Synchronization control of a class of memristor-based recurrent neural networks

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A B S T R A C T

In this paper, we formulate and investigate a class of memristor-based recurrent neural networks. Some sufficient conditions are obtained to guarantee the exponential synchronization of the coupled networks based on drive-response concept, differential inclusions theory and Lyapunov functional method. The analysis in the paper employs results from the theory of differential equations with discontinuous right-hand side as introduced by Filippov. Finally, the validity of the obtained result is illustrated by a numerical example.

1. Introduction

In 1971, Chua [3] presented the logical and scientific basis for the existence of a new two-terminal circuit element called the memristor (a contraction for memory resistor) which has every right to be as basic as the three classical circuit elements already in existence, namely, the resistor, inductor, and capacitor. On 1 May 2008, the Hewlett-Packard (HP) research team proudly announced their realization of a memristor prototype, with an official publication in Nature [34,35]. They also devised a physical model of a memristor called the coupled variable resistor model, which works like a perfect memristor under certain conditions. This new circuit element of memristor shares many properties of resistors and shares the same unit of measurement (ohm). Many researchers focus on the memristor because of its potential applications in next generation computer and powerful brain-like “neural” computer (see [5,11,14,15,26,28,34,35]).

Currently, many researchers attempt to build an electronic intelligence that can mimic the awesome power of a brain by mean of the crucial electronic components-memristors [14,15,34,35]. Lots of research shows that complex electrical response to the ebb and flow of potassium and sodium ions across the membranes of each cell, which allow the synapses to alter their response according to the frequency and strength of signals. It looks maddeningly similar to the response of a memristor would produce, i.e., the behaviour of synapses looks maddeningly similar to a memristor’s response. And there are indications that the memristor exhibits the feature of pinched hysteresis, which means that a lag occurs between the application and the removal of a field and its subsequent effect, just as the neurons in the human brain have. Because of this feature, broad potential applications of the memristor have been identified [5,11,14,15,26], one of which is to apply this device to build a new model of neural networks (memristor-based neural networks) to emulate the human brain [11]. The memristor-based neural networks can remember its past dynamical history, store a continuous set of states, and be “plastic” according to the pre-synaptic and post-synaptic neuronal activity. It will open up new possibilities in the understanding of...
neural processes using memory devices, an important step forward to reproduce complex learning, adaptive and spontaneous behavior with electronic neural networks. Furthermore, in view that memristor-based neural networks will have great application value, at the same time, in consideration of many successful applications of the recurrent neural networks (see [1,2,6,9,10,12,13,17–25,29–33,36–43]), an interesting issue is to investigate the memristor-based recurrent neural networks, which is an ideal model for the case where the memristor-based circuit network process of more efficient learning with a realization of the famous Hebbian rule stating, in a simplified form, that “neurons that fire together, wire together”. Moreover, the analysis of the memristor-based recurrent neural networks is able to reveal crucial features of the dynamics, such as the presence of sliding modes along switching surfaces, the chaos synchronization, and the ability to compute the exact global minimum of the underlying energy function, which make the networks especially attractive for the solution of global optimization problems in real time.

To better understand the memristor-based recurrent neural networks, firstly we describe the circuit of a general class of recurrent neural networks in the Fig. 1. Take the ith subsystem as the unit of analysis in order to simplify illustration, from Fig. 1, the KCL equation of the ith subsystem is written as:

\[
\dot{x}_i(t) = - \frac{1}{C_i} \left[ \sum_{j=1}^{n} \left( \frac{1}{R_{ij}} + \frac{1}{F_j} \right) \times \text{sgn}_j + \frac{1}{R_i} \right] x_i(t) + \frac{1}{C_i} \sum_{j=1}^{n} g_j(x_j(t)) \times \text{sgn}_j + \frac{1}{C_i} \sum_{j=1}^{n} \frac{g_j(x_j(t) - \tau_j(t))}{R_j} \times \text{sgn}_j + \frac{I_i}{C_i},
\]

\[ t \geq 0, \quad i = 1, 2, \ldots, n, \quad (1) \]

where \(x_i(t)\) is the voltage of the capacitor \(C_i\), \(R_{ij}\) denotes the resistor between the feedback function \(g(x_j(t))\) and \(x_i(t)\), \(F_j\) denotes the resistor between the feedback function \(g(x_i(t) - \tau_j(t))\) and \(x_j(t)\), \(\tau_j(t)\) corresponds to the transmission delay and satisfies \(0 \leq \tau_j(t) \leq \tau_i\) (\(\tau_i\) are constants, \(i = 1, 2, \ldots, n\), \(R_i\) represents the parallel-resistor corresponding the capacitor \(C_i\), \(I_i\) is the external input or bias, and

\[
\text{sgn}_j = \begin{cases} 
1, & i \neq j, \\
-1, & i = j.
\end{cases}
\]

From (1),

\[
\dot{x}_i(t) = -\hat{d}_i x_i(t) + \sum_{j=1}^{n} \left( \hat{a}_{ij} g_j(x_j(t)) + \hat{b}_{ij} g_j(x_i(t) - \tau_j(t)) \right) + I_i, \quad t \geq 0, \quad i = 1, 2, \ldots, n, \quad (2)
\]

where

\[
\hat{d}_i = \frac{1}{C_i} \left[ \sum_{j=1}^{n} \left( \frac{1}{R_{ij}} + \frac{1}{F_j} \right) \times \text{sgn}_j + \frac{1}{R_i} \right], \quad \hat{a}_{ij} = \frac{\text{sgn}_j}{C_i R_j}, \quad \hat{b}_{ij} = \frac{\text{sgn}_j}{C_i F_j},
\]

\[
I_i = \frac{I_i}{C_i}.
\]

By replacing the resistors \(R_{ij}\), \(F_j\) and \(R_i\) in the primitive recurrent neural networks (1) or (2) with memristors, whose memductances \(\mathbb{W}_{ij}\), \(\mathbb{M}_{ij}\) and \(\mathbb{P}_i\), respectively, then we can construct the memristor-based recurrent neural networks of the form

\[
\dot{x}_i(t) = -\hat{d}_i(x_i(t)) x_i(t) + \sum_{j=1}^{n} \left( a_{ij} g_i(x_i(t)) + b_{ij} g_i(x_j(t) - \tau_j(t)) \right) + I_i, \quad t \geq 0, \quad i = 1, 2, \ldots, n, \quad (3)
\]

where

\[
\hat{d}_i(x_i(t)) = \frac{1}{C_i} \left[ \sum_{j=1}^{n} \left( \mathbb{W}_{ij} + \mathbb{M}_{ij} \right) \times \text{sgn}_j + \mathbb{P}_i \right], \quad a_{ij}(x_i(t)) = \frac{\mathbb{W}_{ij}}{C_i} \times \text{sgn}_j,
\]

\[
b_{ij}(x_i(t)) = \frac{\mathbb{M}_{ij}}{C_i} \times \text{sgn}_j, \quad I_i = \frac{I_i}{C_i}.
\]

Combining with the pinched hysteresis loop in the current–voltage characteristic of memristors, as a matter of convenience, in this paper we discuss a simple type of the memristor-based recurrent neural networks (3) as follows:

\[
\dot{x}_i(t) = -\hat{d}_i(x_i(t)) x_i(t) + \sum_{j=1}^{n} \left( a_{ij}(x_i(t)) g_i(x_i(t)) + b_{ij}(x_i(t)) g_i(x_j(t) - \tau_j(t)) \right) + I_i, \quad t \geq 0, \quad i = 1, 2, \ldots, n, \quad (4)
\]

where

\[
\hat{d}_i(x_i) = \begin{cases} 
\hat{d}_i, & |x_i(t)| < T_i, \\
\hat{d}_i, & |x_i(t)| > T_i,
\end{cases}
\]
\[ a_{ij}(x_t) = a_{ij}(x_t(t)) = \begin{cases} \dot{a}_{ij}, & |x_t(t)| < T_i, \\ \dot{a}_{ij}, & |x_t(t)| > T_i, \end{cases} \]

\[ b_{ij}(x_t) = b_{ij}(x_t(t)) = \begin{cases} \dot{b}_{ij}, & |x_t(t)| < T_i, \\ \dot{b}_{ij}, & |x_t(t)| > T_i, \end{cases} \]

\[ I_i = \frac{i_i}{C_i}, \]

Fig. 1. The circuit of recurrent neural networks.
in which switching jumps \( T_i > 0 \), \( \hat{d}_i > 0 \), \( \hat{l}_i > 0 \), \( a_{ij}, a_{ji} \), \( b_{ij}, b_{ji} \), \( i, j = 1, 2, \ldots, n \), are constant numbers.

**Remark 1.** There are some existing works about the memristor-based nonlinear circuit networks \([5,11,15,26,28]\). In this paper, we deal with the detailed construction of a general class of memristor-based recurrent neural network model (3) from the aspect of circuit analysis, which can approximately simulate the memristive synapses that they may behave like the real thing about the memristor minds to some extent.

In addition, we note that a plethora of complex nonlinear behaviors including chaos appear even in a simple network of memristor \([5,14,28]\), a detailed analytical study of synchronization control of the basic oscillator is necessary. The synchronization of recurrent neural networks has been gaining increasing research attention because of their potential applications in many real-world systems from a variety of fields such as biology, social systems, linguistic networks, and technological systems \([1,2,6,9,10,12,17–24,29–33,38,40,42,43]\). However, so far, there are very few works dealing with the synchronization control of the memristor-based recurrent neural networks. And the synchronization control of memristor-based recurrent neural networks can be expected to find a quick application in a broad range of fields, both defense and civilian, which need a continuously adaptive optional target tracking. Motivated by the above discussions, based on the works in \([2,5,11,14,15]\), our objective in this paper is to study the exponential synchronization for memristor-based recurrent neural networks (4). By utilizing the drive-response concept, differential inclusions theory, and combining them with Lyapunov functional method, a control law with an appropriate gain matrix is derived to achieve exponential synchronization of the drive-response-based coupled neural networks. The elements of the gain matrix are easily determined by checking a certain Hamiltonian matrix if its eigenvalues lie on the imaginary axis or not instead of arduously solving an algebraic Riccati equation.

The remaining part of the paper consists of four sections. In Section 2, some preliminaries are introduced. In Section 3, the main results are derived. In Section 4, an illustrative example is given to demonstrate the effectiveness of the proposed approach. Finally, concluding remarks are included in Section 5.

2. Preliminaries

Throughout this paper, solutions of all the systems considered in the following are intended in the Filippov’s sense. \([\cdot,\cdot]\) represents the interval. \( \mathcal{C}([-\tau,0],\mathbb{R}^n) \) denotes a Banach space of all continuous functions \( \nu = (\nu_1(s),\nu_2(s),\ldots,\nu_n(s))^T : [-\tau,0] \rightarrow \mathbb{R}^n \), where \( \tau = \max_{t \leq t_0} \{ T_i \} \). \( E_n \) is a \( n \times n \) identity matrix. For vector \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), \( |x| = (|x_1|, |x_2|, \ldots, |x_n|)^T \). \( |x| < T \langle |x| > T \rangle \) represents \( |x_i| < T_i \langle |x_i| > T_i \rangle, i = 1, 2, \ldots, n \), where \( T = (T_1, T_2, \ldots, T_n)^T \). \( T_i > 0 \). And \( \|x\| \) denotes the Euclidean norm, i.e., \( \|x\| = \left( \sum_{i=1}^{n} x_i^2 \right)^{\frac{1}{2}} \). Let \( \hat{d}_i = \max \{ d_i, \hat{d}_i \} \), \( \hat{d}_i = \min \{ d_i, \hat{d}_i \} \), \( \hat{a}_i = \max \{ a_{ij}, \hat{a}_i \} \), \( \hat{a}_i = \min \{ a_{ij}, \hat{a}_i \} \), \( b_{ij} = \max \{ b_{ij}, \hat{b}_j \} \), \( b_{ij} = \min \{ b_{ij}, \hat{b}_j \} \), for \( i, j = 1, 2, \ldots, n \). For matrices \( M = (m_{ij})_{n \times n}, N = (n_{ij})_{n \times n}, M \succ N(M \ll N) \) means that \( m_{ij} > n_{ij} \) for \( i, j = 1, 2, \ldots, n \). And by the interval matrix \( [M,N] \), it follows that \( M \ll N \). For \( \forall \mathcal{L} = (l_{ij})_{n \times n} \in [M,N] \), it means \( M \ll \mathcal{L} \ll N \), i.e., \( m_{ij} < l_{ij} < n_{ij} \) for \( i, j = 1, 2, \ldots, n \).

In addition, the initial conditions of system (4) are given by \( x_i(t) = \psi_i(t) \in \mathcal{C}([-\tau,0],\mathbb{R}^n), i = 1, 2, \ldots, n \).

First, by the theory of differential inclusions, from system (4), we have

\[
\dot{x}_i(t) \in -[d_i, \hat{d}_i]x_i(t) + \sum_{j=1}^{n} \left[ (a_{ij}, \hat{a}_j)g_j(x_j(t)) + (b_{ij}, \hat{b}_j)g_j(x_j(t - \tau_j(t))) \right] + l_i, \quad t \geq 0, \quad i = 1, 2, \ldots, n,
\]

or equivalently, for \( i, j = 1, 2, \ldots, n \), there exist \( \gamma_i \in [d_i, \hat{d}_i], \gamma_j \in [a_{ij}, \hat{a}_j], \gamma_j \in [b_{ij}, \hat{b}_j] \), such that

\[
\dot{x}_i(t) = -\gamma_i x_i(t) + \sum_{j=1}^{n} \left[ \gamma_j g_j(x_j(t)) + \gamma_j g_j(x_j(t - \tau_j(t))) \right] + l_i, \quad t \geq 0, \quad i = 1, 2, \ldots, n.
\]

In this paper, consider system (4+) or (4**+) as the master/drive system and the corresponding slave/response system is as:

\[
\dot{z}_i(t) \in -[d_i, \hat{d}_i]z_i(t) + \sum_{j=1}^{n} \left[ (a_{ij}, \hat{a}_j)g_j(z_j(t)) + (b_{ij}, \hat{b}_j)g_j(z_j(t - \tau_j(t))) \right] + l_i + u_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n,
\]

or equivalently, for \( i, j = 1, 2, \ldots, n \), there exist \( \Gamma_i \in [d_i, \hat{d}_i], \Gamma_j \in [a_{ij}, \hat{a}_j], \Gamma_j \in [b_{ij}, \hat{b}_j] \), such that

\[
\dot{z}_i(t) = -\Gamma_i z_i(t) + \sum_{j=1}^{n} \left[ \Gamma_j g_j(z_j(t)) + \Gamma_j g_j(z_j(t - \tau_j(t))) \right] + l_i + u_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n.
\]

where \( u_i(t) (i = 1, 2, \ldots, n) \) is the appropriate control input that will be designed in order to obtain a certain control objective.

The initial conditions associated with system (5) or (5+) are of the form \( z_i(t) = \phi_i(t) \in \mathcal{C}([-\tau,0],\mathbb{R}^n), i = 1, 2, \ldots, n \). In practical situations, the output signals of the drive system (4+) or (4**+) can be received by the response system (5) or (5+).

In this paper, referring to some relevant works in \([2,6,10]\), we make the following assumptions (A1)–(A3):

(A1) The function \( g_i, i \in \{1,2,\ldots,n\} \), is bounded and satisfies the Lipschitz condition with a Lipschitz constant \( l_r > 0 \), i.e.,
Remark 2. It is obvious that for $i \in \{1, 2, \ldots, n\}$, the set-valued map
$$x_i(t) - \|d_i\|_\infty x_i(t) + \sum_{j=1}^{n} \frac{1}{2} \left[ (\bar{a}_{ij}, \bar{a}_{ij}) g_j(x_j(t)) + (\bar{b}_{ij}, \bar{b}_{ij}) g_j(x_j(t) - g_j(z_j(t))) \right] + I_i, \quad t \geq 0,$$
has nonempty compact convex values. Furthermore, it is upper semi-continuous [4]. Then the local existence of a solution $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ with initial conditions $x_i(0) = \psi_i(s)$, of (4+) is obvious [8]. Moreover, under the condition (A1), if $g_i(\cdot)$ $(i = 1, 2, \ldots, n)$ is bounded, this local solution $x(t)$ can be extended to the interval $[0, +\infty)$ in the sense of Filippov [8].

Let $e(t) = (e_1(t), e_2(t), \ldots, e_n(t))^T$ be the synchronization error, where $e_i(t) = x_i(t) - z_j(t)$, applying the theories of set-valued maps and differential inclusions, then we can get the synchronization error system as follows
\begin{align*}
\dot{e}_i(t) &= -\|d_i\|_\infty e_i(t) + \sum_{j=1}^{n} \frac{1}{2} \left[ (a_{ij}, a_{ij}) g_j(e_j(t) + z_j(t)) + (b_{ij}, b_{ij}) g_j(e_j(t) - g_j(z_j(t))) \right] + I_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n, \tag{6}
\end{align*}
or equivalently, for $i, j = 1, 2, \ldots, n$, there exist $\bar{r}_i \in \{d_i, \bar{d}_i\} \bar{\nu}_j \in \{a_{ij}, a_{ij}\}, \bar{\tau}_j \in \{b_{ij}, \bar{b}_{ij}\},$ such that
\begin{align*}
\dot{e}_i(t) &= -\bar{r}_i e_i(t) + \sum_{j=1}^{n} \bar{\nu}_j (g_j(e_j(t) + z_j(t)) - g_j(z_j(t))) + \sum_{j=1}^{n} \bar{\tau}_j (g_j(e_j(t) - g_j(z_j(t))) - g_j(z_j(t) - g_j(z_j(t)))) - u_i(t), \quad t \geq 0, \quad i = 1, 2, \ldots, n, \tag{6+}
\end{align*}
with initial conditions $v_i(s) = \psi_i(s) - \phi_i(s) \in C([0, \infty), \Omega)$, $i = 1, 2, \ldots, n$.

In the following, the paper aims to design the control input $u_i(t)$ $(i = 1, 2, \ldots, n)$ associated with the state-feedback for the purpose of exponentially synchronizing the unidirectional coupled identical systems (4+) or (4++) and (5) or (5+).

3. Main results

In many real applications, we are interested in designing a memoryless state-feedback controller
\begin{align*}
(u_1(t), u_2(t), \ldots, u_n(t))^T = (\omega h_{b,n}) (e_1(t), e_2(t), \ldots, e_n(t))^T = \Omega e(t), \tag{7}
\end{align*}
where $\omega = (\omega h_{b,n})$ is a constant gain matrix to be determined for synchronizing both the drive system and response system. Furthermore, if a new error $\hat{e}_i(t)$ is defined by $\hat{e}_i(t) = \exp(zt) e_i(t)$, then the synchronization error system (6) or (6+) can be transformed into the following form:
\begin{align*}
\dot{\hat{e}}_i(t) &= -\|d_i\|_\infty \hat{e}_i(t) + \sum_{j=1}^{n} \frac{1}{2} \left[ (a_{ij}, a_{ij}) \phi_j(\hat{e}_j(t)) + (b_{ij}, b_{ij}) \phi_j(\hat{e}_j(t) - g_j(z_j(t))) \right] - \sum_{j=1}^{n} \omega_j \delta_j(\hat{e}_j(t)), \quad t \geq 0, \quad i = 1, 2, \ldots, n, \tag{8}
\end{align*}
or equivalently, for $i, j = 1, 2, \ldots, n$, there exist $\hat{r}_i \in \{d_i, \bar{d}_i\} \hat{\nu}_j \in \{a_{ij}, a_{ij}\}, \hat{\tau}_j \in \{b_{ij}, \bar{b}_{ij}\},$ such that
\begin{align*}
\dot{\hat{e}}_i(t) &= -\hat{r}_i \hat{e}_i(t) + \sum_{j=1}^{n} \hat{\nu}_j \phi_j(\hat{e}_j(t)) + \sum_{j=1}^{n} \hat{\tau}_j \phi_j(\hat{e}_j(t) - g_j(z_j(t))) - \sum_{j=1}^{n} \omega_j \delta_j(\hat{e}_j(t)), \quad t \geq 0, \quad i = 1, 2, \ldots, n, \tag{8+}
\end{align*}
where
$$\phi_j(\hat{e}_j(t)) = \exp(zt) \phi_j(e_j(t)),$$
$$\hat{\phi}_j(\hat{e}_j(t)) = \exp(zt) \phi_j(\hat{e}_j(t) - g_j(z_j(t))),$$
$$\phi_j(e_j(t)) = g_j(e_j(t) + z_j(t)) - g_j(z_j(t)),$$
$$\phi_j(\hat{e}_j(t) - g_j(z_j(t))) = g_j(e_j(t) - g_j(z_j(t))) - g_j(z_j(t) - g_j(z_j(t))).$$

From the definitions of $\phi_j(e_j(t))$ and $\hat{\phi}_j(\hat{e}_j(t))$, then
\[ |\phi_j(e_j(t))| \leq L_j|e_j(t)|, \]
\[ |\phi_j(\dot{e}_j(t))| = |\exp\{zt\} \phi_j(e_j(t))| \leq L_j|\exp\{zt\}e_j(t)| = L_j|\dot{e}_j(t)|.\]

**Definition 1.** System (8) or (8*) is said to be globally asymptotically stable, if there exist a positive definite and radially unbounded function \( V \), and a \( K \)-class function \( K \) defined on \( \mathbb{R}^+ \) such that for any initial condition \( V(s) = (v_1(s), v_2(s), \ldots, v_n(s))^T \in C([-\tau, 0], \mathbb{R}^n) \),
\[ D^+V|_{t=\min(s)} \leq -K_V, \]
where the function \( K \) is said to belong to class \( K \) if \( K(0) = 0 \) and \( K \) is strictly increasing on \( \mathbb{R}^+ \).

The **Definition 1** roots in [4] or [27] (p. 222, Theorem 6.4.6).

For further deriving the exponential synchronization conditions on the control law, the following lemmas are needed.

**Lemma 1.** Define a \( 2n \times 2n \) Hamiltonian matrix
\[
H = \begin{pmatrix}
-D & QQ^T \\
-(K_1 + K_2) - \varepsilon E_n & D^T
\end{pmatrix},
\]
where \( \varepsilon \) is sufficiently small positive constant, \( D = \text{diag}(d_i - \alpha) + \Omega \), \( Q = (\overline{AB}) \), \( K_1 = \text{diag}(L_1^T) \), \( K_2 = \text{diag}\left(\frac{\exp\{2\pi i jL_1^T\}}{L_1}\right) \), \( \overline{A} = (\overline{a}_j)_{n\times n} \), \( \overline{B} = (\overline{b}_j)_{n\times n} \).
If \( -D \) is a stable matrix and Hamiltonian matrix \( H \) has no eigenvalues on the imaginary axis, then the algebraic Riccati equation
\[ -D^TP - PD + PQQ^TP + (K_1 + K_2) + \varepsilon E_n = 0 \]
has a symmetric and positive definite solution \( P \) for a given \( \alpha > 0 \).

**Proof.** The proof is a direct result of the Lemma 4 in [7]. \( \square \)

**Remark 3.** A real matrix \( -D \) is stable if and only if all of its eigenvalues have negative real parts. All eigenvalues of \( -D \) defined in **Lemma 1** can be arbitrarily assigned by appropriately choosing the controller gain matrix \( \Omega \). Especially, choose the gain matrix as a diagonal matrix \( \Omega = \text{diag}(\omega_i) \) and \( \omega_i > \alpha - d_i \), \( i = 1, 2, \ldots, n \), then the eigenvalues of \( -D \) are \(-\omega_i + (d_i - \alpha) \times 0, i = 1, 2, \ldots, n \), which implies that \( -D \) is a stable matrix.

**Lemma 2** [16]. If the origin of \( \dot{e}(t) \) is asymptotically convergent, then the solution \( e(t) \) of system (6) or (6*) is exponentially convergent with a degree \( \alpha \).

**Theorem 1.** Under assumptions (A1)–(A3), if the controller gain matrix \( \Omega \) in (7) is suitably designed such that \(-D \) is a stable matrix and Hamiltonian matrix \( H \) defined in **Lemma 1** for a given \( \alpha > 0 \) has no eigenvalues on the imaginary axis, then the networks (4*) or (4***) and (5) or (5*) are synchronized exponentially with a degree \( \alpha \) at least.

**Proof.** Transform (8) or (8*) into the vector form as:
\[ \dot{\hat{e}}(t) \in -[\overline{D}, D]\hat{e}(t) + [\overline{A}, \overline{A}]\dot{\phi}(\hat{e}(t)) + [\overline{B}, \overline{B}]\dot{\phi}(\hat{e}(t - \tau(t))), \quad t \geq 0, \]

or equivalently, there exist \( \mathcal{D} \in [\overline{D}, D] \), \( \mathcal{A} \in [\overline{A}, \overline{A}] \), \( \mathcal{B} \in [\overline{B}, \overline{B}] \), such that
\[ \dot{\hat{e}}(t) = -\mathcal{D}\hat{e}(t) + \mathcal{A}\dot{\phi}(\hat{e}(t)) + \mathcal{B}\dot{\phi}(\hat{e}(t - \tau(t))), \quad t \geq 0, \]

where
\[
\begin{align*}
\hat{e}(t) & = (\hat{e}_1(t), \hat{e}_2(t), \ldots, \hat{e}_n(t))^T, \\
\dot{\phi}(\hat{e}(t)) & = \left(\hat{\phi}_1(\hat{e}_1(t)), \hat{\phi}_2(\hat{e}_2(t)), \ldots, \hat{\phi}_n(\hat{e}_n(t))\right)^T, \\
\dot{\phi}(\hat{e}(t - \tau(t))) & = \left(\hat{\phi}_1(\hat{e}_1(t - \tau_1(t))), \hat{\phi}_2(\hat{e}_2(t - \tau_2(t))), \ldots, \hat{\phi}_n(\hat{e}_n(t - \tau_n(t)))\right)^T, \\
\mathcal{D} & = \text{diag}(d_i - \alpha) + \Omega, \quad \mathcal{D} = \text{diag}(d_i - \alpha) + \Omega, \\
\mathcal{A} & = (\overline{a}_j)_{n\times n}, \quad \mathcal{A} = (\overline{a}_j)_{n\times n}, \quad \mathcal{B} = (\overline{b}_j)_{n\times n}, \quad \mathcal{B} = (\overline{b}_j)_{n\times n}.
\end{align*}
\]
Since \(-D\) is stable and the Hamiltonian matrix \(H\) has no eigenvalues on the imaginary axis, according to Lemma 1, the algebraic Riccati equation in (9) has a symmetric and positive definite solution \(P\). To confirm that the origin of (10) or (10+) is globally asymptotically convergent, consider the following functional \(V(t)\) in space \(C([-\tau, 0], \mathbb{R})\):

\[
V(t) = \tilde{e}(t)\tilde{P}\tilde{e}(t) + \sum_{j=1}^{n} \frac{\exp\{2\pi \tau_j\}}{1 - \mu_j} \int_{t-\tau_j(t)}^{t} \phi_j^2(\tilde{e}(s)) \, ds.
\]

Calculating the upper right Dini derivative of \(V(t)\) along the trajectory of (10) or (10+), and combining with the fact \(X^T Y + Y^T X \leq X^T X + Y^T Y\) for any matrices \(X\) and \(Y\) with appropriate dimensions, then

\[
D^+ V(t) = \dot{\tilde{e}}(t)^T \tilde{P} \dot{e}(t) + \frac{2}{1 - \mu_j} \int_{t-\tau_j(t)}^{t} \phi_j^2(\tilde{e}(s)) \, ds.
\]

Applying the property of \(\tilde{e}(t)\) and bounding the exponential terms, we have

\[
D^+ V(t) = e(t)^T \tilde{P} e(t) + \sum_{j=1}^{n} \phi_j^2(\tilde{e}(t)) + \sum_{j=1}^{n} \exp\{2\pi \tau_j\} \frac{1}{1 - \mu_j} \phi_j^2(\tilde{e}(t - \tau_j(t)))
\]

By using Assumption 2, we conclude

\[
\dot{\tilde{e}}(t)^T \tilde{P} \dot{e}(t) + \frac{2}{1 - \mu_j} \int_{t-\tau_j(t)}^{t} \phi_j^2(\tilde{e}(s)) \, ds.
\]

Combining with Definition 1, the last inequality \(D^+ V(t) < -\varepsilon \|\dot{e}(t)\|^2\) indicates \(V(t)\) converges to zero asymptotically. By Lemma 2, we conclude \(e(t)\) converges to zero globally and exponentially with a rate of \(\varepsilon\). The proof is completed.

**Remark 4.** In Lemma 1, it is not apparent how one can choose the gain matrix \(\Omega\) such that Hamiltonian matrix \(H\) has no eigenvalues on the imaginary axis. Therefore, it is not simple to find the analytical solutions for the conditions of the Theorem 1. Fortunately, they can be solved numerically by the eigenvalue-solver MATLAB Tool-Box and a trial-and-error procedure. Furthermore, the sufficient conditions in the Theorem 1 would be easily satisfied if \(\text{Re}(\lambda_{\max}(-D))\) (the maximum real part of the eigenvalues of \(-D\)) is more negative by suitably selecting the gain matrix \(\Omega\).

Of course, the gain matrix \(\Omega\) in synchronization control law is not unique. Here we give a tentative method to construct the gain matrix \(\Omega\). A rough technique is as follows: Given a positive constant \(\alpha\) and an arbitrarily sufficiently small positive constant \(\varepsilon\), one can a suitable gain matrix \(\Omega\) by adjusting the stable matrix \(-D\).

In fact, the proposed synchronization law in this paper can be achieved by computer programming, which has the excellent advantages. A specific computational procedure scheme can be designed as follows:

**Step 1:** Given a positive constant \(\alpha\) and an arbitrarily sufficiently small positive constant \(\varepsilon\), choose a suitable gain matrix \(\Omega\) such that \(-D\) is a stable matrix by using any eigenvalue assignment technique.

**Step 2:** Construct the Hamiltonian matrix \(H\) in Lemma 1 and check if \(H\) has no eigenvalues on the imaginary axis. If so, then the procedure goes to Step 4. Otherwise, the procedure continues to Step 3.

**Step 3:** Adjust the value of \(\text{Re}(\lambda_{\max}(-D))\) more negative by selecting a new gain matrix \(\Omega\) and go back to Step 2.

**Step 4:** Obtain the state-feedback controller (7).

Remark 5. In this paper, a piecewise-linear memristor-based recurrent neural network model is given to characterize memristor feature of pinched hysteresis. Such a model is basically a state-dependent nonlinear switching dynamical system. However, for the memristor-based system, since it consists of too many subsystems, it is very difficult to analyze its dynamic behaviors. To proceed dynamic analysis of the memristive switching system, we use differential inclusion to avoid this difficulty. Under the framework of Filippov’s solution, we can turn to analyze a relevant differential inclusion. We obtain some sufficient conditions of the synchronization control of memristor-based recurrent neural networks which are the generalization of those for conventional recurrent neural networks.

4. An illustrative example

Referring to the Example 1 introduced in Zeng and Wang [41] for analyzing and designing associative memories based on recurrent neural networks, consider a neural network with 12 neurons \( n = 12 \) of the configuration given in Fig. 2, after introducing the memristors, we will discuss the following numerical example.

**Example 1.** Consider 12-neuron memristor-based recurrent neural network model with the configuration given in Fig. 2

\[
x(t) = -D(x)x(t) + A(x)g(x(t)) + B(x)g(x(t - \tau(t))) + I, \quad t \geq 0,
\]

where

\[
x(t) = (x_1(t), x_2(t), \ldots, x_{12}(t))^T, \quad I = (I_1, I_2, \ldots, I_{12})^T,
\]

\[
D(x) = \tilde{D}, \quad \text{if} \quad |x(t)| < T, \quad D(x) = \bar{D}, \quad \text{if} \quad |x(t)| > T,
\]

\[
A(x) = \tilde{A}, \quad \text{if} \quad |x(t)| < T, \quad A(x) = \bar{A}, \quad \text{if} \quad |x(t)| > T,
\]

\[
B(x) = \tilde{B}, \quad \text{if} \quad |x(t)| < T, \quad B(x) = \bar{B}, \quad \text{if} \quad |x(t)| > T,
\]

\[
g(x(t)) = (g(x_1(t)), g(x_2(t)), \ldots, g(x_{12}(t)))^T,
\]

\[
g(x(t - \tau(t))) = (g(x_1(t - \tau_1(t))), g(x_2(t - \tau_2(t))), \ldots, g(x_{12}(t - \tau_{12}(t))))^T,
\]

\[
\tau_i(t) = \bar{i}, \quad i = 1, 2, \ldots, 12,
\]

\[
g(\theta) = \frac{1}{2}(|\theta + 1| - |\theta - 1|).
\]

![Fig. 2. Interconnecting structure of a 12-neuron neural network.](image-url)
In fact, system (11) has multiple equilibria, which provide good pattern characterization to be used to analyze and design associative memories. Especially, one needs to design algorithms for huge amounts of desired memory patterns, the memristive neural networks provide great conveniences. Even, when system (11) does not exhibit memristive pinched hysteresis, to store the desired memory patterns in Fig. 3 (For purpose of visualization, 1 represents white, -1 represents black), according to Corollary 5 in [41], if set \( t_i = 1.825 \) \((i = 1, 2, \ldots, 12)\), \( D(x) = E_{12} \), then we can choose desired connection weight matrices

\[
A(x) + B(x) = \begin{pmatrix}
2.325 & 1.825 & 1.225 & -4.575 & 1.525 & -3.05 & 1.225 & 1.825 & 0 & 0 & 0 & 0 \\
1.825 & 2.325 & 1.225 & -4.575 & 1.525 & -3.05 & 1.225 & 1.825 & 0 & 0 & 0 & 0 \\
1.825 & 1.825 & 2.325 & -5.175 & 1.525 & -3.65 & 1.825 & 1.825 & 0 & 0 & 0 & 0 \\
1.825 & 1.825 & 1.825 & -4.675 & 1.525 & -3.65 & 1.825 & 1.825 & 0 & 0 & 0 & 0 \\
1.825 & 1.825 & 1.825 & -5.775 & 2.625 & -3.65 & 1.825 & 1.825 & 0 & 0 & 0 & 0 \\
1.825 & 1.825 & 1.825 & -5.775 & 2.125 & -3.15 & 1.825 & 1.825 & 0 & 0 & 0 & 0 \\
1.825 & 1.825 & 2.425 & -6.375 & 2.125 & -4.25 & 2.925 & 1.825 & 0 & 0 & 0 & 0 \\
1.825 & 2.425 & 2.425 & -6.975 & 2.125 & -4.85 & 2.425 & 2.925 & 0 & 0 & 0 & 0 \\
2.425 & 1.825 & 2.425 & -6.975 & 2.125 & -4.85 & 2.425 & 1.825 & 1.1 & 0 & 0 & 0 \\
2.425 & 2.425 & 2.425 & -6.975 & 2.125 & -4.85 & 2.425 & 1.825 & 0.6 & 0.5 & 0 & 0 \\
2.425 & 2.425 & 2.425 & -6.975 & 2.125 & -4.85 & 2.425 & 1.825 & 0.6 & 0.5 & 0 & 0.5 \\
1.825 & 1.825 & 1.225 & -4.575 & 1.525 & -3.05 & 1.225 & 1.825 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Next, we study the synchronization control of system (11). In order to simplify illustration, let \( \bar{D} = E_{12} \), \( \bar{D} = 1.5E_{12} \), \( I = (1.825, 1.825, \ldots, 1.825)^T \), \( T = (0.8, 0.8, \ldots, 0.8)^T \), \( \hat{A} = \hat{B} = \begin{pmatrix}
3E_6 & 0 & 2E_6 \\
0 & \end{pmatrix} \), and

\[
\hat{A} = \hat{B} = \begin{pmatrix}
1.8 & 0 & 0 & -1.6 & -1.6 & -1.6 & -2 & -2 & -2.6 & -2.6 & -2.6 \\
0 & 1.8 & 0 & -1.6 & -1.6 & -1.6 & -2 & -2 & -2.6 & -2.6 & -2.6 \\
0 & 0 & 1.8 & -1.6 & -1.6 & -1.6 & -2 & -2 & -2.6 & -2.6 & -2.6 \\
-1.6 & -1.6 & -1.6 & 1.8 & 0 & 0 & -2.6 & -2.6 & -2.6 & -2 & -2 \\
-1.6 & -1.6 & -1.6 & 0 & 1.8 & 0 & -2.6 & -2.6 & -2.6 & -2 & -2 \\
-1.6 & -1.6 & -1.6 & 0 & 0 & 1.8 & -2.6 & -2.6 & -2.6 & -2 & -2 \\
-1.3 & -1.3 & -1.3 & -1.5 & -1.5 & -1.5 & 1.7 & 0 & 0 & -0.8 & -0.8 & -0.8 \\
-1.3 & -1.3 & -1.3 & -1.5 & -1.5 & -1.5 & 0 & 1.7 & 0 & -0.8 & -0.8 & -0.8 \\
-1.3 & -1.3 & -1.3 & -1.5 & -1.5 & -1.5 & 0 & 0 & 1.7 & -0.8 & -0.8 & -0.8 \\
-1.5 & -1.5 & -1.5 & -2 & -2 & -2 & -0.8 & -0.8 & -0.8 & 1.3 & 0 & 0 \\
-1.5 & -1.5 & -1.5 & -2 & -2 & -2 & -0.8 & -0.8 & -0.8 & 0 & 1.3 & 0 \\
-1.5 & -1.5 & -1.5 & -2 & -2 & -2 & -0.8 & -0.8 & -0.8 & 0 & 0 & 1.3
\end{pmatrix}
\]

Let parameters \( \alpha = 0.2 \), \( \varepsilon = 0.1 \), and choose the controller gain matrix \( \Omega = 5E_{12} \), then the corresponding Hamiltonian matrix is as

\[
H = \left( \begin{array}{cc}
-Q_1 & Q_2 \\
Q_3 & Q_1^T
\end{array} \right)
\]

Fig. 3. Four desired memory patterns (3 × 4).
with $Q_1 = 5.8\sigma_{12}$, $Q_2 = \begin{bmatrix} 18E_2 & 0 \\ 0 & 8E_2 \end{bmatrix}$. $Q_3 = \text{diag}(-2.3214, -2.5918, -2.9221, -3.3255, -3.8183, -4.4201, -5.1552, -6.053, -7.1496, -8.4891, -10.125, -12.1232).

Obviously, the Hamiltonian matrix $H$ has no eigenvalues on the imaginary axis, and the conditions of Theorem 1 are satisfied, so the coupling of variables can drive the two coupled neural networks exponentially synchronized.

5. Concluding remarks

Chua in 1971 proposed from symmetry arguments that there should be a fourth fundamental element, which he called a memristor, besides the fundamental passive circuit elements: resistor, capacitor and inductor. Although he showed that such an element had many interesting and valuable circuit properties, until now it is still very difficult to present a useful physical model or material example of a memristor. In this paper, we formulate and investigate a preliminary memristor-based recurrent neural network model to imitate the memristive pinched hysteresis. Some sufficient conditions are derived to guarantee the exponential synchronization for the memristor-based recurrent neural networks. Finally, a numerical example is discussed to illustrate the validity of the obtained result. In addition to memory applications, we believe our derived analysis can be further extended to provide valuable design insights and allow an in-depth understanding of key design implications of memristor-based memories.

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