) - \frac{1}{\tau_f} f(x_2(t)) + \frac{2}{\tau_f} \cos(t), \\
\dot{x}_2(t) &= -\frac{1}{\tau_2} x_2(t) - \frac{1}{\tau_f} f(x_1(t)) + \frac{2}{\tau_f} \cos(t).
\end{align*}
\]

Case 1: When \( f(x) = (|x| + 1) - (|x| - 1)/2 \), according to Theorem 3.1, we can conclude that the network (11) is GES in the Lagrange sense. Fig. 1 clearly shows the state trajectories.

Case 2: When \( f(x) = (x + \arctan(x))/2 \), according to Theorem 4.1, we can conclude that the network (11) is GES in the Lagrange sense. Fig. 2 clearly depicts the state trajectories.
6. Conclusion

It is significantive to study global exponential stability in Lagrange sense for periodic neural networks with various activation functions. In this paper, according to the parameters, detailed estimations of global exponential attractive sets are presented without any hypothesis on the existence. It is also verified that outside the global exponential attractive set, there is no periodic state, almost periodic state or chaos attractor. These stability analysis can narrow the search domains of optimization and associative memories, and provide theoretical guidelines for applications.

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References


Ailong Wu received the B.S. and M.S. degrees in applied mathematics from Hubei Normal University, Huangshi, China, in 2007 and 2009, respectively. He is currently a doctoral candidate in Department of Control Science and Engineering, Huazhong University of Science and Technology, Wuhan, China. His current research interests include stability theory, artificial neural networks and switching control.

Zhigang Zeng received the B.S. degree from Hubei Normal University, Huangshi, China, and his M.S. degree from Hubei University, Wuhan, China, in 1993 and 1996, respectively, and his Ph.D. degree from Huazhong University of Science and Technology, Wuhan, China, in 2003.

He is a Professor and Ph.D. advisor in Huazhong University of Science and Technology, China. His current research interests include neural networks, switched systems, computational intelligence, stability analysis of dynamic systems, pattern recognition and associative memories.
Chaojin Fu received the B.S. degree from Hubei Normal University, Huangshi, China, in 1982. He is a Professor and M.S. advisor in Hubei Normal University, China. His current research interests include stability analysis of dynamic systems and differential geometry.

Wenwen Shen received the B.S. degree from Wuhan University of Technology, Wuhan, China, in 2009. She is currently a master candidate in Department of Control Science and Engineering, Huazhong University of Science and Technology, Wuhan, China. Her current research interests include complex systems and complex networks.