Solving the maximum duo-preservation string mapping problem with linear programming

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In this paper, we introduce the maximum duo-preservation string mapping problem (MPSM), which is complementary to the minimum common string partition problem (MCSP). When each letter occurs at most $k$ times in any input string, the version of MPSM is called $k$-MPSM. In order to design approximation algorithms for MPSM, we also introduce the constrained maximum induced subgraph problem (CMIS) and the constrained minimum induced subgraph (CNIS) problem.

We show that both CMIS and CNIS are NP-complete. We also study the approximation algorithms for the restricted version of CMIS, which is called $k$-CMIS ($k \geq 2$). Using Linear Programming method, we propose an approximation algorithm for $2$-CMIS with approximation ratio 2 and an approximation algorithm for $k$-CMIS ($k \geq 3$) with approximation ratio $k^2$. Based on approximation algorithms for $k$-CMIS, we get approximation algorithms for $k$-MPSM with the same approximation ratio.

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1. Introduction

The minimum common string partition problem (MCSP) has been well-investigated as a fundamental problem in computer science [8,12]. Given two finite length strings over the finite letter alphabet, MCSP is to partition strings into identical substrings with the minimum number of partitions. MCSP is also viewed as the problem of finding a letter-preserving bijective mapping $\pi$ from letters in one string $A$ to letters in the other string $B$ with the minimum number of breaks, where a letter-preserving bijective mapping $\pi$ means that each letter in $A$ is mapped into the same letter in $B$ and the mapping is bijective, and a break is a pair of consecutive letters in $A$ that are mapped by $\pi$ to non-consecutive letters in $B$ [12]. In a string, a pair of consecutive letters is called a duo [12].

As an example, let us assume that there is a letter-preserving bijective mapping $\pi$ (see Fig. 1.1) between two strings $A = abcab$ and $B = ababc$. From Fig. 1.1, we can see that $\pi$ has only one break: $ca$ is a duo of $A$, but $\pi(c)\pi(a)$ is not a duo.

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of \( B \). However, the other three duos in \( A \) (\( ab, bc, \ ab \)) are kept by \( \pi \), each of which is called duo-preservation. So, the sum of the number of breaks and the number of duo-preservations is four, which is the length of any input string minus 1.

For a letter-preserving bijective mapping between two strings, on the one hand, the optimization goal can be to minimize the number of breaks that is known as the MCSP problem. On the other hand, the optimization goal can be to maximize the number of duo-preservations. We define the maximization version of the problem as the maximum duo-preservation string mapping problem (MPSM), i.e. the problem of finding a letter-preservation bijective mapping \( \pi \) from one string to the other string with the maximum number of duo-preservations. When each letter occurs at most \( k \) times in any input string, the version of MPSM is called \( k \)-MPSM. The MPSM problem is complementary to the MCSP problem as shown in Section 2. From this complementary relationship, it follows that MPSM is also NP-hard since the MCSP problem is NP-hard [12].

While the MCSP problem has been widely studied, to the best of our knowledge, the MPSM problem has not been addressed before. Specifically, various approximation algorithms have been proposed to solve the \( k \)-MCSP problem, a version of MCSP, where each letter appears at most \( k \) times in any input string. These results are surveyed in Table 1.

Although there are approximation algorithms for MCSP, it is still required to design approximation algorithms for MPSM, because a pair of complementary NP-hard problems may have different approximation cases, i.e. an approximation algorithm for one problem sometimes cannot be used to approximate its complementary problem. For example, the minimum vertex cover problem and the maximum independent set problem are two well-known complementary problems in computer science [11]. For a given graph with \( n \) vertices, the minimum vertex cover problem can be approximated within a ratio of 2, but the maximum independent set is NP-hard to approximate within a factor \( n^{\delta} \), for some \( \delta > 0 \) [10,4,3]. Another pair of complementary problems is the Max-Satisfy problem and the Min- Unsatisfy problem [1,2]. Both the Max-Satisfy problem and the Min-Unsatisfy problem are NP-hard, but their approximation cases are also different. For a system of \( m \) linear equations with \( n \) variables over fractional numbers \( \mathbb{Q} \), the Min-Unsatisfy problem can be approximated within a factor of \( m + 1 \), but the Max-Satisfy problem is NP-hard to approximate within a factor of \( n^\delta \), for some \( \delta > 0 \) [1,2].

We notice that the MPSMP problem can be transformed to a graph optimization problem. We use the example in Fig. 1.1 to explain it. For two strings \( A = \text{abcab} \) and \( B = \text{ababc} \) in Fig. 1.1, we can construct a graph \( G_{AB} \) as follows (see Fig. 1.2). \( G_{AB} \) has three parts \( M_1, M_2, \) and \( M_3 \). Part \( M_1 \) contains four \((a, a)\) nodes. Part \( M_2 \) contains four \((b, b)\) nodes, and part \( M_3 \) contains one \((c, c)\) node. In \( G_{AB} \), there is an edge between \((a, a)\) node at the position \((1, 1)\) of \( M_1 \) and \((b, b)\) node at the position \((1, 1)\) of \( M_2 \), because the first \( a \) and the first \( b \) in \( A \) form a duo \( ab \) and the first \( a \) and the first \( b \) in \( B \) also form a duo \( ab \). Other edges are similarly constructed.

In the graph \( G_{AB} \), the five black nodes are chosen from different rows and different columns in each \( M_i \) part, respectively, because the subgraph induced by these five nodes has the maximum edge number of 3 (dashed lines in Fig. 1.2).

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**Fig. 1.1.** A letter-preserving bijective mapping \( \pi \) for two strings: \( A = \text{abcab}, B = \text{ababc} \).

**Fig. 1.2.** Two strings \( A = \text{abcab} \) and \( B = \text{ababc} \) are transformed into a graph \( G_{AB} \).

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**Table 1**

The approximation ratio summary for \( k \)-MCSP.

<table>
<thead>
<tr>
<th>Paper</th>
<th>2-MCSP</th>
<th>3-MCSP</th>
<th>4-MCSP</th>
<th>( k )-MCSP</th>
</tr>
</thead>
<tbody>
<tr>
<td>[8]</td>
<td>1.5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>[12]</td>
<td>1.1037</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>[6]</td>
<td>3</td>
<td></td>
<td>( \Omega(\log n) )</td>
<td>( O(n^{0.69}) )</td>
</tr>
<tr>
<td>[18]</td>
<td></td>
<td></td>
<td></td>
<td>( O(k^4) )</td>
</tr>
<tr>
<td>[19]</td>
<td></td>
<td></td>
<td></td>
<td>4k</td>
</tr>
</tbody>
</table>

\( ^a \) It is a lower bound.
From the chosen five black nodes, we can get the letter-preserving bijective mapping $\pi$ with the maximum number of duo-preservations in Fig. 11.

In order to study approximation algorithms for the MPSM problem, we introduce the following graph optimization problem.

**Problem 2.1 (CMIS and CNIS).** Given an $m$-partite graph $G$ with $m$ parts: $M_1, \ldots, M_m$, where each $M_i$ has $n_i \times n_i$ vertices and all the vertices are put in an $n_i \times n_i$ matrix, the goal of the constrained maximum induced subgraph problem (CMIS) is to find $n_i$ vertices from each part $M_i$, where $n_i$ vertices are from different rows and different columns, such that the induced subgraph has the maximum number of edges. If all $n_i \leq k$, the restricted version is called $k$-CMIS. On the other hand, we define the minimization version as the constrained minimum induced subgraph (CNIS) problem in an $m$-partite graph.

The contributions of the paper are as follows:

1. We prove that CMIS and CNIS are NP-complete;
2. Based on the randomized rounding technology, we propose an approximation algorithm for $k$-CMIS with approximation ratio $2$;
3. We propose an approximation algorithm for $k$-CMIS ($k \geq 3$) with approximation ratio $k^2$; and
4. We give a polynomial time reduction from the MPSM problem to the CMIS problem. Based on approximation algorithms for $k$-CMIS, we get approximation algorithms for $k$-MPSM with the same approximation ratio;
5. We give a polynomial time reduction from the minimum common string partition problem (MCSP) to the CNIS problem.

Note. Without loss of generality, in this paper we assume that the strings do not contain two consecutive occurrences of the same letter. In the case where two consecutive occurrences of the same letter, the reduced graph optimization problems are CMIS and CNIS problems with edges exist inside the same parts. For CMIS and CNIS problems with edges exist inside the same parts, the approximations results in Section 4 remain correct since in the proofs Section 4 don't use the condition $r \neq s$ for $(v_i^0, v_j^0) \in E$ which implies that edges can exist inside the same parts.

2. Preliminaries

In this section, we first reproduce some formal notations and definitions (Definitions 2.1–2.6) about the MCSP problem from [12], then describe the formal problem statement of the MPSM problem. Since our goal is to design a randomized approximation algorithm based on the randomized rounding technology, the definition of approximation ratio of a randomized approximation algorithm is introduced.

**Definition 2.1 (Duo).** A duo is an ordered pair of letters that occur consecutively in a string [12].

**Definition 2.2 (Partition).** A partition of a string $A$ is a sequence of strings $P = (P_1, P_2, \ldots, P_m)$ whose concatenation is equal to $A$, that is $P_1 \cdots P_m = A$, where the strings $P_i \ (1 \leq i \leq m)$ are called the blocks of $P$ and $m$ is called the number of blocks [12].

**Definition 2.3 (Common partition).** Given a partition $P = (P_1, P_2, \ldots, P_m)$ of a string $A = a_1, \ldots, a_n$ and a partition $Q = (Q_1, \ldots, Q_m)$ of a string $B = b_1, \ldots, b_n$, we say that the pair $(P, Q)$ is a common partition $\pi$ of $A$ and $B$ if $Q$ is a permutation of $P$. The common partition $\pi$ can be naturally interpreted as a bijective mapping from $A$ to $B$, such that, for each $j \ (1 \leq j \leq m)$, the letters from $P_j$ are mapped from left to right to the corresponding letters from $Q_j \ (1 \leq j \leq m)$ [12].

**Definition 2.4 (Break).** A break is a pair of letters that are consecutive in string $A$ but are mapped by $\pi$ to letters that are not consecutive in string $B$. Obviously, the block number of a partition is equal to its break number plus $1$ [12].

**Definition 2.5 (Letter-preserving bijective mapping).** A letter-preserving bijective mapping is a bijective mapping $\pi$ from letters of string $A$ to letters of the other string $B$ such that any letter in $A$ is mapped into the same letter in $B$ [12]. Duo-preservation means that a duo of $A$ is kept by $\pi$ in $B$.

**Problem 2.1 (Minimum common string partition problem).** The minimum common string partition problem (MCSP) is to find a common partition of two strings $A$ and $B$ with the minimum number of blocks. MCSP is also viewed as the problem of finding a letter-preserving bijective mapping from letters in one string to letters in the other string with the minimum number of breaks. The restricted version of MCSP, where each letter occurs at most $k$ times in each input string, is denoted by $k$-MCSP [12].
Definition 2.6 (Related strings). Two strings $A$ and $B$ are related if every letter appears the same number of times in $A$ and $B$ [12].

Obviously, two strings have a common partition iff they are related [12].

Lemma 2.7. Two strings $A$ and $B$ have a letter-preserving bijective mapping iff they are related.

Proof. From the definition of the letter-preserving bijective mapping, there is a one-to-one correspondence between letters in $A$ and $B$. Thus, every letter appears the same number of times in $A$ and $B$. Hence, $A$ and $B$ are related.

Problem 2.2 (MPSM). The maximum duo-preservation string mapping problem (MPSM) is the problem of finding a letter-preserving bijective mapping $\pi$ from string $A$ to string $B$ with the maximum number of duo-preservations, where $A$ and $B$ have the same length. The restricted version of MPSM, where each letter occurs at most $k$ times in each input string, is denoted by $k$-MPSM.

Note that two input strings are related in the MPSM problem.

Theorem 2.8. MPSM and MCSP are complementary.

Proof. Suppose $\pi$ is the letter-preserving bijective mapping between two strings $A$ and $B$ in MPSM and MCSP. Then, the number of duos of $A$ is $n-1$. Let $n_b$ be the number of breaks and $n_d$ be the number of duo-preservations. Thus, $n_b + n_d = n - 1$. So, the MPSM problem is complementary to the MCSP problem.

In the following, we give an example.

In Fig. 11, the letter-preserving bijective mapping $\pi'$ has a break: $ca$ is a duo, but $\pi'(c)\pi'(a)$ is not a duo. Thus, the number of blocks is two. However, for other three duos: $ab$, $bc$, $ab$, they are kept by $\pi'$. So, the sum of the number of blocks and the number of pairs of consecutive letters that are kept by $\pi'$ is five, which is the length of one input strings.

Definition 2.9 (Approximation ratio). Let $OPT(I)$ denote the optimum solution for an instance $I$ of the maximization problem. Let $R(I)$ denote the expected value of the output solution $R(I)$ of a randomized approximation algorithm $R$. For some $r \geq 1$, if $OPT(I) \leq r$, for any instance $I$, then the randomized approximation algorithm $R$ is called the algorithm of approximation ratio of $r$ [16].

3. The CMIS and CNIS problem are NP-complete

In this section, we prove that the CMIS and CNIS problems are NP-complete. We give a polynomial time reduction from the MPSM problem to the CMIS problem.

Lemma 3.1. There exist a polynomial time reduction from the MPSM problem to the CMIS problem.

Proof. Given two related strings $X = x_1x_2\ldots x_n$ and $Y = y_1y_2\ldots y_n$, let $m$ be the number of different letters in $X$ and $S = \{a_1, \ldots, a_m\}$ be the unduplicated letter set. Let $n_1$ be the number of appearances of letter $a_1$ in $X$. Thus, $n_1 + \cdots + n_m = n$.

In the following, we construct an instance $G_{XY}$ of CMIS, which has $m$ parts. For each $a_i$, we construct one part $M_i$, which has $n_i \times n_i$ nodes. Let $(a_1^{11}, a_1^{12}, \ldots, a_1^{1n_i})$ be all $a_1$s in $X$ by their appearance order. Let $(a_i^{11}, a_i^{22}, \ldots, a_i^{2n_i})$ be all $a_i$s in $Y$ by their appearance order. For each $a_i^{1h}$ and $a_i^{2\ell}$ ($1 \leq h, \ell \leq n_i$), we construct one node $(a_i^{1h}, a_i^{2\ell})$ in the $h$-th row and $\ell$-th column of $M_i$. Edges only exist between nodes from different parts. There is an edge between $(a_i^{1h}, a_i^{2\ell})$ and $(a_j^{1r}, a_j^{2s})$ iff $a_i^{1h}a_j^{1r}$ is a duo in $X$ and $a_i^{2\ell}a_j^{2s}$ is a duo in $Y$. For the graph $G_{XY}$, the goal of the CMIS problem is to find $n_i$ vertices at different rows and different columns from each part $M_i$ such that the subgraph induced by the chosen $n_i$ nodes $(n = n_1 + \cdots + n_m)$ has the maximum number of edges.

The number of vertices in $G_{XY}$ is $n_1^2 + n_2^2 + \cdots + n_m^2$ which is $O(n^2)$. Thus, the number of edges in $G_{XY}$ is at most $O(n^4)$. So, the reduction is of polynomial time complexity.

Fig. 12 is a reduction example. The following Theorem 3.2 shows that the MPSM problem for strings $X$ and $Y$ is related to the CMIS problem in the graph $G_{XY}$.

Theorem 3.2. For the graph $G_{XY}$, an induced subgraph by $n$ nodes, of which $n_i$ nodes are chosen from different rows and different columns in each $M_i$ part respectively, has the maximum number of edges iff $X$ and $Y$ have a bijective mapping with the maximum number of duo-preservations.
Fig. 3.3. Two strings $A = \text{abcab}$ and $B = \text{ababc}$ are transformed into a graph $\tilde{G}_{AB}$.

**Proof.** Since each node in the graph $G_{XY}$ denotes that a letter in $X$ is mapped to a letter in $Y$, these $n$ nodes denote a bijective mapping $\pi$ from $X$ to $Y$. Since each edge in $G_{XY}$ denotes a duo-preservation, the number of edges in the induced subgraph by these $n$ nodes is the number of duo-preservations in the bijective mapping $\pi$. Thus, the maximum number of edges in the induced subgraph by these $n$ nodes is the maximum number of duo-preservations in a bijective mapping. The other direction of the theorem is trivial. \square

Thus, by Theorem 3.2 an approximation algorithm for the MPSM problem can be achieved by designing an approximation algorithm for the CMIS problem with the same approximation ratio. If each letter in $X$ and $Y$ appears at most $k$ times, then each part in $G_{XY}$ has at most $k \cdot k$ nodes by the above reduction process. Thus, the $k$-MPSM problem can be reduced to the $k$-CMIS problem with the same approximation ratio.

On the other hand, because the MPSM is NP-hard and it is obvious that CMIS is in NP, we get the following conclusion.

**Theorem 3.3.** The CMIS problem is NP-complete.

The CMIS problem and the CNIS problem are complementary, we get the following conclusion.

**Theorem 3.4.** The CNIS problem is NP-complete.

It is easy to know that if we modify the reduction process of Lemma 3.1, we can get a reduction from the MCSP problem to the constrained minimum induced subgraph (CNIS) problem. In the reduction process of Lemma 3.1, we construct the graph $G_{XY}$, which has the same vertices of $G_{XY}$, but the construction of edges is modified as follows: there is an edge between $(a_1^{i_1}, a_2^{j_1})$ and $(a_1^{i_2}, a_2^{j_2})$ iff $a_1^{i_1}a_1^{i_2}$ is a duo in $X$ and $a_2^{j_1}a_2^{j_2}$ is not a duo in $Y$. Then, the modified reduction is a reduction from the MCSP problem to the constrained minimum induced subgraph (CNIS) problem. Thus, we get the following conclusion.

**Lemma 3.5.** There exists a polynomial time reduction from the MCSP problem to the constrained minimum induced subgraph (CNIS) problem.

For example, for two strings $A = \text{abcab}$ and $B = \text{ababc}$, we can construct a graph $\tilde{G}_{AB}$ as follows (see Fig. 3.3).

In the graph $\tilde{G}_{AB}$, the subgraph induced by the five black nodes has the minimum edge number of 1 (dashed line in Fig. 3.3).

The following conclusion shows that the MCSP problem for strings $X$ and $Y$ is related to the CNIS problem in the graph $\tilde{G}_{XY}$.

**Theorem 3.6.** For the graph $\tilde{G}_{XY}$, a subgraph induced by $n$ nodes, of which $n_i$ nodes are chosen from different rows and different columns in each $M_i$ part respectively, has the minimum number of edges iff $X$ and $Y$ have a bijective mapping with the minimum number of breaks.

**Proof.** Since each node in the graph $\tilde{G}_{XY}$ denotes that a letter in $X$ is mapped to a letter in $Y$, these $n$ nodes denote a bijective mapping $\pi$ from $X$ to $Y$. Since each edge in $\tilde{G}_{XY}$ must produce a break, the number of edges in the induced subgraph by $n$ nodes is the number of breaks in the bijective mapping $\pi$. Thus, the minimum number of edges in the subgraph induced by $n$ nodes is the minimum number of breaks in a bijective mapping. The other direction of the theorem is trivial. \square
4. Approximation algorithms for $k$-CMIS ($k \geq 2$)

In order to study approximation algorithms, Raghavan and Thompson introduced a randomized rounding method in [21]. Since then, the randomized rounding method has been widely used to design approximation algorithms for many NP-hard problems ([5,15,22,20,13,14,16,17], etc.). The general idea behind the randomized rounding method is: (1) An NP-hard problem is first transformed into a 0–1 Integer Programming (IP) problem. (2) Then, it is relaxed to a Linear Programming (LP) problem. (3) For the optimal solution to LP, the value of each variable is rounded to 0 or 1 by some specific method. Thus, one approximation solution to some specific NP-hard problem can be achieved if the challenges of steps (1) and (3) can be overcome.

In this section, we will design approximation algorithms for the CMIS problem based on the Linear Programming (LP) technology.

First, we give the 0–1 Integer Programming (IP) formulation for the CMIS problem. Suppose $G$ is an $m$-partite graph with $m$ parts: $M_1, \ldots, M_m$, where the vertices in each $M_i$ are put in $n_i \times n_i$ matrix. For each node $v_i^p$ ($1 \leq i \leq n_r, 1 \leq p \leq n_r$) in $M_r$ ($1 \leq r \leq m$), where $v_i^p$ is at the $i$-th row and $p$-th column, let $x_i^p$ be a 0–1 decision variable, that $x_i^p = 1$ means that $v_i^p$ is chosen, otherwise $v_i^p$ is not chosen. Thus, the 0–1 IP formulation for the CMIS problem is as follows:

Maximize $\sum_{(y_i^p, y_j^q) \in E} x_i^p x_j^q$ \hspace{1cm} (IP$_1$)

subject to $\sum_{i=1}^{m} x_i^p = 1$, for $r = 1, \ldots, m$ \hspace{1cm} (1)

$\sum_{p=1}^{m} x_i^p = 1$, for $r = 1, \ldots, m$ \hspace{1cm} (2)

$x_i^p \in \{0, 1\}$, for $r = 1, \ldots, m$

Constraints (1) and (2) guarantee that only the nodes at different rows and different columns from each part are chosen. For example, for the graph $G_{AB}$ in Fig. 1.2, we have the following 0–1 IP formulation:

Maximize $A_{11} B_{11} + A_{12} B_{12} + B_{12} C + A_{21} B_{21} + A_{22} B_{22}$

subject to $A_{11} + A_{12} = 1$; $A_{21} + A_{22} = 1$;

$A_{11} + A_{21} = 1$; $A_{12} + A_{22} = 1$;

$B_{11} + B_{12} = 1$; $B_{21} + B_{22} = 1$;

$B_{11} + B_{21} = 1$; $B_{12} + B_{22} = 1$; $C = 1$;

$A_{ij}, B_{ij} \in \{0, 1\}$, where $i = 1$ or 2, $j = 1$ or 2

Where $A_{ij}$ ($1 \leq i, j \leq 2$) is the decision variable corresponding to the node at the $i$-th row and $j$-th column in $M_1$, $B_{ij}$ ($1 \leq i, j \leq 2$) is the decision variable corresponding to the node at the $i$-th row and $j$-th column in $M_2$, and $C$ is the decision variable corresponding to the node in $M_3$.

Using the common relaxation method for IP formulation [16], we build the following LP formulation for the CMIS problem:

Maximize $\sum_{(y_i^p, y_j^q) \in E} z_{i p, j q}$ \hspace{1cm} (LP$_1$)

subject to $z_{i p, j q} \leq x_i^p$, for all $r$ \hspace{1cm} (3)

$z_{i p, j q} \leq x_j^q$, for all $r$ \hspace{1cm} (4)

$\sum_{i=1}^{m} x_i^p = 1$, for $r = 1, \ldots, m$ \hspace{1cm} (3')

$\sum_{p=1}^{m} x_i^p = 1$, for $r = 1, \ldots, m$ \hspace{1cm} (4')

$0 \leq z_{i p, j q} \leq 1$, for all $r$ \hspace{1cm} (5)

$0 \leq x_i^p \leq 1$, for all $r$ \hspace{1cm} (5')

$0 \leq x_j^q \leq 1$, for all $r$
For example, the LP formulation for the graph $G_{AB}$ in Fig. 1.2 is as follows:

Maximize $z_1 + z_2 + z_3 + z_4 + z_5$
subject to
$z_1 \leq A_{11}$, \quad $z_1 \leq B_{11}$;
$z_2 \leq A_{12}$, \quad $z_2 \leq B_{12}$;
$z_3 \leq A_{21}$, \quad $z_3 \leq C$;
$z_4 \leq A_{22}$, \quad $z_4 \leq B_{21}$;
$z_5 \leq A_{32}$, \quad $z_5 \leq B_{22}$;
$A_{11} + A_{12} = 1$; \quad $A_{21} + A_{22} = 1$;
$A_{12} + A_{22} = 1$; \quad $A_{12} + A_{22} = 1$;
$B_{11} + B_{12} = 1$; \quad $B_{21} + B_{22} = 1$;
$B_{12} + B_{22} = 1$; \quad $B_{12} + B_{22} = 1$;
$C = 1$;
$A_{ij}, B_{ij}, z_r \in [0, 1]$ for all $i, j, r$

First, we design the approximation algorithm for the 2-CMIS problem based on its LP formulation (see Algorithm 4.1):

1: By solving the Linear Programming formulation $LP_1$ for the 2-CMIS problem, we get an optimum solution $x_{i_medium}^p$, $z_{i_medium}$ for all $r$.
2: Randomized rounding:
3: for $r = 1$ to $m$ do
4: \hspace{1em} if $n_r = 2$ then
5: \hspace{2em} Let $X_i = (x_i^{11}, x_i^{22})$. Let $Y^1_i = (x_i^{11}, x_i^{21})$ and $Y^2_i = (x_i^{22}, x_i^{21})$
6: \hspace{1em} $Y^1_i$ is chosen with probability $\frac{\sqrt{q}}{\sqrt{2q} + \sqrt{2q}}$ (i = 1, 2).
7: \hspace{2em} When $Y^1_i$ is chosen, we set $x_i^{11} = x_i^{22} = 1$ and $x_i^{12} = x_i^{21} = 0$; When $Y^2_i$ is chosen, we set $x_i^{11} = x_i^{12} = 1$ and $x_i^{21} = x_i^{22} = 0$.
8: \hspace{2em} end if
9: \hspace{1em} if $n_r = 1$ then
10: \hspace{2em} We set $x_i^{11} = 1$.
11: \hspace{2em} end if
12: end for
13: Output those nodes whose variables are set to 1. Let $z_{i_medium}^{p} = x_i^{p} x_i^{q}$. The edge number of induced subgraph by these output nodes is $\sum_{(i, j) \in E} z_{i, j}^{p} z_{i, j}^{q}$.

Algorithm 4.1: The approximation algorithm for the 2-CMIS problem.

It is known that Linear Programming can be solved in a polynomial time (see [7]). Thus, Algorithm 4.1 is of polynomial time complexity.

**Theorem 4.1.** Algorithm 4.1 is of expected approximation ratio 2 for 2-CMIS.

**Proof.** Let $P_r(X)$ denote the probability of the event $X$. Let $\tilde{E}(X)$ denote the expected value of the event $X$. For each $r$, let $A_{ij}$ denote the event that $Y_j$ is chosen ($j = 1, 2$).

By the constraints (3) and (4), we get $z_{i_medium}^{p} = \min\{x_i^{p}, x_i^{q}\}$. By the constraints (3') and (4'), we get $x_i^{11} + x_i^{22} = 1$, $x_i^{11} + x_i^{21} = 1$ and $x_i^{12} + x_i^{22} = 1$. So, we get $x_i^{11} = x_i^{22}$ and $x_i^{12} = x_i^{21}$.

Thus, for any $i, p$ (i, $p = 1$ or 2), when $i = p$, $P_r(x_i^{p} = 1) = P_r(x_i^{11} = 1, x_i^{22} = 1) = \frac{\sqrt{q}}{\sqrt{2q} + \sqrt{2q}} = \frac{\sqrt{q}}{\sqrt{2q} + \sqrt{2q}}$; when $i \neq p$, $P_r(x_i^{p} = 1) = P_r(A_{22}) = \frac{\sqrt{q}}{\sqrt{2q} + \sqrt{2q}} = \frac{\sqrt{q}}{\sqrt{2q} + \sqrt{2q}}$.

Since $\frac{\sqrt{q}}{\sqrt{2q} + \sqrt{2q}} \leq \sqrt{\frac{x_i^{11} + x_i^{22}}{2}} \leq \frac{1}{\sqrt{2}}$, we get $\sqrt{x_i^{11} + \sqrt{2q}} \leq \sqrt{q}$. Thus $P_r(x_i^{p} = 1) \geq \frac{1}{\sqrt{2}} \sqrt{x_i^{p}}$. Similarly, we can get $P_r(x_i^{q} = 1) \geq \frac{1}{\sqrt{2}} \sqrt{x_i^{q}}$.

So $P_r(z_{i_medium}^{p} = 1) = P_r(x_i^{p} x_i^{q} = 1) = P_r(x_i^{p} = 1)P_r(x_i^{q} = 1) \geq \frac{1}{\sqrt{2}} \sqrt{x_i^{p}} \sqrt{x_i^{q}}$. 

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can be achieved with the same approximation ratio as Algorithm 4.1. Thus, we can get the following conclusion.

**Corollary 4.2.**

Let \( OPT \) denote the optimum solution. Let \( OPT(LP_1) \) denote the optimum solution of \( IP_1 \) formulation for \( I \). Let \( OPT(LP_1) \) denote the optimum solution of \( LP_1 \) formulation for \( I \).

So \( \tilde{E}(A(I)) = \tilde{E}\left( \sum_{(v^p, v^q) \in E} z'_{ip \cdot jq} \right) = \sum_{(v^p, v^q) \in E} \tilde{E}(z'_{ip \cdot jq}) = \sum_{(v^p, v^q) \in E} Pr(z'_{ip \cdot jq} = 1) \geq \frac{1}{2} \sum_{(v^p, v^q) \in E} z'_{ip \cdot jq} = \frac{1}{2} OPT(LP_1). \)

Thus \( \frac{OPT(I)}{E(A(I))} = \frac{OPT(LP_1)}{E(A(I))} \leq \frac{OPT(LP_1)}{OPT(LP_1)} = \frac{1}{2}. \)

Hence, Algorithm 4.1 is of approximation ratio 2 for 2-CMIS.

Since the 2-MPSM problem can be reduced to 2-CMIS problem, an approximation algorithm for the 2-MPSM problem can be achieved with the same approximation ratio as Algorithm 4.1. Thus, we can get the following conclusion.

**Corollary 4.2.** There is an approximation algorithm with expected approximation ratio 2 for 2-MPSM.

Second, we design the approximation algorithm for the 3-CMIS problem based on its \( LP_1 \) formulation (see Algorithm 4.2):

1: By solving the Linear Programming formulation \( LP_1 \) for the 3-CMIS problem, we get an optimum solution \( x^{ip \cdot jq} \) for all \( r \).
2: Randomized rounding:
3: for \( r = 1 \) to \( m \) do
4: if \( n_r = 3 \) then
5: Let \( X_r = \begin{pmatrix} x_{11}^{1r} & x_{12}^{1r} & x_{21}^{1r} & x_{22}^{1r} & x_{31}^{1r} & x_{32}^{1r} & x_{33}^{1r} \\ x_{11}^{2r} & x_{12}^{2r} & x_{21}^{2r} & x_{22}^{2r} & x_{31}^{2r} & x_{32}^{2r} & x_{33}^{2r} \end{pmatrix} \).
6: Let \( \bar{F}_1 = (x_{11}^{1r}, x_{12}^{2r}, x_{21}^{3r}) \) and \( \bar{F}_2 = (x_{12}^{1r}, x_{22}^{2r}, x_{31}^{3r}) \) and \( \bar{F}_3 = (x_{13}^{1r}, x_{23}^{2r}, x_{33}^{3r}) \) and \( S_1 = x_{11}^{1r} + x_{12}^{2r} + x_{21}^{3r} \) and \( S_2 = x_{12}^{1r} + x_{22}^{2r} + x_{31}^{3r} \) and \( S_3 = x_{13}^{1r} + x_{23}^{2r} + x_{33}^{3r} \).
7: \( \bar{F}_1 \) is chosen with probability \( \frac{1}{\sqrt{x_{11}^{1r} + x_{12}^{2r} + x_{21}^{3r}}} (i = 1, 2, 3). \)
8: \( \bar{F}_2 \) is chosen with probability \( \frac{1}{\sqrt{x_{12}^{1r} + x_{22}^{2r} + x_{31}^{3r}}} (i = 1, 2, 3). \)
9: When \( \bar{F}_1 \) is chosen, we set \( x_{11}^{1r} = x_{12}^{2r} = x_{21}^{3r} = 1 \); When \( \bar{F}_2 \) is chosen, we set \( x_{12}^{1r} = x_{22}^{2r} = x_{31}^{3r} = 1 \); When \( \bar{F}_3 \) is chosen, we set \( x_{13}^{1r} = x_{23}^{2r} = x_{33}^{3r} = 1 \);
10: end if
11: if \( n_r = 2 \) then
12: We set \( x_r^{ip} = 1 \) by the method in Algorithm 4.1.
13: end if
14: if \( n_r = 1 \) then
15: We set \( x_r^{ip} = 1 \).
16: end if
17: end for
18: Output those nodes whose variables are set to 1. Let \( z_{ip \cdot jq} = x_{ip}^{jq}. \) The edge number of induced subgraph by these output nodes is \( \sum_{(v^p, v^q) \in E} z'_{ip \cdot jq}. \)

**Algorithm 4.2:** The approximation algorithm for the 3-CMIS problem.

It is known that Linear Programming can be solved in a polynomial time (see [7]). Thus, Algorithm 4.2 is of polynomial time complexity.
Theorem 4.3. Algorithm 4.2 is of expected approximation ratio 9 for 3-CMIS.

Proof. Let \( Pr(X) \) denote the probability of the event \( X \). Let \( \tilde{E}(X) \) denote the expected value of the event \( X \). For each \( r \), let \( A_j \) denote the event that \( Y_j \) is chosen (\( j = 1, 2, 3 \)).

By the constraints (3) and (4), we get \( z_{i,p}^{s_j} = \min[x_i^{s_j}, x_j^{s_j}] \). By the constraints (3') and (4'), we get \( S_1 + S_2 + S_3 = x_1^{11s} + x_1^{12s} + x_1^{13s} + x_2^{21s} + x_2^{22s} + x_2^{31s} + x_2^{32s} + x_3^{33s} = 3 \).

When \( n_r = 3 \), for any \( i, p \), if \( x_i^{j,p} \in Y_j \), then \( Pr(x_i^{j,p} = 1) = Pr(A_j) = \frac{\sqrt{S_j}}{\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}} \geq \frac{\sqrt{S_j}}{\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}} \leq \frac{\sqrt{S_1} + \sqrt{S_2} + \sqrt{S_3}}{3} = 1 \), we get \( 3 \). Thus, \( Pr(x_i^{j,p} = 1) \geq 3 \). When \( n_r = 2 \), by the proof of Theorem 4.1, for any \( i, p \), we have \( Pr(x_i^{j,p} = 1) \geq \frac{1}{2} \sqrt{x_i^{j,p}} \). Thus, in any case, we get \( Pr(x_i^{j,p} = 1) \geq \frac{1}{2} \sqrt{x_i^{j,p}} \).

Similarly, we can get \( Pr(x_i^{j,p} = 1) \geq \frac{1}{3} \sqrt{x_i^{j,p}} \).

So \( Pr(z_{i,p}^{s_j} = 1) = Pr(x_i^{j,p}, x_j^{s_j} = 1) \)

\[ = \frac{1}{9} \sqrt{x_i^{j,p}} \sqrt{x_j^{s_j}} \]

\[ \geq \frac{1}{9} \min[x_i^{j,p}, x_j^{s_j}] \]

\[ \geq \frac{1}{9} z_{i,p}^{s_j} \cdot \]

For any instance \( I \), let \( A(I) \) denote the output solution of the approximation algorithm. Let \( OPT(I) \) denote the optimum solution. Let \( OPT(LP_1) \) denote the optimum solution of \( LP_1 \) formulation for \( I \). Let \( OPT(LP_1) \) denote the optimum solution of \( LP_1 \) formulation for \( I \).

So \( \tilde{E}(A(I)) = \tilde{E} \left( \sum_{(v_i^p, v_j^p) \in E} z_{i,p}^{s_j} \right) \)

\[ = \sum_{(v_i^p, v_j^p) \in E} \tilde{E}(z_{i,p}^{s_j}) \]

\[ = \sum_{(v_i^p, v_j^p) \in E} Pr(z_{i,p}^{s_j} = 1) \]

\[ \geq \frac{1}{9} \sum_{(v_i^p, v_j^p) \in E} z_{i,p}^{s_j} \]

\[ \geq \frac{1}{9} \cdot OPT(LP_1) \].

Thus \( \frac{OPT(LP_1)}{E(A(I))} \leq \frac{OPT(LP_1)}{E(A(I))} \leq 9 \).

Hence, Algorithm 4.2 is of approximation ratio 9 for 3-CMIS. \( \square \)

Since the 3-MPSM problem can be reduced to 3-CMIS problem, an approximation algorithm for the 3-MPSM problem can be achieved with the same approximation ratio as Algorithm 4.2. Thus, we can get the following conclusion.

Corollary 4.4. There is an approximation algorithm with expected approximation ratio 9 for 3-MPSM.

Similar to Algorithm 4.2, we can design an approximation algorithm for \( k \)-CMIS with approximation ratio \( k^2 \). The algorithm details can be found in Algorithm 4.3.

Thus, using the similar proof method of Theorem 4.3, it is easy to show the following conclusion (in the proof of Theorem 4.3, 3 and 9 are replaced with \( k \) and \( k^2 \) respectively).

Theorem 4.5. Algorithm 4.3 is of expected approximation ratio \( k^2 \) for \( k \)-CMIS (\( k \geq 4 \)).
Algorithm 4.3: The approximation algorithm for the k-CMIS problem.

**Note.** In Algorithm 4.3, the elements of $\overline{Y}_i (1 \leq i \leq k)$ are at different rows and different columns. There are $k!$ possible these vectors. We choose $k$ diagonal elements as these $\overline{Y}_i$. In order to improved the approximation ratio $k^2$, it is required to use better methods for choosing these $\overline{Y}_i$. We leave it as an open problem.

Since the $k$-MPSM problem can be reduced to $k$-CMIS problem, an approximation algorithm for the $k$-MPSM problem can be achieved with the same approximation ratio as Algorithm 4.3. Thus, we can get the following conclusion.

**Corollary 4.6.** There is an approximation algorithm with expected approximation ratio $k^2$ for $k$-MPSM ($k \geq 4$).

5. Conclusion

In this paper, we have proved that CMIS and CNIS are NP-complete. We have also proposed a 2-approximation algorithm for 2-CMIS and a $k^2$-approximation algorithm for k-CMIS ($k \geq 3$), which are based on the randomized rounding technology. Based on approximation algorithms for k-CMIS, we get approximation algorithm for k-MPSM with the same approximation ratio.

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