Quantized $\mathcal{H}_\infty$ filter design of interconnected continuous-time delay systems

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SUMMARY

A generalized approach to the problem of quantized filtering is investigated by designing a set of decentralized $\mathcal{H}_\infty$ filters for a class of linear interconnected continuous-time systems with unknown-but-bounded couplings and interval delays and where the quantizer has arbitrary form. An LMI-based method using a decentralized quantized filter is designed at the subsystem level to render the global filtered system delay-dependent asymptotically stable with guaranteed $\gamma$-level. It is established that this setting encompasses several special cases of interest including interconnected delay-free systems, single time-delay systems and single systems. We illustrate the theoretical developments by numerical simulations. Copyright © 2012 John Wiley & Sons, Ltd.

1. INTRODUCTION

Quantization in control systems has been an active research topic in recent years [1, 2]. Control problems under different types of quantizations in both, linear and nonlinear cases have been examined. The need for quantization arises when digital networks are part of the feedback loop and this eventually gives rise to packet dropouts or data transfer rate limitations [3]. On the other hand, signal processing and signal quantization always exist in computer-based control systems [4], and therefore, recent research studies have been reported on the analysis and design problems for control systems involving various quantization methods, see [5–10] and the references cited therein.

Recently in [1], a study of quantized and delayed state-feedback control systems under constant bounds on the quantization error and the time-varying delay was reported. In [5], a quantizer taking value in a finite set is defined, and then, quantized feedback stabilization for linear systems is considered. In [6], the problem of stabilizing an unstable linear system by means of quantized state feedback, where the quantizer takes value in a countable set, is addressed. It should be noted that the approach in [5] relies on the possibility of making discrete on line adjustments of quantizer parameters, which were extended in [8] for more general nonlinear systems with general types of quantizers involving the states of the system, the measured outputs, and the control inputs. On the basis of [8], stabilization of discrete-time LTI systems with quantized measurement outputs is reported in [11]. Further related results are reported in [9, 10]. On another research front, decentralized stability and feedback stabilization of interconnected systems have been the topic of recurring interests, and recent relevant results have been reported in [12–17]. Filters are the most essential...
building blocks of signals processing, and the problem of filtering with quantized signals has been considered in [18–21]. In addition, results on quantization pertaining to the present work can be found in [22, 23].

In this paper, building on the pioneering work reported in [5, 8], the approach that it is possible to vary some parameters of the quantizer in real time, on the basis of collected data for a class of interconnected continuous-time systems, is adopted. An improved technique is developed on the problem of quantized feedback stabilization from a generalized setting for a class of linear interconnected continuous-time systems with unknown-but-bounded couplings and interval delays. This is attained by designing a decentralized quantized \( H_\infty \) output-feedback control wherein the quantizer has arbitrary form that satisfies a quadratic inequality constraint, the introduction of which avoids the difficulty of ‘zooming-in’ and ‘zooming-out’ procedure in [8]. An LMI-based method is designed at the subsystem level to render the closed-loop system delay-dependent asymptotically stable with guaranteed \( H_\infty \) level. It is established that this setting encompasses several special cases of interest including interconnected delay-free systems, single time-delay systems and single systems. In this way, the potential applicability of the hybrid quantized feedback control techniques is expanded to classes of time-delay systems. It must be emphasized that several types of quantizers can satisfy a quadratic inequality constraint, and in turn, this paved the way to construct control laws explicitly. The theoretical developments are illustrated by numerical simulations.

**Notations:** In the sequel, the Euclidean norm \( \| \cdot \| \) is used for vectors in the \( n \)-dimensional vector space \( \mathbb{R}^n \) and we denote by \( || || \) the corresponding induced matrix norm in \( \mathbb{R}^{n \times n} \). We use \( W^T \) and \( W^{-1} \) to denote the transpose and the inverse of any square matrix \( W \), respectively. We use \( W > 0 \) to denote a symmetric positive definite (negative semidefinite matrix \( W \) and \( I_j \) to denote the \( n_j \times n_j \) identity matrix. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. In symmetric block matrices or complex matrix expressions, we use the symbol \( \bullet \) to represent a term that is induced by symmetry.

**Lemma 1.1 (The S Procedure [24])**
Denote the set \( Z = \{ z \} \) and let \( \mathcal{F}(z), \mathcal{V}_1(z), \mathcal{V}_2(z), \ldots, \mathcal{V}_k(z) \) be some functionals or functions. Define domain \( D \) as

\[
D = \{ z \in Z : \mathcal{V}_1(z) \geq 0, \mathcal{V}_2(z) \geq 0, \ldots, \mathcal{V}_k(z) \geq 0 \}
\]

and the two following conditions:

(i) \( \mathcal{F}(z) > 0, \forall z \in D \),
(ii) \( \exists \varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \ldots, \varepsilon_k \geq 0 \) such that \( \mathcal{S}(\varepsilon, z) = \mathcal{F}(z) - \sum_{j=1}^{k} \varepsilon_j \mathcal{V}_j(z) > 0 \forall z \in Z \)

Then (ii) implies (i).

**Lemma 1.2 (The Integral Inequality [25])**
For any constant matrix \( 0 < \Sigma \in \mathbb{R}^{n \times n} \), scalar \( \tau_* < \tau(t) < \tau^+ \) and vector function \( \dot{x} : [t - \tau^+, t - \tau_*] \to \mathbb{R}^n \) such that the following integration is well-defined, then it holds that

\[
-(\tau^+ - \tau_*) \int_{t-\tau^+}^{t-\tau_*} \dot{x}^T(s) \Sigma \dot{x}(s) \, ds \leq -[x(t - \tau_*) - x(t - \tau^+)]^T \Sigma [x(t - \tau_*) - x(t - \tau^+)]
\]

Lemma 1.2 is frequently called the ‘integral inequality’ and it is derived from Jensen’s inequality [25]. Sometimes, the arguments of a function are omitted when no confusion arises.
2. PROBLEM STATEMENT

Consider a nonlinear interconnected time-delay system composed of a finite number $N$ of coupled subsystems represented by:

\[
\dot{x}(t) = Ax(t) + A_d x(t - \tau) + c(t, x(t)) + \Gamma w(t) x
\]

\[
y(t) = C x(t) + C_d x(t - \tau)
\]

\[
z(t) = G x(t) + G_d x(t - \tau)
\]

where $x = [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^n$, $y = [y_1^T, \ldots, y_N^T]^T \in \mathbb{R}^m$, $m = \sum_{j=1}^N m_j$ is the measured output of the overall system, $z = [z_1^T, \ldots, z_N^T]^T \in \mathbb{R}^p$, $p = \sum_{j=1}^N p_j$ is the vector of state combination to be estimated and $w = [w_1^T, \ldots, w_N^T]^T \in \mathbb{R}^p$, $p = \sum_{j=1}^N p_j$ is the disturbance input which belongs to $L_2[0, \infty)$. The model matrices are $A = \text{diag}(A_1, \ldots, A_N)$, $A_j \in \mathbb{R}^{n_j \times n_j}$, $\Gamma = \text{diag}(\Gamma_1, \ldots, \Gamma_N)$, $\Gamma_j \in \mathbb{R}^{p_j \times p_j}$, $C = \text{diag}(C_1, \ldots, C_N)$, $C_j \in \mathbb{R}^{m_j \times n_j}$, $C_d = \text{diag}(C_d_1, \ldots, C_d_N)$, $C_{d_j} \in \mathbb{R}^{m_j \times p_j}$, $G = \text{diag}(G_1, \ldots, G_N)$, $G_j \in \mathbb{R}^{p_j \times m_j}$ and $G_d = \text{diag}(G_d_1, \ldots, G_d_N)$, $G_{d_j} \in \mathbb{R}^{m_j \times p_j}$, which describe the nominal system. The function

\[
c(t, x(t)) = [c_1^T(t, x(t)), \ldots, c_N^T(t, x(t))]
\]

\[
c_j(t) = \sum_{\ell \neq j}^{N} A_{j\ell} x_{\ell}(t) + A_{dj} x_j(t - \tau_j(t))
\]

is a vector function piecewise-continuous in its arguments. In the sequel, we assume that this function is uncertain and the available information is that, in the domains of continuity $G$, it satisfies the global quadratic inequality

\[
c^T(t, x(t)) c(t, x(t)) \leq x^T(t) \tilde{R}^T \Phi^{-1} \tilde{R} x(t) + x^T(t - \tau) \tilde{S}^T \Psi^{-1} \tilde{S} x(t - \tau)
\]

where $\tilde{R} = [\tilde{R}_1, \ldots, \tilde{R}_N]^T$, $\tilde{R}_j \in \mathbb{R}^{r_j \times n_j}$ and $\tilde{S} = [\tilde{S}_1^T, \ldots, \tilde{S}_N^T]^T$, $\tilde{S}_j \in \mathbb{R}^{r_j \times n_j}$ are constant matrices such that $c(t, 0) = 0$ and $x = 0$ is an equilibrium of system (1). With focus on the structural form of system (1), the $j$th subsystem model can be described by

\[
\dot{x}_j(t) = A_j x_j(t) + A_{dj} x_j(t - \tau_j(t)) + c_j(t) + \Gamma_j w_j(t)
\]

\[
y_j(t) = C_j x_j(t) + C_{dj} x_j(t - \tau_j(t))
\]

\[
z_j(t) = G_j x_j(t) + G_{dj} x_j(t - \tau_j(t))
\]

where $x_j(t) \in \mathbb{R}^{n_j}$, $w_j(t) \in \mathbb{R}^{p_j}$, $z_j(t) \in \mathbb{R}^{p_j}$ and $y_j(t) \in \mathbb{R}^{r_j}$ are the subsystem state, disturbance input, linear combination of states and measured output, respectively and $c_j(t)$ is given by (4). The factors $\tau_j$, $j \in \{1, \ldots, N\}$ are unknown time-delay factors satisfying

\[
0 < \phi_j \leq \tau_j(t) \leq \varphi_j, \quad \dot{\tau}_j(t) \leq \eta_j
\]

where the bounds $\varphi_j$, $\hat{\varphi}_j$, $\eta_j$ are known constants to guarantee smooth growth of the state trajectories. Following (5), the function $c_j \in \mathbb{R}^{n_j}$ is a piecewise-continuous vector function in its arguments and consequently the quadratic inequality

\[
c^T(t, x(t)) c_j(t, x(t)) \leq \phi_j^2 x^T(t) \tilde{R}_j^T \Phi^{-1} \tilde{R}_j x(t) + \psi_j^2 x^T(t - \tau) \tilde{S}_j^T \tilde{S} x(t - \tau)
\]

is satisfied, where $\phi_j > 0$, $\psi_j > 0$ are bounding parameters such that $\Phi = \text{diag} \{\phi_1^{-2} I_{r_1}, \ldots, \phi_N^{-2} I_{r_N}\}$ and $\Psi = \text{diag} \{\psi_1^{-2} I_{r_1}, \ldots, \psi_N^{-2} I_{r_N}\}$ where $I_{m_j} \in \mathbb{R}^{m_j \times m_j}$ represents identity matrix. From (5) and (8), it is always possible to find matrices $\Phi$ and $\Psi$ such that

\[
c^T(t, x(t)) c_j(t, x(t)) \leq x^T(t) \Phi^{-1} R x(t) + x^T(t - \tau) \Psi^{-1} S x(t - \tau)
\]
where \( R = \text{diag}\{R_1, \ldots, R_N\} \), \( \Phi = \text{diag}\{\delta_1 I_{r_1}, \ldots, \delta_N I_{r_N}\} \) and \( \delta_j = \phi_j^{-2} \) and \( S = \text{diag}\{S_1, \ldots, S_N\} \), \( \Psi = \text{diag}\{\kappa_1 I_{r_1}, \ldots, \kappa_N I_{r_N}\} \) and \( \kappa_j = \phi_j^{-2} \).

**Remark 2.1**

The class of systems described by (1)–(3) subject to delay-pattern (7) is frequently encountered in modeling several physical systems and engineering applications including large space structures, multi-machine power systems, cold mills, transportation systems, water pollution management, to name a few [26]. It is significant to observe from (8) that the local function \( c_j(\cdot) \) depends on the full state and delayed state vectors \( x(t), x(t-\tau) \), and therefore, inequality (8) for \( j = 1, \ldots, N \) represents a set of coupling relations that has to be manipulated simultaneously to achieve the overall desired objective. In general, the vector \( c(t) = \sum_{j=1}^{N} c_j(t) \) represents the interaction pattern among the subsystems wherein the component vector \( c_j(t) \) depends on the current and delayed states of the form (4). Under the interconnected structural identity

\[
\sum_{j=1}^{N} \sum_{\ell \neq j} A_{j\ell} x_{\ell}(t) + A_{dj\ell} x_{\ell}(t - \tau_{\ell}(t)) = \sum_{j=1}^{N} \sum_{\ell \neq j} A_{\ell j} x_j(t) + A_{d\ell j} x_j(t - \tau_j(t))
\]

Of interest in this paper is the issue of signal quantization, where we think of a quantizer as a device that converts a real-valued signal into a piecewise constant one taking on a finite set of values and wherein it is possible to vary some parameters of the quantizer in real time, on the basis of collected data.

It has been a common practice [27] to rearrange the terms in a convenient way so as to reflect within the \( j \)th-subsystem the appropriate components leading to the bounding inequality (8) with adjustable bounding parameters \( \phi_j, \psi_j \). Note in (1) and (8) that the subsystem delay with local and coupling patterns are emphasized and in numerical simulations, all the subsystems have to be treated simultaneously. An overall feasible solution of system \( S \) is only guaranteed if the feasible solutions of subsystems \( S_j \) are attained. Thus, the rationale behind inequality (8) is to help in inducing decentralized computations.

In the course of filtering and/or control design, it is often considered that the process output is passed directly to the controller/filter. A control input signal/state estimate is generated and in turn passes it directly back to the process. In many applications, it turns out that the interface between the controller/filter and the process features some additional information-processing devices. Of interest herein is the issue of signal quantization.

Our objective in this paper is to address a generalized approach to examine the problem of quantized filtering for a class of linear interconnected continuous-time systems. In this approach, we think of a quantizer as a device that converts a real-valued signal into a piecewise constant one, taking on a finite set of values and wherein it is possible to vary some parameters of the quantizer in real time, on the basis of collected data. We seek to design a decentralized \( \mathcal{H}_{\infty} \) filter for a class of linear interconnected continuous-time systems with unknown-but-bounded couplings and interval delays.

## 3. LOCAL QUANTIZERS

In the sequel, we treat a **quantizer** as a device in the filtering loop that converts a real-valued signal into a piecewise constant one. We adopt the definition of a local (subsystem) quantizer with general form as introduced in [8]. Let \( f_j \in \mathbb{R}^s, \ j = 1, \ldots, N \) be the variable being quantized. A **local quantizer** is defined as a piecewise constant function \( Q_j : \mathbb{R}^s \rightarrow D_j \), where \( D_j \) is a finite subset of \( \mathbb{R}^s \) representing the collection of quantization regions. This leads to a partition of \( \mathbb{R}^s \) into a finite number of quantization regions of the form \( \{ f_j \in \mathbb{R}^s : Q(f_j) = j \}, \ j \in D_j \). These quantization regions are not assumed to have any particular shape. It is crucial to recognize that the subsequent analysis is independent of the structure of the quantizer employed. In the simulation part, we will need a characterization of the function \( Q_j(\cdot) \) including uniform quantizer [5, 8] and static logarithmic quantizer [2].
4. COMPUTING THE FILTER GAINS

In this section, new criteria will be developed for LMI-based characterization of decentralized quantized filtering. To formulate the filtering problem, a set of local filters is considered of the following form:

\[
\dot{\xi}_j(t) = A_{fj} \xi_j(t) + B_{fj} Q_j (y_j - C_j \xi_j(t)) \\
\gamma_j(t) = C_{fj} \xi_j(t)
\]

(10)

where for \( j \in \{1, \ldots, N\} \), \( \xi_j(t) \in \mathbb{R}^{n_j} \) is the filter state vector, \( \gamma_j(t) \in \mathbb{R}^{q_j} \) is the estimate of performance output vector and \( A_{fj}, B_{fj}, C_{fj} \) are constant filter matrices to be designed. Applying local quantized filter (10) to subsystem (6), the following augmented subsystem is obtained:

\[
\dot{\xi}_j(t) = A_{aj} \xi_j(t) + A_{adj} \xi_j(t - \tau_j(t)) + \Gamma_j \gamma_j(t) + B_{fa} \Delta_j(y_j) \\
\sigma_j(t) = G_{aj} \xi_j(t) + G_{adj} \xi_j(t - \tau_j(t))
\]

(11)

where

\[
A_{aj} = \begin{bmatrix} A_{fj} & 0 \\ B_{fj} C_j & A_{fj} \end{bmatrix}, \quad A_{adj} = \begin{bmatrix} A_{adj} \\ B_{fj} C_{adj} \end{bmatrix}
\]

(12)

In the quantized filtering strategy to be developed successively, the prediction error is defined as

\[
\epsilon_j(t) = y_j(t) - C_j \xi_j(t)
\]

(13)

and the local quantization error denoted by

\[
\Delta_j(y_j) = \epsilon_j(t) - Q_j(\epsilon_j(t))
\]

(14)

Based on (5), we have

\[
\bar{\gamma}(t, x(t)) \leq \xi(t) \bar{\xi}(t) + \xi(t - \tau) \bar{\xi}(t - \tau)
\]

(15)

where \( \bar{R} = \text{diag}\{R_1, \ldots, R_N\} \) with \( R_j \in \mathbb{R}_{+} \) and \( \bar{S} = \text{diag}\{S_1, \ldots, S_N\} \) with \( S_j \in \mathbb{R}^{2n_j \times 2n_j} \). In terms of \( \tilde{\xi} = [\xi_1^T, \ldots, \xi_N^T]^T \in \mathbb{R}^{2n}, \sigma = [\sigma_1^T, \ldots, \sigma_N^T]^T \in \mathbb{R}^p \), the global quantized filtered system has the form

\[
\dot{\tilde{\xi}}(t) = A_{a} \tilde{\xi}(t) + A_{a\alpha} \tilde{\xi}(t - \tau(t)) + \Gamma_j \gamma(t) + B_{fa} \Delta_j(y) \\
\sigma(t) = G_a \tilde{\xi}(t) + G_{a\alpha} \tilde{\xi}(t - \tau(t))
\]

(16)

where \( A_{a} = \text{diag}\{A_{a1}, \ldots, A_{aN}\} \), \( A_{a\alpha} = \text{diag}\{A_{a1\alpha}, \ldots, A_{aN\alpha}\} \), \( \Gamma_j = \text{diag}\{\Gamma_{aj}, \ldots, \Gamma_{aj}\} \), \( \gamma_j \in \mathbb{R}^{2n_j \times p_j} \), \( B_{fa} = \text{diag}\{B_{fa1}, \ldots, B_{faN}\} \), \( B_{fa\alpha} = \text{diag}\{B_{fa1\alpha}, \ldots, B_{faN\alpha}\} \), \( \gamma_j \in \mathbb{R}^{2n_j \times m_j} \), \( \gamma_j \in \mathbb{R}^{2n_j \times m_j} \), \( \gamma_j \in \mathbb{R}^{2n_j \times m_j} \), \( \gamma_j \in \mathbb{R}^{2n_j \times m_j} \), \( \gamma_j \in \mathbb{R}^{2n_j \times m_j} \), \( \gamma_j \in \mathbb{R}^{2n_j \times m_j} \). We seek to establish tractable conditions for designing the subsystem filter matrices \( A_{fj}, B_{fj}, C_{fj} \) that guarantee that the global filtered system (16) is delay-dependent asymptotically stable with disturbance attenuation level \( \gamma = \sum_{j=1}^{N} \gamma_j \) for all \( c(t, x(t)) \) and achieve an induced \( L_2 \)-disturbance attenuation \( \gamma \) according to the following definition

**Definition 4.1**

Given \( \gamma_j > 0, j \in \{1, \ldots, N\} \), the filtering error system (16) is said to be delay-dependent robustly asymptotically stable with an induced \( L_2 \) disturbance attenuation \( \gamma \) if it is delay-dependent asymptotically stable for all admissible uncertainties satisfying (8) and under zero initial conditions

\[
\|\sigma(t)\|_2 < \gamma \|w(t)\|_2 \quad \forall 0 \neq w(t) \in L_2[0, \infty)
\]
Introduce the Lyapunov–Krasovskii functional:

\[ V(t) = \sum_{j=1}^{N} V_j(t) \]

\[ V_j(t) = V_{jo}(t) + V_{ja}(t) + V_{jc}(t) + V_{je}(t) + V_{jm}(t) + V_{jn}(t) \]

\[ V_{jo}(t) = \xi_j(t) P_j \xi_j(t), \quad V_{ja}(t) = \int_{t}^{t_{\varphi_j}} \xi_j(s) Q_j \xi_j(s) \, ds, \]

\[ V_{jm}(t) = \varphi_j \int_{t-\tau_j(t)}^{t} \xi_j(\alpha) W_j \xi_j(\alpha) \, d\alpha, \quad V_{jn}(t) = (Q_j - \varphi_j) \int_{t-\tau_j(t)}^{t} \xi_j(\alpha) S_j \xi_j(\alpha) \, d\alpha, \]

\[ V_{je}(t) = \int_{t-\tau_j(t)}^{t} \xi_j(\alpha) Z_j \xi_j(\alpha) \, ds, \]

\[ V_{jc}(t) = \int_{t}^{t_{\varphi_j}} \xi_j(s) Z_j \xi_j(s) \, ds, \]

\[ V_{je}(t) = \int_{t}^{t_{\varphi_j}} \xi_j(s) R_j \xi_j(s) \, ds \]

(17)

where \( P_j, 0 < W_j, 0 < Q_j, 0 < R_j, 0 < S_j, 0 < Z_j \) are weighting matrices of appropriate dimensions and can be expressed as follows:

\[ P_j = \text{diag} \left[ P_{1j}, P_{3j} \right], \quad P_{1j} > 0, \quad P_{3j} > 0 \]

\[ Q_j = \begin{bmatrix} Q_{1j} & Q_{2j} \\ \bullet & Q_{3j} \end{bmatrix}, \quad R_j = \begin{bmatrix} R_{1j} & R_{2j} \\ \bullet & R_{3j} \end{bmatrix}, \quad S_j = \begin{bmatrix} S_{1j} & S_{2j} \\ \bullet & S_{3j} \end{bmatrix} \]

\[ W_j = \begin{bmatrix} W_{1j} & W_{2j} \\ \bullet & W_{3j} \end{bmatrix}, \quad Z_j = \begin{bmatrix} Z_{1j} & Z_{2j} \\ \bullet & Z_{3j} \end{bmatrix} \]

(18)

Note for the global system that \( P = \text{diag}(P_1, \ldots, P_N), \quad Q = \text{diag}(Q_1, \ldots, Q_N), \quad R = \text{diag}(R_1, \ldots, R_N), \quad S = \text{diag}(S_1, \ldots, S_N), \quad W = \text{diag}(W_1, \ldots, W_N), \quad Z = \text{diag}(Z_1, \ldots, Z_N) \).

The following theorem provides an LMI-based stability result for the global filtered system (11) and (12), \( j \in \{1, \ldots, N\} \)

**Theorem 4.1**

Given the bounds \( \varphi_j > 0, \quad \varphi_j > 0, \quad \eta_j > 0, \quad \gamma_j > 0 \) and the filter matrices \( A_{fj}, B_{fj}, C_{fj}, \in \{1, \ldots, N\} \). The global filtered system (16) is delay-dependent asymptotically stable with \( L_2 \)-performance bound \( \gamma = \sum_{j=1}^{N} \gamma_j \) if for \( j = 1, \ldots, N \) there exist weighting matrices \( 0 < P_j, 0 < Q_j, 0 < W_j, 0 < S_j, 0 < Z_j \), \( 0 < R_j \) and scalars \( \delta_j > 0, \quad \kappa_j > 0 \) tolerate uncertain nonlinear interconnections and perturbations with vector degree

\[ v = \text{diag}(v_1 I_{r_1}, \ldots, v_N I_{r_N}), \quad v_j = \delta_j + \kappa_j \]

if the following problem is feasible

\[ \min \sum_{j=1}^{N} v_j + \gamma_j, \quad \text{subject to} \quad \Pi_j = \begin{bmatrix} \Pi_{1j} & \Pi_{2j} & \Pi_{4j} \\ \bullet & \Pi_{3j} & 0 \\ \bullet & \bullet & \Pi_{5j} \end{bmatrix} < 0 \]

(19)
if the following problem is feasible

\[ \text{min} \sum_{j=1}^{N} v_j + \gamma_j \]

subject to

\[ \tilde{\Pi}_j = \begin{bmatrix} \varphi_j \Pi_{js} & (q_j - \varphi_j) \Pi_{jt} & \mathcal{P}_j \Gamma_j + \mathcal{P}_j B_{fa}j \\ 0 & 0 & 0 \\ \varphi_j \Pi_{jt} & (q_j - \varphi_j) \Pi_{jt} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} < 0 \]

Theorem 4.2

Given the bounds \( \varphi_j > 0 \), \( q_j > 0 \), \( \eta_j > 0 \), \( \gamma_j > 0 \) and the filter matrices \( A_{fa}j \), \( B_{fa}j \), \( C_{fa}j \).

The augmented system (11) is delay-dependent asymptotically stable with \( H_{\infty} \)-performance bound \( \gamma = \sum_{j=1}^{N} \gamma_j \) if for \( j = 1, \ldots, N \) there exist weightings matrices \( 0 < \mathcal{X}_j \), \( 0 < \Lambda_{1j} \), \( 0 < \Lambda_{2j} \), \( 0 < \Lambda_{3j}, 0 < \Lambda_{4j} \), \( 0 < \Lambda_{5j} \) and scalars \( \delta_j > 0 \), \( \kappa_j > 0 \), \( \gamma_{1j} \), \( \gamma_{2j} \) and scalars \( \delta_j > 0 \), \( \kappa_j > 0 \) tolerate uncertain nonlinear interconnections and perturbations with vector degree

\[ \nu = \text{diag}\{v_1 I_{r_1}, \ldots, v_N I_{r_N}\}, \quad v_j = \delta_j + \kappa_j \]

Employing (18) with some algebraic manipulations, the following theorem establishes expressions for the filter gains

Proof

The proof is given in the Appendix. Now, to optimize the performance and robustness with respect to the tolerance vector \( \nu = \text{diag}\{v_1 I_{r_1}, \ldots, v_N I_{r_N}\} \), the minimum value of the sum of \( \gamma_j \) and \( v_j \) is sought as required by (19). Thus the globally asymptotic stable of the equilibrium \( x = 0 \) of the filtering error system (16) for all \( c(t, x) \) satisfying (5) is attained. This completes the proof. \( \square \)
Moreover, the local gain matrix is given by
\[ A_j = \mathcal{Y}_j \mathcal{X}_j^{-1}, \quad B_j = \mathcal{Y}_j \mathcal{X}_j^{-1}, \quad C_j \]

**Proof**

See the Appendix. \(\square\)

**Remark 4.1**

To emphasize the generality of our approach, some special cases are identified in the sequel:

1. **Interconnected delay-free systems.** Here, we consider the class of nominally linear systems structurally composed of \(N\)-coupled delay-free subsystems, and the model of the \(j\)th subsystem is described by

\[
\begin{align*}
\dot{x}_j(t) &= A_j x_j(t) + B_j u_j(t) + c_j(t) + \Gamma_j w_j(t) \\
z_j(t) &= G_j x_j(t) \\
y_j(t) &= C_j x_j(t)
\end{align*}
\]

where for \(j \in \{1, \ldots, N\}\), the coupling vector \(c_j(k)\) is a piecewise-continuous vector function in its arguments and satisfies the quadratic inequality

\[ c_j^T(t, \cdot) c_j(t, \cdot) \leq \phi_j x^T(t) E_j^T E_j x(t) \]  

where \(\phi_j > 0\) are adjustable bounding parameters and \(E_j \in \mathbb{R}^{n_j \times n_j}\) are constant matrices.

By setting \(\mathcal{S}_j = 0, \ G_{dj} = 0, \ C_{dj} = 0, \ F_{dj} = 0\) in Theorem 4.2, the following result could be derived:

**Corollary 4.1**

The global filtered system (16) is delay-dependent asymptotically stable with \(\mathcal{L}_2\)-performance bound \(\gamma = \sum_{j=1}^{N} \gamma_j\) if for \(j = 1, \ldots, N\) there exist weighting matrices \(0 < P_{1j}, \ 0 < P_{2j}, \ 0 < P_{3j}, \ 0 < Q_j, \ 0 < S_j, \ 0 < R_j, \ 0 < \mathcal{W}_j, \ 0 < \mathcal{Z}_j, \ \mathcal{X}_j, \ \mathcal{Y}_j\) and scalars \(\phi_j > 0, \ \alpha_j > 0, \ \gamma_j > 0\) satisfying the following LMI

\[
\mathcal{P}_j = \begin{bmatrix}
\mathcal{P}_{1j} & \mathcal{P}_{2j} \\
\mathcal{P}_{2j}^T & -E_j^T E_j
\end{bmatrix} < 0
\]
where $\Pi_{1j}$ is given in (20). Moreover, the filter gains are given by

$A_{fj} = \mathcal{P}_{aj}^{-1} \gamma_j, \quad B_{fj} = \mathcal{P}_{aj}^{-1} \chi_j, \quad C_{fj}$

2. Single time-delay system. Consider the single linear time-delay system

$$\begin{align*}
\dot{x}(t) &= A x(t) + A_d x(t - \tau(t)) + B u(t) + \Gamma w(t) \\
\dot{z}_j(t) &= G x(t) + G_d x(t - \tau(t)) \\
y_j(t) &= C x(t) + C_d x(t - \tau(t))
\end{align*}$$

where $0 < \varphi \leq \tau(t) \leq \rho$, $\dot{t}(t) \leq \eta$. By setting $j = 1$ in Theorem 4.2, the corresponding design result follows:

**Corollary 4.2**

Given the bounds $\varphi > 0$, $\rho > 0$, $\eta > 0$, $\gamma > 0$, the global filtered system (16) is asymptotically stable with $\mathcal{L}_2$-performance bound $\gamma$ if there exist weighting matrices $0 < \mathcal{P}_1, \mathcal{P}_2, 0 < \mathcal{Q}, 0 < \mathcal{S}, 0 < \mathcal{R}, 0 < \mathcal{W}, 0 < \mathcal{Z}, \chi, \gamma$ and scalars $\phi > 0, \psi > 0, \alpha > 0$ tolerate uncertain nonlinear interconnections and perturbations with degree $\nu = \delta + \kappa$ if the following problem is feasible

$$\begin{align*}
\min & \quad \nu + \gamma, \quad (30) \\
\tilde{\Pi} &= \begin{bmatrix} \tilde{\Pi}_1 & \tilde{\Pi}_2 \\ \Pi_3 & \end{bmatrix} < 0 \\
\tilde{\Pi}_1 &= \begin{bmatrix} \tilde{\Pi}_{1o} \gamma_j \mathcal{W} \mathcal{V} \\
\mathcal{Q} - \mathcal{S} - \mathcal{W} \\
\mathcal{S} \\
\Pi_{m} & -I \\
\end{bmatrix}, \\
\tilde{\Pi}_2 &= \begin{bmatrix} 0 & \tilde{\Pi}_{23} & \tilde{\Pi}_{24} \\
0 & 0 & 0 \\
\mathcal{S} & 0 & 0 \\
\end{bmatrix}.
\end{align*}$$
\[ \hat{\Pi}_o = \begin{bmatrix} \mathcal{P}_1 A_a & 0 \\ \mathcal{P}_2 A_a + \lambda C & \gamma' \end{bmatrix} + Q + R + Z + \begin{bmatrix} A^T \mathcal{P}_1 & A^T \mathcal{P}_2 + C^T \lambda' \gamma' \\ 0 & \gamma' \end{bmatrix} + \phi E' E \]

\[ \Pi_m = (1 - \eta) Z + 2S - \psi E_d E_d \]

\[ \hat{\Pi}_{23} = \begin{bmatrix} 0 \\ \mathcal{X} \end{bmatrix}, \quad \hat{\Pi}_{24} = \begin{bmatrix} \mathcal{P}_1 \Gamma \\ \mathcal{P}_2 \Gamma \end{bmatrix}, \quad \Pi_3 = \begin{bmatrix} -S - R & 0 & 0 & 0 \\ \bullet & -\alpha I & 0 & 0 \\ \bullet & \bullet & -I & 0 \\ \bullet & \bullet & \bullet & -\gamma^2 I \end{bmatrix} \]

(32)

where \( \Pi_3 \) is given in (20). Moreover, the filter gains are given by

\[ A_f = \mathcal{P}_3^{-1} \gamma, \quad B_f = \mathcal{P}_3^{-1} \mathcal{X}, \quad C_f \]

3. Single linear system. Consider the single linear system

\[ \dot{x}(t) = Ax(t) + Bu(t) + \Gamma w(t) \]

\[ z_j(t) = G x(t) \]

\[ y_j(t) = C x(t) \]

(34)

for which we will use quantized output measurements. By setting \( j = 1 \) and \( \mathcal{S} \equiv 0, \mathcal{G}_d \equiv 0, \mathcal{C}_d \equiv 0, \mathcal{F}_d \equiv 0 \) in Theorem 4.2, the following design result could be obtained:

Corollary 4.3

Given the bound \( \gamma > 0 \), the global filtered system (16) is asymptotically stable with \( \mathcal{L}_2 \)-performance bound \( \gamma \) if there exist weighting matrices \( 0 < \mathcal{P}_1, \mathcal{P}_2, 0 < \mathcal{P}_3, 0 < Q, 0 < \mathcal{S}, 0 < R, 0 < \mathcal{W}, 0 < \mathcal{Z}, \mathcal{X}, \mathcal{Y} \) and scalars \( \phi > 0, \alpha > 0 \) tolerate uncertain nonlinear interconnections and perturbations with degree \( \delta \) if the following problem is feasible

\[ \min \quad \delta + \gamma, \]

\[ \hat{\Pi} = \begin{bmatrix} \hat{\Pi}_1 & \hat{\Pi}_2 & \hat{\Pi}_3 \\ \bullet & \bullet & \bullet \end{bmatrix} < 0 \]

(35)

\[ \hat{\Pi}_1 = \begin{bmatrix} \mathcal{P}_1 A_a & 0 \\ \mathcal{P}_2 A_a + \lambda C & \gamma' \end{bmatrix} + Q + R + Z \]

\[ \hat{\Pi}_2 = \begin{bmatrix} \mathcal{P}_1 \Gamma \\ \mathcal{P}_2 \Gamma \end{bmatrix} \]

\[ \hat{\Pi}_3 = \begin{bmatrix} \mathcal{P}_1 \Gamma \\ \mathcal{P}_2 \Gamma \end{bmatrix} \]

(36)

\[ \Pi_{mm} = Z + 2S, \quad \hat{\Pi}_{23} = \begin{bmatrix} 0 \\ \mathcal{X} \end{bmatrix}, \quad \hat{\Pi}_{24} = \begin{bmatrix} \mathcal{P}_1 \Gamma \\ \mathcal{P}_2 \Gamma \end{bmatrix} \]

(37)

where \( \Pi_3 \) is given in (20). Moreover, the filter gains are given by

\[ A_f = \mathcal{P}_3^{-1} \gamma, \quad B_f = \mathcal{P}_3^{-1} \mathcal{X}, \quad C_f \]
4. It is to be noted in Theorem 4.1 and Corollary 4.1 that there are several DOFs to achieve the desired stabilization with guaranteed performance, particularly because both the off-line gain computation and the on-line quantized feedback are decentralized. This is a salient feature of the developed results of this paper, which is not shared by several published results [12, 14]. In the absence of quantizer, the main results of this paper provide an improved design approach over the published work in the area of interconnected time-delay systems [13–17].

5. QUANTIZED FILTER DESIGN

Next, we move to examine the stability and desired disturbance attenuation level of the augmented filtered system (11) in the presence of the quantization error. In the sequel, we adopt the definition of a local (subsystem) quantizer with general form as introduced in [8]. Let $f_j \in \mathbb{R}^s$, $j = 1, \ldots, n_s$ be the variable being quantized. A local quantizer is defined as a piecewise constant function $Q_j : \mathbb{R}^s \rightarrow D_j$, where $D_j$ is a finite subset of $\mathbb{R}^s$. This leads to a partition of $\mathbb{R}^s$ into a finite number of quantization regions of the form $\{f_j \in \mathbb{R}^s : Q_j(f_j) = d_j, d_j \in D_j\}$. These quantization regions are not assumed to have any particular shape. We assume that there exist positive real numbers $M_j$ and $\Delta_j$ such that the following conditions hold:

1. If
   $$|f_j| \leq M_j$$
   then
   $$|Q_j(f_j) - f_j| \leq \Delta_j$$
   (38)

2. If $|f_j| > M_j$, then $|Q_j(f_j)| > M_j - \Delta_j$

We note that condition (38) provides a bound on the quantization error when the quantizer does not saturate. Condition (39) gives a way to detect the possibility of saturation. In the sequel, $M_j$ and $\Delta_j$ will be referred to as the range of $Q_j$ and the quantization error, respectively. Henceforth, we assume that $Q(x) = 0$ for $x$ in some neighborhood of the origin. The foregoing requirements are met by the quantizer with rectangular quantization regions [5, 7].

Proceeding further, we employ the Lyapunov–Krasovskii functional (17) and consider that the gains $K_{o_j}$ are obtained from application of Theorem 4.1. The following theorem establishes the main design result for subsystem $S_j$.

**Theorem 5.1**

Given the bounds $\mu_j > 0$, $\mu_j > 0$. If the local quantizer $M_j$ is selected large enough with respect to $\Delta_j$ while adjusting the local scalar $\alpha_j$ so as to satisfy the inequality

$$M_j > 4\Delta_j \frac{||P_j B_j K_{o_j}||}{\lambda_m(\Lambda_j)} ||C_j + \alpha_j C_d||$$

(40)

Then, the augmented filtered system (11) is delay-dependent asymptotically stabilizable with $L_2$-performance bound $\gamma_j$ by decentralized quantized filters (10)

**Proof**

Because

$$\frac{y_j(t)}{\mu_j} = \frac{C_j x_j(t) + C_d x_j(t - \tau_j(t))}{\mu_j}$$

is quantized before being passed to the feedback, we obtain by using the properties of local quantizer (38) and (39) that whenever $|y_j(t)| \leq M_j \mu_j$, the inequality

$$|y_j(t) - Q_j \left( \frac{y_j(t)}{\mu_j} \right) | \leq \Delta_j$$

(41)
holds true. Extending on Theorem 4.1, it follows that
\[
J_j \leq \int_0^\infty \frac{1}{\tau_j} \int_{t_{j-1}}^t \left\{ \eta_j(t, s) \bar{\mathcal{X}}_j(t, s) + 2x_j^T \mathcal{M}_j(t, y_j) \right. \\
- x_j^T \Lambda_j x_j + \varrho_j \bar{\chi}_j(t) W_j \bar{\chi}_j(t) + z_j^T(s) z_j(s) - \gamma_j^2 w_j^T(s) w_j(s) \left\} \, ds \tag{42}
\]
where \( \bar{\mathcal{X}}_j \) corresponds to \( \mathcal{X}_j \) except that \( \mathcal{X}_{aj} \to \mathcal{X}_j + \Lambda_j \) with \( \Lambda_j > 0 \) being an arbitrary matrix. Proceeding as before, we focus on the integrand in (42) and manipulating in view of Theorem 4.1 to get
\[
\eta_j^T(t, s) \bar{\mathcal{X}}_j(t, s) + 2x_j^T \mathcal{M}_j(t, y_j) - x_j^T \Lambda_j x_j \\
+ \varrho_j \bar{\chi}_j(t) W_j \bar{\chi}_j(t) + z_j^T(s) z_j(s) - \gamma_j^2 w_j^T(s) w_j(s) \\
\leq \chi_j^T(t, s) \bar{\mathcal{X}}_j(t, s) + \varrho_j \bar{\chi}_j(t) W_j \bar{\chi}_j(t) + z_j^T(s) z_j(s) \\
- \gamma_j^2 w_j^T(s) w_j(s) - \frac{1}{2} \lambda_m(\Lambda_j) \left( |x_j| - 4\Delta_j \frac{||\mathcal{P}_j B_j \mathcal{K}_{\sigma_j}||}{\lambda_m(\Lambda_j)} \right) \tag{43}
\]
It follows from (40) that we can always find a scalar \( \beta_j \in (0, 1) \) such that
\[
M_j > 4\Delta_j \frac{||\mathcal{P}_j B_j \mathcal{K}_{\sigma_j}||}{\lambda_m(\Lambda_j)} ||C_j + \alpha_j \mathcal{C}_{dj}|| \times \frac{1}{1 - \beta_j} \tag{44}
\]
This is equivalent to
\[
\frac{1}{1 - \beta_j} \times 4\Delta_j \frac{||\mathcal{P}_j B_j \mathcal{K}_{\sigma_j}||}{\lambda_m(\Lambda_j)} ||C_j + \alpha_j \mathcal{C}_{dj}|| \mu_j < M_j \mu_j \tag{45}
\]
Therefore, for any \( \mu_j \neq 0 \), we can find a scalar \( \mu_j > 0 \) such that
\[
\frac{1}{1 - \beta_j} \times 4\Delta_j \frac{||\mathcal{P}_j B_j \mathcal{K}_{\sigma_j}||}{\lambda_m(\Lambda_j)} ||C_j + \alpha_j \mathcal{C}_{dj}|| \mu_j < M_j \mu_j \tag{46}
\]
At the extreme case \( |y_j| = 0 \), we set \( \mu_j = 0 \) so that the output of the local quantizer is considered zero and therefore (46) holds true. This, in turn, implies that we can always select \( \mu_j \) so that (46) is satisfied, (43) holds and hence,
\[
J_j \leq \chi_j^T(t, s) \hat{\mathcal{N}}_j \chi_j(t, s) - \frac{1}{2} \beta_j \lambda_m(\Lambda_j) \frac{|x_j|}{||C_j + \alpha_j \mathcal{C}_{dj}||} |y_j| \tag{47}
\]
where \( \hat{\mathcal{N}}_j \) is given by (19) and (20) for some vector \( \chi_j(t, s) \). The rest of the proof follows from Theorem 4.1. \( \square \)

**Remark 5.1**

For the case of decentralized state feedback control \( u_j(t) = K_j x_j(t), \ j = 1, \ldots, n_x \), then Theorem 5.1 specializes to the following corollary
Corollary 5.1
Given the bounds $\varphi_j > 0$, $\mu_j > 0$. If the local quantizer $M_j$ is selected large enough with respect to $\Delta_j$ while adjusting the local scalar $\alpha_j$ so as to satisfy the inequality

$$M_j > 4\Delta_j \frac{||P_j B_j K_{o_j}||}{\lambda_m(\Lambda_j)}$$  \hspace{1cm} (48)$$

Then, the family of subsystems described by (1) and (3) is delay-dependent asymptotically stabilizable with $L_2$-performance bound $\gamma_j$ by decentralized quantized state-feedback controller

$$u_j(t) = \mu_j K_j Q_j \left( \frac{x_j(t)}{\mu_j} \right) \text{, } j = 1, ..., n_s$$

Remark 5.2
By the mean-value theorem and following [28], it can be shown that $\lambda_m(P_j)||x_j||^2 \leq V_j \leq \varphi_j ||x_j||^2$ where

$$\varphi_j = \left[ \lambda_M(P_j) + \varphi_j [\lambda_M(Z_j) + \lambda_M(W_j)] 
+ 3\varphi_j^2 \left( \lambda_M(A_j^1 A_j) + \left( \lambda_M(A_j^1 A_j^2) \right) \right) \right]$$

On the basis of the results of [8], we define the local ellipsoids

$$B_{o_j}(\mu_j) := \{ x_j : x_j^TP_j x_j \leq \lambda_m \left( P_j \mu_j^2 \right) \}$$
$$B_{s_j}(\mu_j) := \{ x_j : x_j^TP_j x_j \leq \lambda_m \left( P_j \Delta_j^2 (1 + \sigma_j)^2 \mu_j^2 \right) \}$$
$$D_j := 2 \frac{||P_j B_j K_{o_j}||}{\lambda_m(\Lambda_j)} ||C_j + \alpha_j C_d_j||$$

In the ‘zooming-in’ stage, it can be inferred that $B_{s_j}(\mu_j) \subset B_{o_j}(\mu_j)$ are invariant regions for system (16) given $\sigma_j > 0$. Moreover, all solutions of (16) that start in $B_{o_j}(\mu_j)$ enter $B_{s_j}(\mu_j)$ in finite time.

Remark 5.3
It is crucial to recognize that the local scalar $\alpha_j$ plays a basic role in steering the trajectories of (16) towards the final ellipsoid $B_{s_j}(\mu_j)$. This is a distinct feature of quantized time-delay systems.

Remark 5.4
It should be noted that Theorem 5.1 and Corollary 5.1 provide several DOFs to determine the desired stability with guaranteed performance. This is a salient feature of the developed results of this paper, because both the off-line gain computation and the on-line quantized feedback are decentralized.

6. EXAMPLE

A discrete multi-reach water pollution model of the type (6) is considered. The model of each reach (subsystem) represents two aggregate bio-strata, the first one is for algae and the other is for ammonia products. The data values are taken from [4]. We wish to design a set of quantized filters
for this system based on Theorem 4.2. The model is described by

**Subsystem 1:**

\[
A_1 = \begin{bmatrix}
-0.3 & 0.1 \\
-0.4 & -0.2 \\
\end{bmatrix},
A_{d1} = \begin{bmatrix}
0.6 & 0 \\
0.2 & 0.3 \\
\end{bmatrix},
\Gamma_1 = \begin{bmatrix}
0.2 \\
0.3 \\
\end{bmatrix},
G_1 = \begin{bmatrix}
0.01 & 0.02 \\
\end{bmatrix},
\]

\[
G_{d1} = \begin{bmatrix}
0.05 & 0.05 \\
\end{bmatrix},
C_1 = \begin{bmatrix}
0.1 & 0.3 \\
\end{bmatrix},
C_{d1} = \begin{bmatrix}
0.03 & 0.04 \\
\end{bmatrix},
\]

\[
R_1 = \begin{bmatrix}
1.5 & 0 \\
0 & 1.2 \\
\end{bmatrix},
S_1 = \begin{bmatrix}
0.5 & 0 \\
0 & 0.2 \\
\end{bmatrix},
\]

**Subsystem 2:**

\[
A_2 = \begin{bmatrix}
-0.1 & 0.2 \\
0.3 & -0.4 \\
\end{bmatrix},
A_{d2} = \begin{bmatrix}
-0.5 & 0.1 \\
0 & -0.4 \\
\end{bmatrix},
\Gamma_2 = \begin{bmatrix}
0.1 & 0.5 \\
\end{bmatrix},
G_2 = \begin{bmatrix}
0.06 & 0.02 \\
\end{bmatrix},
\]

\[
G_{d2} = \begin{bmatrix}
0.04 & 0.02 \\
\end{bmatrix},
C_2 = \begin{bmatrix}
0.2 & 0.5 \\
\end{bmatrix},
C_{d2} = \begin{bmatrix}
0.05 & 0.01 \\
\end{bmatrix},
\]

\[
R_2 = \begin{bmatrix}
2.1 & 0 \\
0 & 0.8 \\
\end{bmatrix},
S_2 = \begin{bmatrix}
0.2 & 0 \\
0 & 0.4 \\
\end{bmatrix},
\]

**Subsystem 3:**

\[
A_3 = \begin{bmatrix}
-0.2 & 0.1 \\
0.6 & -0.3 \\
\end{bmatrix},
A_{d3} = \begin{bmatrix}
0.4 & 0 \\
0 & 0.4 \\
\end{bmatrix},
\Gamma_3 = \begin{bmatrix}
0.2 & 0.8 \\
\end{bmatrix},
G_3 = \begin{bmatrix}
0.03 & 0.04 \\
\end{bmatrix},
\]

\[
G_{d3} = \begin{bmatrix}
0.05 & 0.05 \\
\end{bmatrix},
C_3 = \begin{bmatrix}
0.4 & 0.4 \\
\end{bmatrix},
C_{d3} = \begin{bmatrix}
0.02 & 0.03 \\
\end{bmatrix},
\]

\[
R_3 = \begin{bmatrix}
1.1 & 0 \\
0 & 2.2 \\
\end{bmatrix},
S_3 = \begin{bmatrix}
0.1 & 0 \\
0 & 0.1 \\
\end{bmatrix},
\]

In addition to the coupling matrices

\[
\tilde{R}_1 = \begin{bmatrix}
1.7 & 0.1 \\
0.1 & 2.2 \\
\end{bmatrix},
\tilde{R}_2 = \begin{bmatrix}
3.6 & 0.2 \\
0.7 & 4.4 \\
\end{bmatrix},
\tilde{R}_3 = \begin{bmatrix}
4.2 & 0.3 \\
0.8 & 4.2 \\
\end{bmatrix},
\]

\[
\tilde{S}_1 = \begin{bmatrix}
2.2 & 0.4 \\
0.9 & 5.4 \\
\end{bmatrix},
\tilde{S}_2 = \begin{bmatrix}
6.2 & 0.7 \\
1.1 & 6.3 \\
\end{bmatrix},
\tilde{S}_3 = \begin{bmatrix}
6.8 & 1.3 \\
1.8 & 8.2 \\
\end{bmatrix}
\]

and the delay bounds

\[
\varphi_1 = 0.6, \ \varphi_1 = 2.0, \ \eta_1 = 1.4, \ \varphi_2 = 0.8, \ \varphi_1 = 2.8, \ \eta_1 = 1.2,
\]

\[
\varphi_3 = 0.7, \ \varphi_1 = 2.4, \ \eta_1 = 1.3
\]

To illustrate the effectiveness of our quantized filter design, we initially consider the nominal control design in which no actuator failures will occur. Then, by using the data values \(\Sigma_1 = 1, \ \Sigma_2 = 1,\)

![Figure 1. State patterns of subsystem 1.](image-url)
\[ \Sigma_3 = 1, \quad \Gamma_1 = 0, \quad \Gamma_2 = 0, \quad \Gamma_3 = 0 \] and solving problem using MATLAB-LMI solver, it is found that the feasible solution of Theorem 4.2 yields the following control gain matrices and performance levels.

![Figure 2. State patterns of subsystem 2.](image1)

![Figure 3. State patterns of subsystem 3.](image2)

![Figure 4. First state patterns of subsystem 1 with and without quantization.](image3)
\[ \gamma_1 = 2.5109, \]
\[ A_{f1} = \begin{bmatrix} -0.281 & -0.095 \\ -0.501 & -0.195 \end{bmatrix}, \quad B_{f1} = \begin{bmatrix} 0.191 \\ 0.602 \end{bmatrix}, \quad C_{f1}^T = \begin{bmatrix} 0.334 \\ -0.572 \end{bmatrix}. \]
\[ \gamma_2 = 1.9775, \]
\[ A_{f2} = \begin{bmatrix} -0.171 & -0.488 \\ -0.243 & -0.498 \end{bmatrix}, \quad B_{f2} = \begin{bmatrix} 0.333 \\ 0.476 \end{bmatrix}, \quad C_{f2}^T = \begin{bmatrix} -0.478 \\ 0.136 \end{bmatrix}. \]
\[ \gamma_3 = 2.1088, \]
\[ A_{f3} = \begin{bmatrix} -0.317 & -0.508 \\ -0.601 & -0.086 \end{bmatrix}, \quad B_{f3} = \begin{bmatrix} 0.254 \\ 0.711 \end{bmatrix}, \quad C_{f3}^T = \begin{bmatrix} 0.402 \\ -0.298 \end{bmatrix}. \]

Simulation results of the algae (first state) and ammonia products (second state) trajectories and their estimates for the three cited modes are generated and plotted in Figures 1–3. From these results, the

Figure 5. First state patterns with two quantization ranges

Figure 6. First state patterns with two quantization errors.
Applying Lemma 1.1, we get the problem into that of designing a decentralized $\mathcal{H}_\infty$ filter for a class of linear interconnected continuous-time systems with unknown-but-bounded couplings and interval delays and where the quantizer has arbitrary form. An LMI-based method using a decentralized quantized filter has been constructed at the subsystem level to render the global filtered system delay-dependent asymptotically stable with guaranteed $\gamma$-level. It has been established that this approach encompasses several special cases of interest including interconnected delay-free systems, single time-delay systems and single systems. The theoretical developments have been illustrated by numerical simulations.

7. CONCLUSIONS

A generalized approach to the problem of quantized filtering has been developed by casting the problem into that of designing a decentralized $\mathcal{H}_\infty$ filter for a class of linear interconnected continuous-time systems with unknown-but-bounded couplings and interval delays and where the quantizer has arbitrary form. An LMI-based method using a decentralized quantized filter has been constructed at the subsystem level to render the global filtered system delay-dependent asymptotically stable with guaranteed $\gamma$-level. It has been established that this approach encompasses several special cases of interest including interconnected delay-free systems, single time-delay systems and single systems. The theoretical developments have been illustrated by numerical simulations.

APPENDIX

Proof of stability
Recall that $V(t) = \sum_{j=1}^{N} V_j(t)$. A straightforward computation gives the time-derivative of $V_j$ along the solutions of (11) with $\omega_j(t) \equiv 0$ as

$$
\dot{V}_j(t) = 2\xi_j(t)P_j \dot{\xi}_j(t) + \xi_j(t)[Q_j + R_j + Z_j] \dot{\xi}_j(t)
- \xi_j(t - \varphi_j)Q_j x_j(t - \varphi_j) - (1 - \xi_j(t - \tau_j(t)))Z_j \xi_j(t - \tau_j(t)) - \xi_j(t - \varphi)R_j \xi_j(t - \varphi) + \xi_j(t) [\varphi_j W_j + (\varphi_j - \varphi_j)^2 S_j] \dot{\xi}_j(t) - \int_{t-\varphi_j}^{t} \dot{\xi}_j(\alpha) W_j \dot{\xi}_j(\alpha) d\alpha
- \int_{t-\varphi}^{t-\varphi_j} \dot{\xi}_j(\alpha) S_j \dot{\xi}_j(\alpha) d\alpha
\leq 2\xi_j(t)P_j \dot{\xi}_j(t) + \xi_j(t)[Q_j + R_j + Z_j] \dot{\xi}_j(t)
- \xi_j(t - \varphi_j)Q_j \xi_j(t - \varphi_j) - (1 - \mu_j) \xi_j(t - \tau_j)Z_j \xi_j(t - \tau_j) - \xi_j(t - \varphi)R_j \xi_j(t - \varphi)
+ \xi_j(t) [\varphi_j W_j + (\varphi_j - \varphi_j) S_j] \dot{\xi}_j(t)
- \varphi_j \int_{t-\varphi_j}^{t} \dot{\xi}_j(\alpha) W_j \dot{\xi}_j(\alpha) d\alpha
- (\varphi_j - \varphi_j) \int_{t-\varphi}^{t-\varphi_j} \dot{\xi}_j(\alpha) S_j \dot{\xi}_j(\alpha) d\alpha
$$

(49)

Applying Lemma 1.1, we get

$$
- \varphi_j \int_{t-\varphi_j}^{t} \dot{\xi}_j(\alpha) W_j \dot{\xi}_j(\alpha) d\alpha \leq \begin{bmatrix} \xi_j(t) \\ \xi_j(t - \varphi) \end{bmatrix}^{T} \begin{bmatrix} -W_j & W_j \\ \varphi_j W_j & -W_j \end{bmatrix} \begin{bmatrix} \xi_j(t) \\ \xi_j(t - \varphi) \end{bmatrix}
$$

(50)
Similarly,

\[-(\varphi_j - \varphi_j) \int_{t-\varphi_j}^{t-\varphi_j} \xi_j^T(\alpha) S_j \hat{\xi}_j(\alpha) d\alpha \]

\[= -(\varphi_j - \varphi_j) \left[ \int_{t-\varphi_j}^{t-\varphi_j} \xi_j^T(\alpha) S_j \hat{\xi}_j(\alpha) d\alpha + \int_{t-\varphi_j}^{t-\tau_j} \hat{\xi}_j^T(\alpha) S_j \hat{\xi}_j(\alpha) d\alpha \right] \]

\[\leq -(\tau_j - \varphi_j) \left[ \int_{t-\tau_j}^{t-\tau_j} \hat{\xi}_j^T(\alpha) S_j \hat{\xi}_j(\alpha) d\alpha \right] - (\varphi_j - \tau_j) \left[ \int_{t-\varphi_j}^{t-\tau_j} \hat{\xi}_j^T(\alpha) S_j \hat{\xi}_j(\alpha) d\alpha \right] \]

\[\leq -\left( \int_{t-\tau_j}^{t-\varphi_j} \hat{\xi}_j^T(\alpha) d\alpha \right) S_j \left( \int_{t-\tau_j}^{t-\varphi_j} \hat{\xi}_j(\alpha) d\alpha \right) - \left( \int_{t-\varphi_j}^{t-\tau_j} \hat{\xi}_j^T(\alpha) d\alpha \right) \]

\[\leq -\left[ \xi(t - \varphi_j) - \xi(t - \tau_j) \right] S_j \left[ \xi(t - \varphi_j) - \xi(t - \tau_j) \right] - \left[ \xi(t - \tau_j) - \xi(t - \varphi_j) \right] S_j \left[ \xi(t - \tau_j) - \xi(t - \varphi_j) \right] \]

(51)

From (49)–(51) and (16) with Schur complements and incorporating (8) via Lemma 1.1, we have

\[\dot{V}_j(t) \leq \zeta_j^T(t) \Xi_j \zeta_j(t),\]

\[\zeta_j(t) = \left[ \xi_j^T(t) \xi_j^T(t - \varphi_j) \xi_j^T(t - \tau_j) \hat{\xi}_j^T(t - \varphi_j) \hat{\xi}_j^T(t - \tau_j) \right]^T \]

(52)

where \(\Xi_j\) corresponds to \(\Pi_j\) in (19) with \(G_{adj} \equiv 0, \ G_{adj} \equiv 0\) and Schur complement operations. If \(\Pi_j < 0\) so is \(\Xi_j < 0\), leading to \(\dot{V}_j(t) \leq -\omega_j ||\zeta_j||^2\). This establishes the internal asymptotic stability.

Next, we consider the performance measure

\[J_j = \int_0^\infty \left( z_j^T(s)z_j(s) - \gamma_j^2 w_j^T(s)w_j(s) \right) ds \]

For any \(w_j(t) \in L_2(0, \infty) \neq 0\) and zero initial condition \(x(0) = 0\) (hence \(V_j(0) = 0\), we have

\[J_j \leq \int_0^\infty \left( z_j^T(s)z_j(s) - \gamma_j^2 w_j^T(s)w_j(s) + \dot{V}_j(x) \right)_{(11)} ds \]

where \(\dot{V}_j(x)_{(11)}\) is the Lyapunov derivative along the state trajectories of system (11). Proceeding, we get

\[z_j^T(s)z_j(s) - \gamma_j^2 w_j^T(s)w_j(s) + \dot{V}_j(s)_{(11)} = \omega_j^T(s) \widehat{\Xi}_j \omega_j(s),\]

\[\omega_j(s) = \left[ \zeta_j^T(s) \hat{\xi}_j^T(s) \right]^T \]

(53)

where \(\widehat{\Xi}_j\) corresponds to \(\widehat{\Pi}_j\) given by (19) by Schur complements. If \(\widehat{\Pi}_j < 0\), it is readily seen from (53) by Schur complements that

\[z_j^T(s)z_j(s) - \gamma_j^2 w_j^T(s)w_j(s) + \dot{V}_j(s)_{(11)} < 0 \]

for arbitrary \(s \in [t, \infty)\), which implies for any \(w_j(t) \in L_2(0, \infty) \neq 0\) that \(J_j < 0\) or equivalently

\[J = \sum_{j=1}^{n_z} J_j < 0.\]

This in turn leads to \(||z_j(t)||_2 < \gamma_j ||w_j(t)||_2\) for all \(j = 1, \ldots, n_z\).

**Computing the gains**

To compute that the feedback gains, we apply Schur complements and rewrite \(\widehat{\Xi}\) as

\[\widehat{\Pi}_j = \begin{bmatrix} \widehat{\Pi}_{oj} & \widehat{\Pi}_{cj} & \widehat{\Pi}_{wj} \\ \cdot & -\widehat{\Pi}_{sj} & 0 \\ \cdot & -\widehat{\Pi}_{wj} & \cdot \end{bmatrix} < 0 \]

(54)
Then we define $\delta_j = \phi_j^{-1}$, $\kappa_j = \psi_j^{-1}$ and

$$X_j = P_j^{-1} = \text{diag} \left[ X_{1j} \; \; X_{3j} \right]$$

Applying the congruent transformation

$$T_j = \text{diag} \left[ X_j \; \; X_j \; \; X_j \; \; I_j \; \; I_j \; \; I_j \; \; I_j \right]$$

along with the linearizations

$$\Lambda_{1j} = X_j Q_j X_j, \; \; \Lambda_{2j} = X_j W_j X_j, \; \; \Lambda_{3j} = X_j S_j X_j,$$

$$\Lambda_{4j} = X_j Z_j X_j, \; \; \Lambda_{5j} = X_j R_j X_j, \; \; \gamma_{2j} = B_{faj} X_3,$$

$$\Lambda_{7j} = X_j G_{1j}^t \Phi_j + \Gamma_j + B_j K_{oij}, \; \; \Pi_{fe} = X_j C_{d1j}^t B_{faj}^t$$

$$\gamma_{1j} = A_{fj} X_3$$

Using the algebraic matrix inequalities $-W_j^{-1} \preceq -2X_j + \Lambda_{2j}$, $-S_j^{-1} \preceq -2X_j + \Lambda_{3j}$ in addition to the matrix definitions (20), we obtain LMI (19) by Schur complements. This concludes the proof.

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