On the large-maturity smile for the Heston model

Carole Bernard,* Zhenyu Cui,† Don McLeish‡

December 20, 2011.

Abstract

Reformulating the results of del Baño Rollin, Ferreiro-Castilla, and Utzet [3], we give necessary and sufficient conditions for the moments of the stock price in the Heston model to exist and extend Theorem 2.1 of [5]. Forde and Jacquier [5] provide necessary conditions for the moments to exist when $\kappa > \rho \sigma$. Although this assumption is satisfied on Equity markets (because the correlation is generally negative), it does not hold for FX-related derivatives. Furthermore we show that the application of the Gärtner-Ellis theorem attempted in [5] fails to obtain the asymptotic behavior of calls or puts with large maturity when $\kappa > \rho \sigma$ (the case investigated in their paper). Nevertheless it can be used for put options when $\kappa \leq \rho \sigma$. To show this, we give a detailed classification of the cases when the rate function is essentially smooth under both the original and the share measures. This classification complements and corrects [6] and [5].

Key Words: Moment explosion, Heston model, Asymptotics for large maturity, Essential smoothness, Large deviations principle.

JEL Classification: C02, C63, G12, G13
Mathematics Subject Classification (2010): 60G44, 91B70, 91B25

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* C. Bernard, Department of Statistics and Actuarial Science, University of Waterloo, 200 University Avenue West, Waterloo, ON, N2L3G1. 
c3bernar@uwaterloo.ca

† Z. Cui, Department of Statistics and Actuarial Science, University of Waterloo, cuizhyu@gmail.com

‡ D. McLeish, Department of Statistics and Actuarial Science, University of Waterloo, 
dlmcleis@uwaterloo.ca
1 Introduction

In the first part of this paper we obtain the limiting scaled cumulant generating function for the log stock price in the Heston model, and thus provide necessary and sufficient conditions for the moments of stock price to exist in the Heston model. We obtain results for both the case \( \kappa > \rho \sigma \) (considered in Forde and Jacquier [5], Jacquier, Keller-Ressel and Mijatovic [8]) and also the case \( \kappa \leq \rho \sigma \), which is new to the literature. The case \( \kappa \leq \rho \sigma \) is usually referred to as the “large correlation regime” and is of particular interest to model the Foreign Exchange market (Jacquier and Martini [9]). Proposition 2.1 gives the moment generating function of the log asset price \( X_t \) on its convergence domain. Proposition 2.2 concerns the limiting behavior of the convergence domain in Lemma 2.1. These properties are needed to study moment explosion behaviors of the stock price under Heston model, see Andersen and Piterbarg [1].

Denote by \( X \) the log spot in the Heston model. In the second part of this note, we give a rigorous classification of the essential smoothness property of the family of random variables \( (X_t/t \pm E_{\lambda}/t)_{t \geq 1} \), where \( E_\lambda \) is an exponential random variable independent of \( X \), with parameter \( \lambda \). The essential smoothness of these random variables is needed to apply the Gärtner-Ellis theorem, in order to establish the large deviation principle (LDP) needed in its application (Dembo and Zeitouni [4]). Forde and Jacquier [5] attempt to characterize the asymptotic implied volatility in the Heston model in the case \( \kappa > \rho \sigma \) by concluding that the family of random variables \( (X_t/t \pm E_{\lambda}/t)_{t \geq 1} \) satisfies the LDP from the fact that the LDP holds for \( (X_t)_{t \geq 1} \). In the second part of this paper, we give a complete answer to this question by classifying cases when the LDP persists and those where it does not. This classification complements and corrects the note [6]. It shows that the technique used in [5] fails in their case \( \kappa > \rho \sigma \), since the essential smoothness required to apply Gärtner-Ellis theorem in order to obtain the asymptotic behavior of calls or puts with large maturity fails. However we show it does apply for put options when \( \kappa \leq \rho \sigma \). Recently Jacquier, Keller-Ressel and Mijatovic [8] again apply essential smoothness to study the asymptotic implied volatility for affine models with jumps.

For convenience we adopt the same notations as [5]. Denote by \( X_t \) the log asset price
log($S_t$) in the Heston model and by $Y_t$ its underlying variance process.

\[
\begin{align*}
  dX_t &= -\frac{1}{2}Y_t dt + \sqrt{Y_t} \left( \rho dW_1(t) + \sqrt{1-\rho^2} dW_2(t) \right) \\
  dY_t &= \kappa (\theta - Y_t) dt + \sigma \sqrt{Y_t} dW_1(t)
\end{align*}
\]

(1)

where $\kappa$, $\theta$ and $\sigma$ are positive, $|\rho| < 1$, and $W_1$ and $W_2$ are independent standard Brownian motions under a given probability measure $P$. Furthermore $Y_0 > 0$. The Feller condition $\frac{2\kappa \theta}{\sigma^2} - 1 \geq 0$ is assumed in [5] but can be relaxed here because we are studying properties of the cumulant generating functions of the process but not its path properties. The cumulant generating function remains well-defined even when the Feller condition is violated.

In Section 2, we extend Theorem 2.1 of [5] for $\kappa > \rho \sigma$ to the general case when $\kappa > 0$. In Section 3 we give a complete classification of the essential smoothness property needed to apply Gärtner-Ellis theorem for the cases when $(X_t)_{t\geq 1}$ is perturbed by an independent exponential random variable $E_{\lambda}$. We consider the general case, assuming only $\kappa > 0$. The last section corrects important parts of the proof of Theorem 2.1 of [5].

2 Limiting cumulant generating function in Heston model

Recall that $S_t = e^{X_t}$. Denote by $M_t(p)$ the $p^{th}$ moment of $S_t$.

\[
M_t(p) := E[S_t^p] = E[e^{pX_t}].
\]

The formula of the moment generating function (mgf) of $X_t$ given by Hurd and Kuznetsov [7] (and used in [5]) does not hold for its whole convergence domain. Proposition 2.1 provides an expression of the mgf, essentially due to del Baño Rollin, Ferreiro-Castilla, and Utzet [3], valid on the full convergence domain. To do so we first need to define two auxiliary functions $q$ and $r$.  

Define the function $q$, analytic in a neighborhood of 0 on the complex plane $\mathbb{C}$ by

$$q(x) = \sum_{n=1}^{\infty} \frac{B_{2n}4^n(4^n - 1)}{(2n)!} x^{n-1} = 1 - \frac{1}{3}x + \frac{2}{15}x^2 - \frac{17}{315}x^3 + \ldots$$

$$= \begin{cases} \frac{\tanh(\sqrt{x})}{\sqrt{x}} & \text{for } x \geq 0 \\ \frac{\tan(\sqrt{-x})}{\sqrt{-x}} & \text{for } x < 0 \end{cases}$$

where $B_{2n}$ are the Bernoulli numbers. Also define

$$r(x) = \sum_{n=0}^{+\infty} \frac{x^n}{(2n)!} = \begin{cases} \cosh(\sqrt{x}) & \text{if } x \geq 0 \\ \cos(\sqrt{-x}) & \text{if } x < 0 \end{cases}$$

Proposition 2.1. Mgf of $X_t$

On its convergence domain $(l^*_t, u^*_t)$, for all values of $\kappa > 0$ and $|\rho| < 1$, the mgf of $X_t$ is $M_t(p)$ where

$$\log \left( \frac{M_t(p)}{S_0^p} \right) = \frac{\kappa \theta}{\sigma^2} t(\kappa - \sigma \rho p) - \frac{2\kappa \theta}{\sigma^2} \log(f(p)) - \frac{Y_0(p - p^2)}{2} q \left( \frac{t^2}{4} g(p) \right)$$ (2)

where

$$f(p) := r \left( \frac{t^2}{4} g(p) \right) \left[ 1 + \frac{(\kappa - \sigma \rho p)}{2} q \left( \frac{t^2}{4} g(p) \right) \right].$$

and $g(p) = (\kappa - \sigma \rho p)^2 - \sigma^2(p(p-1))$. Let $p^+$ and $p^-$ be the roots of $g(p) = 0$,

$$p^\pm = \frac{\sigma - 2\kappa \rho \pm \sqrt{\sigma^2 + 4\kappa^2 - 4\rho \sigma \kappa}}{2(1 - \rho^2)\sigma},$$

where $p^+ > 1$ if $\kappa \neq \rho \sigma$, $p^+ = 1$ if $\kappa = \rho \sigma$ and $p^- < 0$ for all $\kappa > 0$.

The left abscissa of convergence $l^*_t$ of $M_t(p)$ is the largest root $p < p^-$ of the following equation

$$1 + \frac{(\kappa - \sigma \rho p)}{2} q \left( \frac{t^2}{4} g(p) \right) = 0.$$ (3)

The right abscissa of convergence $u^*_t$ is the smallest positive root of the same equation (3).

Proof. Expression (2) in Proposition 2.1 is based on the equation (3) of del Baño Rollin, Ferreiro-Castilla, and Utzet [3] that we have simplified using the notation $q(\cdot)$. Note that
$g(p)$ is a quadratic function and $g(p) > 0$ if and only if $p \in (p^-, p^+)$. Since $g(0) = \kappa^2$ and $g(1) = (\kappa - \sigma \rho)^2$, we have that $p^- < 0$ and that $p^+ > 1$ if $\kappa \neq \rho \sigma$ and $p^+ = 1$ if $\kappa = \rho \sigma$. This extends the proof of Forde and Jacquier [5] where they only assume $\rho \sigma < \kappa$ (see Appendix B of their paper). Note also that for all $\kappa > 0$, the expression (2) is well-defined and finite at $p = 0$ and $p = 1$. It follows that the domain of convergence always includes the interval $[0, 1]$. It is also clear that (2) approaches infinity at the roots of $f(p) = 0$. This proves the second part of Proposition 2.1.

For the next proposition, we refer to Theorem 3.1 in [3] for the result and the proof.

**Proposition 2.2. Properties of the Abscissas of Convergence**

Assume $|\rho| < 1$. Given $t > 0$, denote by $\alpha_t^-$ and $\alpha_t^+$ the respective roots of $g(p) = -\frac{4\pi^2}{t^2}$ and $\beta_t^-$ and $\beta_t^+$ the respective roots of $g(p) = -\frac{\pi^2}{t^2}$. The abscissas of convergence $l_{t}^*$ and $u_{t}^*$ satisfy the following properties

- If $\kappa \geq \rho \sigma$ then $u_{t}^* \in [p^+, \alpha_t^+]$.
- If $\kappa < \rho \sigma$, then
  - for $t < \frac{2}{\sigma \rho p^+ - \kappa}$, then $u_{t}^* \in (p^+, \beta_t^+)$,
  - for $t \geq \frac{2}{\sigma \rho p^+ - \kappa}$, then $u_{t}^* \in (1, p^+]$.
- For all $\kappa > 0$, $l_{t}^* \in (\alpha_t^-, p^-)$.

To derive from Propositions 2.1 and 2.2 the asymptotic results of Forde and Jacquier [5], we first prove the following lemma.

**Lemma 2.1. Asymptotic Behavior of the Convergence Domain**

1. For all $\kappa > 0$, the left abscissa of convergence $l_{t}^*$ satisfies $l_{t}^* \rightarrow p^-$ from the left as $t \rightarrow +\infty$.

2. The right abscissa of convergence $u_{t}^*$ satisfies
   - if $\kappa \geq \rho \sigma$, $u_{t}^* \rightarrow p^+$ from the right as $t \rightarrow +\infty$.
   - if $\kappa < \rho \sigma$, $u_{t}^* \rightarrow 1$ from the right as $t \rightarrow +\infty$. 

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Proof. The lemma follows from the properties of the abscissas of convergence recalled in Proposition 2.2. The proof consists of two steps.

Proof that \( l^*_t \to p^- \) from the left: This follows directly from Proposition 2 and the fact that \( \alpha^- \to p^- \).

Proof that \( u^*_t \) converges: In the case \( \kappa \geq \rho \sigma \) then \( u^*_t \in [p^+, \alpha^+_t] \) and convergence to \( p^+ \) follows from the convergence of \( \alpha^+_t \) to \( p^+ \). Consider the case \( \kappa < \rho \sigma \). Put \( \tau(x) = \sqrt{|x|}q(x) = \begin{cases} \tanh(\sqrt{x}) & \text{for } x \geq 0 \\ \tan(-\sqrt{-x}) & \text{for } x < 0 \end{cases} \). Then \( 1 - \tau^2(x) \sim 4e^{-2\sqrt{x}} \) as \( x \to +\infty \). \( u^*_t \) is the smallest positive root of the equation (3). For \( t \) sufficiently high, it is clear that \( u^*_t \in (0, p^+) \). For \( 0 < p < p^+ \), (3) can be rewritten as

\[
\varepsilon_t(p)(\kappa - \sigma \rho p)^2 - \sigma^2 p(p - 1) = 0
\]  

(4)

where \( \varepsilon_t(p) = 1 - \tau^2 \left( \frac{t^2}{4} g(p) \right) = 1 - \tanh^2(\frac{1}{2} \sqrt{g(p)}) \). Note that (4) is similar to \( g(p) \) but with leading coefficient \( \varepsilon_t(p) \) varying rather than equal to 1. Indeed with \( g_\varepsilon(p) = \varepsilon(\kappa - \sigma \rho p)^2 - \sigma^2 p(p - 1) \) we have \( g_1(p) = g(p) \). Here \( 1 \geq \varepsilon_t(p) \geq 0 \) and

\[
\varepsilon_t(p) \sim 4e^{-t\sqrt{g(p)}} \quad \text{as} \quad \frac{t^2}{4} g(p) \to \infty.
\]

Denote the larger root of the equation \( g_\varepsilon(p) = 0 \) by

\[
p^+_{\varepsilon} = \frac{\sigma - 2\kappa \varepsilon \rho + \sqrt{\sigma^2 - 4\kappa \sigma \varepsilon \rho + 4\kappa^2 \varepsilon}}{2\sigma(1 - \varepsilon \rho^2)}
\]  

(5)

so in particular \( p^+_{1\varepsilon} = p^+ = \frac{\sigma - 2\kappa \rho + \sqrt{\sigma^2 - 4\kappa \sigma \rho + 4\kappa^2}}{2\sigma(1 - \varepsilon \rho^2)} \) and \( p^+_{0\varepsilon} = 1 \). By differentiating\(^1\) the function \( \frac{\partial}{\partial \varepsilon} g_\varepsilon(p^+_{\varepsilon}) = 0 \), and solving for \( \frac{\partial}{\partial \varepsilon} p^+_{\varepsilon} \) we obtain

\[
\frac{\partial}{\partial \varepsilon} p^+_{\varepsilon} = \frac{(\kappa - \sigma \rho p^+_{\varepsilon})^2}{\sigma(2\sigma(1 - \varepsilon \rho^2)p^+_{\varepsilon} - \sigma + 2\varepsilon \rho \kappa)} = \frac{(\kappa - \sigma \rho p^+_{\varepsilon})^2}{\sigma \sqrt{\sigma^2 - 4\kappa \sigma \varepsilon \rho + 4\kappa^2 \varepsilon}} \geq 0
\]

by (5). Since \( \frac{\partial}{\partial \varepsilon} p^+_{\varepsilon} \geq 0 \), \( p^+_{\varepsilon} \) is a non-decreasing function of \( \varepsilon \) in the interval \([0, 1]\). We prove convergence to 1 by contradiction. Suppose there is a subsequence of values of \( t \to \infty \) over which \( u^*_t \) converges to some value \( u \) where \( p^+ > u > 1 \) and \( g(u) > 0 \). Then since \( \varepsilon_t(u) \sim 4e^{-t\sqrt{g(u)}} \to 0 \), as \( t \to \infty \), \( u \) must be a solution to the equation \( g_0(u) = 0 \) and since it is the larger of the roots, \( u = 1 \). This contradicts the assumption \( u > 1 \). The only other

\(^1\)Using the chain rule with \( \frac{\partial}{\partial \varepsilon} g_\varepsilon(p^+_{\varepsilon}) = \frac{\partial}{\partial \varepsilon} h(\varepsilon, k(\varepsilon)) \) for appropriate functions \( h \) and \( k \).
possibility is \( u = p^+ \) so that \( g(u) = 0 \). Notice that \( 1 - \tau^2(x) \) is a non-increasing function of \( x \) for \( x > 0 \) which implies \( \varepsilon_t(p) \) is a decreasing function of \( t \). Thus, for \( s > t \), it follows that \( \varepsilon_s(u^+_t) < \varepsilon_t(u^+_t) \) and since \( p_\varepsilon \) is non-decreasing in \( \varepsilon \), \( u^+_s < u^+_t \). However \( u^+_t < p^+ \) for all \( t \). Since \( u_t \) is monotonically decreasing in \( t \) for large \( t \) and since \( u^+_t < p^+ \), the only possible limit is 1.

\( \square \)

**Theorem 2.1. Extended Theorem 2.1 of Forde and Jacquier [5]**

With \( V_t(p) = \log E(e^{p(X_t-x_0)}) \) and

\[
 u^* := \lim_{t \to \infty} u^*_t = \begin{cases} 
 p^+ & \text{if } \kappa \geq \rho \sigma \\
 1 & \text{if } \kappa < \rho \sigma 
\end{cases}
\]

we have

\[
 V(p) := \lim_{t \to \infty} \frac{1}{t} V_t(p) = \begin{cases} 
 \frac{\kappa \theta}{\sigma^2} \left( \kappa - \sigma pp - \sqrt{g(p)} \right) & \text{if } p^- \leq p < u^* \\
 \frac{\kappa \theta}{\sigma^2} (\kappa - \sigma pp) & \text{if } p = u^* = p^+ \text{ when } \kappa > \rho \sigma \\
 -2 \frac{\kappa \theta}{\sigma^2} (\sigma p - \kappa) & \text{if } p = u^* = 1 \text{ and } \kappa < \rho \sigma \\
 0 & \text{if } p = u^* = 1 \text{ and } \kappa = \rho \sigma \\
 \infty & \text{otherwise}
\end{cases}
\]

**Proof.** This follows directly from the expression for the cgf

\[
 V_t(p) = \frac{\kappa \theta}{\sigma^2} t (\kappa - \sigma pp) - \frac{2 \kappa \theta}{\sigma^2} \log(f(p)) - \frac{Y_0(p - p^2)}{\frac{1}{2} + (\kappa - \sigma pp) q \left( \frac{t^2}{4} g(p) \right)} + p \ln(S_0)
\]

since for \( p \in [p^-, u^*] \), \( g(p) \geq 0 \) and

\[
 \lim_{t \to \infty} \frac{1}{t} \left[ \frac{\kappa \theta}{\sigma^2} t (\kappa - \sigma pp) - \frac{2 \kappa \theta}{\sigma^2} \log(f(p)) - \frac{Y_0(p - p^2) q \left( \frac{t^2}{4} g(p) \right)}{\frac{1}{2} + (\kappa - \sigma pp) q \left( \frac{t^2}{4} g(p) \right)} \right] = \frac{\kappa \theta}{\sigma^2} (\kappa - \sigma pp) - \lim_{t \to \infty} \frac{2 \kappa \theta}{\sigma^2} \frac{1}{t} \log(f(p)) - \lim_{t \to \infty} \frac{\frac{1}{2} Y_0(p - p^2) q \left( \frac{t^2}{4} g(p) \right)}{1 + (\kappa - \sigma pp) \frac{1}{2} q \left( \frac{t^2}{4} g(p) \right)}.
\]

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But for $g(p) > 0$, as $t \to \infty$, we have

$$\frac{1}{t} \log(f(p)) = \frac{1}{t} \log \left( \cosh \left( \frac{t}{2} \sqrt{g(p)} \right) + (\kappa - \sigma \rho p) \frac{\sinh \left( \frac{t}{2} \sqrt{g(p)} \right)}{\sqrt{g(p)}} \right) \sim \frac{1}{t} \log \left( \frac{1}{2} e^{\frac{t}{2} \sqrt{g(p)}} \left( 1 + (\kappa - \sigma \rho p) \frac{1}{\sqrt{g(p)}} \right) \right) \sim \frac{\sqrt{g(p)}}{2}$$

and since $q \left( \frac{t^2}{2} g(p) \right) = \mathcal{O}(\frac{1}{t})$,

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left( e^{p(X_t-x_0)} \right) = \frac{\kappa \theta}{\sigma^2} (\kappa - \sigma \rho p) - \frac{2 \kappa \theta \sqrt{g(p)}}{\sigma^2} = \frac{\kappa \theta}{\sigma^2} \left( \kappa - \sigma \rho p - \sqrt{g(p)} \right).$$

When $p = p^-$ or $p = p^+$ so that $g(p) = 0$, it is trivial. For example when $p = p^+$, (6) becomes

$$\frac{1}{t} \left[ \frac{\kappa \theta}{\sigma^2} t(\kappa - \sigma \rho p^+) - \frac{2 \kappa \theta}{\sigma^2} \log \left( 1 + (\kappa - \sigma \rho p^+) \frac{t}{2} \right) - \frac{Y_0(p^+ - p^{+2})}{t} \frac{2}{\frac{t}{2} + (\kappa - \sigma \rho p^+)} \right] \sim \frac{\kappa \theta}{\sigma^2} (\kappa - \sigma \rho p^+).$$

The case $p = u^* = 1$ when $\kappa \leq \rho \sigma$ is also trivial. \hfill $\square$

### 3 Classification of essential smoothness

Recently, Forde et al. [6] point out that there is a gap in the proof of Corollary 2.4 in [5]. The issue comes from verifying the conditions needed to apply the Gärtner-Ellis theorem. Precisely, to obtain asymptotics of call prices with a large maturity, the problem comes from establishing the “essential smoothness” of $\tilde{\Lambda}^{-1}(\cdot)$ under the share measure $^2 \tilde{P}$.

$$\tilde{\Lambda}^{-1}(p) = \lim_{t \to +\infty} \frac{1}{t} \log \mathbb{E}_{\tilde{P}}[e^{tp((X_t-x_0)/t-E_1/t)}],$$

\footnote{We use the notation $\tilde{P}$ for the share measure as in [6] because the notation $P^*$ in Forde and Jacquier [5] for the share measure can be confusing. Indeed in the original paper of Forde and Jacquier [5], $V^*$ refers to the Fenchel-Legendre transform under $P$ and not to the limiting function of the cumulant generating function under $P^*$, which is denoted by $V_S$ in the original paper. Here we adopt the notation of [6] and use $\tilde{\Lambda}$, $\tilde{\Lambda}^{1\pm}$ and $\Lambda^{1\pm}$.}
where the expectation is taken under the share measure \( \tilde{P} \) defined by \( d\tilde{P}/dP \big|_t = \exp(X_t - x_0) \) and where \( E_1 \) is an exponential random variable with parameter 1 independent of \( X_t \).

Similarly to obtain asymptotics of put prices with a large maturity, the difficulty reduces to establishing the essential smoothness of

\[
\Lambda^{1+}(p) = \lim_{t \to +\infty} \frac{1}{t} \log E[e^{tp((X_t-x_0)/t+E_1/t)}]
\]

under \( P \) where \( E_1 \) is an independent exponential random variable with parameter 1 independent of \( X_t \) under \( P \). \( \Lambda^{1-}(p) \) and \( \tilde{\Lambda}^{1+}(p) \) are defined similarly.

**Proposition 3.1. Classifications for Essential Smoothness**

1. \( \tilde{\Lambda}^{1-}() \) fails to be essentially smooth for any \( \kappa > 0 \).

2. \( \tilde{\Lambda}^{1+}() \) is essentially smooth if and only if \( 0 < \kappa \leq (2\rho + \sqrt{2})\sigma \).

3. \( \Lambda^{1+}() \) is essentially smooth if and only if \( 0 < \kappa \leq \rho\sigma \).

4. \( \Lambda^{1-}() \) is essentially smooth if and only if \( 0 < \kappa \leq (\sqrt{2} - \rho)\sigma \).

**Proof.** Observe that the limiting cumulant generating function \( \tilde{\Lambda}(p) \) under \( \tilde{P} \) verifies

\[
\tilde{\Lambda}(p) = \lim_{t \to +\infty} \frac{1}{t} \log E_{\tilde{P}}[e^{tp((X_t-x_0)/t)]} = \lim_{t \to +\infty} \frac{1}{t} \log E[e^{X_t-x_0}e^{p(X_t-x_0)}]
\]

\[
= \lim_{t \to +\infty} \frac{1}{t} \log E(e^{(p+1)(X_t-x_0)}) = V(p+1).
\]

The domain of convergence of \( \tilde{\Lambda} \) is \( D_{\tilde{\Lambda}} := \{ p \in \mathbb{R} : \tilde{\Lambda}(p) < +\infty \} = [p^- - 1, u^* - 1] \) (by Theorem 1 above\(^3\)). Furthermore,

\[
\tilde{\Lambda}(p) = \frac{\kappa\theta}{\sigma^2} \left( \kappa - \sigma p(p+1) - \sqrt{g(p+1)} \right).
\]

To obtain the convergence domain of \( \tilde{\Lambda}^{1\pm} \) and \( \Lambda^{1\pm} \), we apply Lemma 2.1 of [6] with \( \lambda = 1 \) or \( \lambda = -1 \). Then,

\[
D_{\tilde{\Lambda}^{1-}} = D_{\tilde{\Lambda}} \cap (-1, +\infty) = (-1, u^* - 1],
\]

\[^3\]There are typos in the expressions \( p_{\pm}^- \) and \( p_{\pm}^+ \) in Corollary 3.1 of Forde and Jacquier [5]. In particular, their \( p_{\pm}^\pm \) do not satisfy \( p_{\pm}^\pm = p_{\pm} - 1 \).
We also have that
\[ D_{\tilde{\Lambda}^1} = D_{\tilde{\Lambda}} \cap (-\infty, 1) = \begin{cases} [p^- - 1, u^* - 1] & \text{if } u^* < 2 \\ [p^- - 1, 1) & \text{if } u^* \geq 2 \end{cases} \]

If \( \kappa \leq \rho \sigma \), \( u^* = 1 < 2 \). In the case \( \kappa > \rho \sigma \), \( p^- < 0 \) and so \( p^- - 1 < -1 \). Since \( \tilde{\Lambda}'(-1) < +\infty \), therefore \( \tilde{\Lambda}^1 \) is not steep at \(-1\). In this case, \( u^* = p^+ < 2 \) if and only if \( g(p^+) = 0 > g(2) = \sigma^2 (\frac{\xi}{\sigma} - (2\rho - \sqrt{2}))\left(\frac{\xi}{\sigma} - (2\rho + \sqrt{2})\right) \) and this is equivalent to the condition \( \frac{\xi}{\sigma} < 2\rho + \sqrt{2} \). Therefore \( \tilde{\Lambda}^1 \) is steep at both ends of the convergence domain if and only if \( \kappa \leq \sigma(2\rho + \sqrt{2}) \).

Similarly we observe that
\[ D_{\Lambda^1} = D_{\tilde{\Lambda}^1} \cap (-1, +\infty) = \begin{cases} [p^-, u^*] & \text{if } p^- > -1 \\ (-1, u^*) & \text{if } p^- \leq -1 \end{cases} \]

It is easy to check that \( g(-1) < 0 \) (that is \(-1 < p^-\)) if and only if \( \kappa < \sigma(\sqrt{2} - \rho) \). Therefore \( \Lambda^1 \) is steep at both boundaries if and only if \( \kappa \leq \sigma(\sqrt{2} - \rho) \) (because it is also steep at \(-1\) when \( p^- = -1 \), which happens when \( \kappa = \sigma(\sqrt{2} - \rho) \)). Finally,
\[ D_{\Lambda^1} = D_{\tilde{\Lambda}^1} \cap (-\infty, 1) = [p^-, 1) \]

and \( \Lambda^1 \) is steep at both boundaries if and only if \( \kappa \leq \rho \sigma \).

Based on the results of Carr and Madan [2], to derive the asymptotics of the put option prices with large maturity, one uses
\[ \frac{1}{S_0 e^{xt}} \mathbb{E}[(S_0 e^{xt} - S_t)^+] = P(X_t - x_0 + E_1 < xt), \]

where \( E_1 \) is an exponential random variable with parameter 1 independent of \( X_t \) under \( P \). Therefore one needs the essential smoothness of \( \Lambda^1 \) under \( P \) to verify the conditions of the Gärtner-Ellis theorem as in Forde and Jacquier [5]. Using item (3) of Proposition 3.1, it is essentially smooth only when \( \kappa \leq \rho \sigma \) (which is not considered by Forde and Jacquier [5]).

To derive the asymptotic behavior of call option prices, one needs
\[ 1 - \frac{1}{S_0} \mathbb{E}[(S_t - S_0 e^{xt})^+] = \tilde{P}(X_t - x_0 - E_1 \leq xt), \]
where $E_1$ is an exponential random variable with parameter 1 independent of $X_t$ under $\tilde{P}$. Here we need to check the essential smoothness of $\tilde{\Lambda}^{1-}$ under $\tilde{P}$. Item (1) of Proposition 3.1 shows that it is never essentially smooth.

It is correctly stated in [6], Remark 2.2(a) following Lemma 2.1 that the essential smoothness of $X_t/t + E_1/t$ does not follow. However when closing the gap in the proof of Proposition 2.3, they make use of the incorrect inequality (2.8) in [6]. It is clear that the opposite inequality

$$1_{\alpha>0}(1 - e^{-\lambda\alpha}) \geq 1_{\alpha>0}(1 - e^{-\alpha}).$$

should hold for $\lambda > u_+ > 1$ instead of (2.8) of [6]. Then (2.9) of [6] does not hold and it is not valid to use the “sandwich” argument as in their proof.

4 Further comments

In this last section, we correct some results in [5] for readers who would like to use these results. The equation (3.9) in [5] should be

$$V'(p) = \frac{\kappa\theta}{\sigma} \left( -\rho + \frac{2\rho\kappa - \sigma + 2p\sigma(1 - \rho^2)}{2\sqrt{g(p)}} \right)$$

where $g(p) = (\kappa - \rho\sigma p)^2 - \sigma^2(p^2 - p)$. Moreover,

$$V''(p) = \frac{\kappa\theta ((\sigma - 2\kappa\rho)^2 + 4\kappa^2(1 - \rho^2))}{4(g(p))^2}.$$

Since $\rho \in (-1, 1)$ it is clear that $V''(p) > 0$ for all $p \in (p^-, p^+)$, and that $V'$ is continuous and strictly increasing from $-\infty$ at $p = p^-$ to $= \infty$ at $p = p^+$. Therefore there exists a unique root $p^*(x)$ to

$$V'(p^*(x)) = x.$$

This equation can be rewritten as

$$\left( x + \frac{\kappa\theta\rho}{\sigma} \right) \frac{2\sigma}{\kappa\theta} \sqrt{g(p^*(x))} = 2\rho\kappa - \sigma + 2p^*(x)\sigma(1 - \rho^2).$$
The solution given in (3.10) of [5] is not correct. There is only one solution $p^*(x)$:

$$p^*(x) = \frac{\sigma - 2\kappa \rho}{2\sigma (1 - \rho^2)} + \frac{(\kappa \theta \rho + x\sigma)\sqrt{\sigma^2 + 4\kappa^2 - 4\rho \sigma \kappa}}{2\sigma (1 - \rho^2) \sqrt{\kappa^2 \theta^2 + 2x \sigma \kappa \theta \rho + x^2 \sigma^2}}$$

The expression following the inequality (3.11) should be corrected to

$$p^*(0) = \frac{\sigma - 2\kappa \rho + \rho \sqrt{\sigma^2 + 4\kappa^2 - 4\rho \sigma \kappa}}{2\sigma (1 - \rho^2)}$$

which is valid for both positive and negative values of $\rho$.

References


